THE COHOMOTOPY AND UNIFORM COHOMOTOPY GROUPS

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Introduction. The investigations of homotopy classes of maps of an n-sphere S^n into a given topological space X lead us to the concept of homotopy group which was introduced by W. Hurewicz. Dually to this, by the consideration of homotopy classes of maps of X into S^n we arrive at the concept of cohomotopy group which was introduced by K. Borsuk [1][1]. In 1949, E. Spanier [10] proved that if (X,A) is a compact pair with $\dim(X-A)<2n-1$, then the homotopy classes of maps of (X, A) into (S^n, p) form an abelian group, and he investigated these cohomotopy groups in detail.

Homotopy groups can be defined for any topological space, while for cohomotopy groups the dimensional restriction stated above seems to be essential. But the condition of compactness is undesirable. Therefore, it will be interesting to show that cohomotopy groups are able to be defined for a wider class of spaces.

The purpose of this paper is to generalize the theory due to E. Spanier concerned with cohomotopy groups to the case of paracompact spaces.

The generalizations are able to do in two directions. One of these is based on the concept of usual homotopy and the other is based on that of uniform homotopy which was used by C. H. Dowker [3]. More precisely, we shall show that if (X, A) is a paracompact pair with $\dim (X-A)<2n-1$, then both homotopy classes $\pi^n(X, A)$ and uniform homotopy classes $\pi^n_u(X, A)$ of maps of (X, A) into (S^n, p) form abelian groups. Thus, for a paracompact pair, we have two kinds of homomorphism sequences which will be called cohomotopy and uniform cohomotopy sequences respectively. These results are obtained in §§ 1-8.

In § 9 we shall establish the general limiting process. This will be an important tool, because, by means of this process, many results with respect to complexes will be extended to cases of paracompact spaces. By application of this process to the cohomotopy sequence we have the decomposition theorems (Theorem 9. 5) of cohomotopy and u-cohomotopy sequences into direct spectrums. Thus the exactness of these sequences which is the most important property is reduced to that of cohomotopy sequence for any simplicial complex. The proof of exactness is stated in § 10.

The relations between cohomotopy sequence and Čech cohomology sequence

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1). Numbers in brackets refer to the references cited at the end of this paper.

based on infinite coverings are discussed in § 11. As direct consequences from these we have the generalized Hopf's classification theorems (Theorem 11.3 and 11.4).

1. Preliminaries. Let $\mathfrak{X}$ be the set of sequences of real numbers $y=(y_i)$ ($i=1, 2, \cdots\cdots$), where y's are zero except for a finite set of integers i. $\mathfrak{X}$ is metrized by

$$\text{dis}\,(y, y') = \left(\sum_{i=1}^{\infty} (y_i - y_i')^2\right)^{1/2}.$$

DEFINITION 1.1. The sets below are defined by the corresponding conditions on the right.

$$S^n = (y \in \mathfrak{X} \mid y_i = 0 \text{ for } i > n+1, \sum_{1 \leqq i \leqq n+1} y_i^2 = 1),$$

$$E^{n+1} = (y \in \mathfrak{X} \mid y_i = 0 \text{ for } i > n+1, \sum_{1 \leqq i \leqq n+1} y_i^2 \leqq 1),$$

$$E_+^{n+1} = (y \in S^n \mid y_{n+1} \geqq 0), \quad E_-^{n+1} = (y \in S^n \mid y_{n+1} \leqq 0),$$

$$E_+^0 = p = (1, 0, \cdots\cdots 0, \cdots\cdots), \quad E_-^0 = \bar{p} = (-1, 0, \cdots\cdots 0, \cdots\cdots).$$

