A REMARK ON ABSOLUTE CESÀRO SUMMABILITY OF FOURIER SERIES

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Recently T. Tsuchikura [3] has proved that the summability $|C, r|$ of the Fourier series of $L^p(0, 2\pi)$, $(1<p\leqq 2)$, is of local property for $r>1/p$. On the other hand, Bosanquet and Kestelman [1] showed that the summability $|C, 1|$ is not of local property for the Fourier series of integrable functions. In the present paper we shall prove the following result.

THEOREM. *The summability* $|C,1/p|$ *at a point of the Fourier series of functions of* $L^p(0, 2\pi)$, $(p\geqq 1)$, *is not of local property.*

Proof. For $p = 1$, the theorem is the result of Bosanquet and Kestelman. Let $p > 1$, then it suffices to show the existence of a function $f(x)\in L^p(0, 2\pi)$ which is zero in the interval $1 < x < 2\pi$ and the Fourier series of which is not summable at the point $x = \pi$. Suppose that the Fourier series of any function $f(x)\in L^p(0, 2\pi)$ which vanishes in $1 < x < 2\pi$ is summable $|C, 1/p|$ at $x = \pi$, then we have [2] that

$$(1)\qquad \sum_{n=1}^{\infty}\frac{|a_n\cos n\pi + b_n \sin n\pi|}{n^{1/p}}=\sum_{n=1}^{\infty}\frac{|a_n|}{n^{1/p}}=\frac{1}{\pi}\sum_{n=1}^{\infty}\left|\int_0^1 f(x)\frac{\cos nx}{n^{1/p}}\,dx\right|<\infty,$$

where

$$(2)\qquad f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx);$$

that is, for every function of $L^p(0, 1)$, it follows that

$$(3)\qquad \sum_{k=1}^{\infty}\left|\int_0^1 f(x)\frac{\cos nx}{n^{1/p}}\,dx\right|<\infty.$$

This shows that the sequence $\sum_{k=1}^{n}\frac{\cos kx}{k^{1/p}}$ $(n=1, 2,\cdots)$ is weakly convergent as a sequence of linear functionals on $L^p(0, 1)$, hence there exists a constant M by the theorem of Banach-Steinhaus such that

$$(4)\qquad \left\{\int_0^1\left|\sum_{k=1}^{n}\frac{\cos kx}{k^{1/p}}\right|^{p/(p-1)}dx\right\}^{1-1/p}\leqq M,\quad (n=1,2,\cdots).$$

Since the series $\sum_{n=1}^{\infty}\frac{\cos nx}{n^{1/p}}$ is covergent save for $x = 0$, we have by Fatou's lemma that

$$(5)\qquad \left\{\int_0^1\left|\sum_{n=1}^{\infty}\frac{\cos nx}{n^{1/p}}\right|^{p(p-1)}dx\right\}^{1-1/p}\leqq M.$$

It is known [4, p.117], however, that

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{1/p}} \sim x^{\frac{1}{p}-1} = x^{-(p-1)/p} \quad (x\to 0), \tag{6}$$

and this contradicts the inequality (5). Thus the proof of the theorem is completed.

REMARK. Using the equality [4, p. 2]

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = \frac{1}{2}\log\frac{1}{2(1-\cos x)}, \quad (x \neq 0)$$

the method of our proof is valid for $p = 1$.

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