DEFINITION 1.2. A pair (X, A) is a topological space and a closed subset A. If X is paracompact, then A is also, and such a pair (X, A) is called a paracompact pair. A map f of a pair (X, A) into a pair (Y, B) is a continuous function from X to Y which maps A into B, and will be denoted by

$$f : (X, A) \longrightarrow (Y, B).$$

Let I denote the closed real number interval $0 \leqq t \leqq 1$ with the customary topology. If $f, g : (X, A) \longrightarrow (Y, B)$, then f is homotopic to g (denoted by $f \simeq g$), if there is a map

$$F : (X \times I, A \times I) \longrightarrow (Y, B)$$

such that

$$\left.\begin{aligned} F(x, 0) &= f(x) \\ F(x, 1) &= g(x) \end{aligned}\right\} \quad \text{for all } x \in X.$$

If $f(x) = g(x)$ for all $x \in A$, then f is homotopic to g relative to A ($f \simeq g$ rel. A), if f is homotopic to g and if the homotopy

$$F : (X \times I, A \times I) \longrightarrow (Y, B)$$

can be chosen so that

$$F(x, t) = f(x) \quad \text{for all } x \in A, t \in I.$$

It is obvious that homotopy and homotopy relative to A are both proper equivalence relations so that they divide the set of continuous maps of (X, A) into (Y, B) into disjoint equivalence classes called homotopy classes or homotopy classes relative to A, respectively. If $f : (X, A) \longrightarrow (Y, B)$, then $\{f\}$ denotes its homotopy class and $\{f\}_A$ denotes its homotopy class relative to A.

DEFINITION 1.3. Let (Y, B) be a metric pair, i. e. Y is a metric space. For maps $f, g: (X, A) \to (Y, B)$, f is uniformly homotopic to g (denote by $f \overset{u}{\simeq} g$) if there exists a homotopy F between f and g which satisfies the following condition: for a given real number $\varepsilon > 0$, there is a number $\delta > 0$ such that if $|t - t'| < \delta$ then

$$\operatorname{dis}(F(x, t), F(x, t')) < \varepsilon$$

holds for any point $x \in X$. Such F is called a u-homotopy.

Uniform homotopy is also proper equivalence. For a map $f: (X, A) \to (Y, B)$, $\{f\}_u$ denote its uniform homotopy class.

Throught the present paper all spaces considered will be assumd to be normal space.

2. Some properties of dimension. DEFINITION 2. 1. A covering of a space X by open sets is said to be of order less than or equal to n, if no collection of $n+2$ distinct elements of this covering have a point in common. A space X is said to have dimension less than or equal to n (denoted by $\dim X \leqq n$), if every finite open covering of X has a finite open refinement whose order is less than or equal to n.

LEMMA 2.1. *If* $\dim X \leqq n$ *and* A *is closed in* X, *then* $\dim A \leqq n$.

For a proof see [6, p. 14].

LEMMA 2.2. *If* X *is a paracompact space, and* Y *is a compact space with* $\dim X \leqq n$ *and* $\dim Y \leqq m$, *then* $\dim (X \times Y) \leqq n + m$.

This is proved in [7].

3. Nerves and canonical maps. DEFINITION 3. 1. Let $\{U_\alpha\}$ be a locally finite covering[2] of a space X. By the nerve of the covering $\{U_\alpha\}$ we mean a simplicial complex which is a geometric realization of $\{U_\alpha\}$. A simplicial complex is topologized by the weak topology in the sence of J. H. C. Whitehead [11, §15 ; 12, §5]. Let K be the nerve of $\{U_\alpha\}$. Then the vertices $\{u_\alpha\}$ of K correspond with the sets $\{U_\alpha\}$ in one-to-one, and finite vertices $u_\alpha, \cdots, u_\gamma$ of K form a simplex in K if and only if $U_\alpha \cap \cdots \cap U_\gamma \neq \phi$. Let A be a closed subset of X. Let L denote a subcomplex of K consisting of simplices $u_\alpha \cdots u_\gamma$ of K such that $U_\alpha \cap \cdots \cap U_\gamma \cap A \neq \phi$. Then L may be regarded as the nerve of the covering $\{U_\alpha \cap A\}$ of A.

DEFINITION 3. 2. Let $\mathfrak{U} = \{U_\alpha\}$ be a locally finite covering of X, and K the nerve of $\mathfrak{U}$, and L be the subcomplex corresponding to a closed subset $A \subset X$. A mapping $\phi : (X, A) \to (K, L)$ is said to be canonical if, for each

2) By a covering we mean an open covering and by a locally finite covering mean a neighborhood finite covering in the sense of Lefschetz [6, p.13].

vertex u of K, the inverse image of the open star of u with respect to K under the map ϕ is contained in the corresponding element U of $\mathfrak{U}$.

LEMMA 3.3. *For a locally finite covering* $\mathfrak{U}$ *of* X, *there exists a canonical map* $\phi:(X, A) \to (K, L)$.

For a proof see [7, Lemma 2].

Let $f:(X, A) \to (Y, B)$ be a map of a pair (X, A) into a simplicial pair (Y, B) (i. e. Y is a simplicial complex and B is a closed subcomplex of Y). Let $\mathfrak{U}$ be the covering of Y consisting of the point sets open stars of the vertices of Y. Then $\mathfrak{U}' = f^{-1}(\mathfrak{U})$ is the covering of X consisting of inverse images of elements of $\mathfrak{U}$. If u is a vertex of Y its star will be denote by U and $f^{-1}(U)$ will be denote by U'. If $\mathfrak{U}''$ is a locally finite refinement of $\mathfrak{U}'$, choose for each vertex u'' of the nerve K of $\mathfrak{U}''$ a vertex u of Y such that U'' is contained in U'. The correspondence $u'' \to u$ is a simplicial map of K into Y and if it is extended linearly over the simplexes of K, is a continuous map

$$\tau:(K, L) \to (Y, B).$$

LEMMA 3.4. *Using the above notation* $h:(X, A) \to (K, L)$ *is canonical then* $f \simeq \tau h$. *If* Y *is a finite complex*[2], *then* $f \overset{u}{\simeq} \tau h$.

This lemma is easily proved (cf. [9, Lemma 3.4]).

LEMMA 3.5. *Let* (X, A) *be a pair such that* $\dim F \leqq n$ *for any closed subset* $F \subset X-A$. *Given a continuous map* $f:(X,A) \to (Y,B)$ *into a finite simplicial pair* (Y, B), *and given an open* $(n+1)$*-simplex* σ *of* Y *whose closure doesn't meet* B, *there is a map* $g:(X, A) \to (Y, B)$ *such that* $f \overset{u}{\simeq} g$ *rel.* $f^{-1}(Y-\sigma)$ *and* $g(X) \subset Y-\sigma$.

PROOF. Notice that in the proof of Lemma 3.5 of [10], as a homotopy $F:(M \times I, N \times I) \to (\bar{\sigma}, \dot{\sigma})$ between f' and $f \mid N$ relative to N we can choose a u-homotopy F relative to N. Then the proof of the lemma is quite parallel with that of Lemma 3.5. [10].

LEMMA 3.6. *Let* (X, A) *be a paracompact pair with* $\dim(X-A) \leqq n$. *If* F *is any closed subset of* $X \times I - A \times I$, *then* $\dim F \leqq n+1$.

PROOF. Using of our Lemma 2.3 instead of Lemma 2.3 [10], this lemma is proved by the same fashion as the proof of Lemma 3.6 [10].

4. Deformations in spheres. In order to define the group operaton for uniform cohomotopy groups, it is necessary to rewrite the lemmas in § 4 of [10] by means of uniform homotopy.

DEFINITION 4.1. A subset A of a space X is called a deformation retract of X, if there is a homotopy

$$F : (X \times I,\ A \times I) \longrightarrow (X,\ A)$$

such that

$$\left.\begin{array}{l} F(x,0) = x \\ F(x,i) \in A \end{array}\right\} \text{ for all } x \in X$$

$$F(x,t) = x \qquad \text{for all } x \in A,\ t \in I.$$

In this case F is termed a retracting deformation of X onto A.

If X is a metric space, and a retracting deformation F of X onto A is a uniform homotopy, then A is said to be a uniform deformation retract (u-deformation retract) and such F is called a retracting uniform deformation (retracting u-deformation).

LEMMA 4.2. *Let A be an u-deformation retract of a metric space X and let $f : (X, A) \to (Y, B)$ be an open and uniformly continuous map of (X, A) onto a metric pair (Y, B) which maps $X-A$ homeomorphically onto $Y-B$.*

Then B is a u-deformation, retract of Y.

PROOF. Let $g : (X \times I,\ A \times I) \to (Y \times I,\ B \times I)$ be the map defined by $g(x, t) = (f(x), t)$. Since f is an open and uniformly continuous map, g is also. It is clear that g maps $X \times I - A \times I$ homeomorphically onto $Y \times I - B \times I$. Let F be a retracting deformation of X onto A. Define a map

$$F' : (Y \times I,\ B \times I) \to (Y,\ B)$$

by $\qquad F'(y, t) = (fFg^{-1})(y, t).$

Then F' is single valued and continuous, and this is a retracting deformation of Y onto B (cf. Proof of Lemma 4.2 [9]).

Since f, g are uniformly continuous and F is uniform homotopy, it is clear that F' is a uniform homotopy.

From Lemma 4. 2 we have the following corollaries. Proofs of these are similar with that of Corollary 4.3 and 4.4 [10], and so we shall omit the proofs.

COROLLARY 4. 3. *In the product space $S^n \times S^n$, the subset $(S^n \times p) \cup (p \times S^n)$ is a u-deformation retract of $S^n \times S^n - (\bar{p}, \bar{p})$ if $n \geqq 1$.*

COROLLARY 4. 4. *In the product space $S^n \times S^n \times S^n$, the subset*

$$(S^n \times S^n \times p) \cup (S^n \times p \times S^n) \cup (p \times S^n \times S^n)$$

is a u-deformation retract of $S^n \times S^n \times S^n - (\bar{p}, \bar{p}, \bar{p})$ if $n \geqq 1$.

5. Definitions of additions. Let α, β be two maps of a pair (X, A) into a pair (Y, y) consisting of a space Y and a point $y \in Y$. Then $\alpha \times \beta$ will denote the map of (X, A) into $(Y \times Y, (y, y))$ defined by

$$(\alpha \times \beta)(x) = (\alpha(x), \beta(x)).$$

DEFINITION 5.1 Let $f : (X, A) \to (Y \times Y, (y, y))$. A homotopy

$$F : (X \times I,\ A \times I) \to (Y \times Y,\ (y, y))$$

will be called a normalizing homotopy for f, if

$$\left.\begin{array}{l} F(x, 0) = f(x) \\ F(x,1) \in (Y\times y) \cup (y\times Y) \end{array}\right\} \text{ for all } x \in X.$$

The map $f' : (X,A) \to [(Y\times y) \cup (y\times Y), (y,y)]$ defined by

$$f'(x) = F(x, 1)$$

is called a normalization of f.

If Y is a compact metric space and the homotopy F in the above definition is a uniform homotopy, then F will be called a normalizing uniform homotopy (normalizing u-homotopy) for f and the map f' is called a uniform normalization (u-normalization) of f.

In the sequel $Y \vee Y$ will denote the space $(Y\times y) \cup (y\times Y)$.

Let $\qquad \Omega : [Y\vee Y, (y,y)] \to (Y,y)$

be defined by

$$\begin{array}{ll} \Omega\,(y',y) = y' & \text{for } (y',y) \in Y\times y, \\ \Omega\,(y,y'') = y'' & \text{for } (y,y'') \in y\times Y. \end{array}$$

DEFINITION 5.2. Let α, β: $(X, A) \to (Y, y)$ and assume that $\alpha\times\beta$: $(X,A) \to (Y\times Y, (y,y))$ can be normalized. Let f: $(X,A) \to (Y\vee Y, (y,y))$ be a normalization of $\alpha\times\beta$. The sum with respect to f (denoted by $\alpha<f>\beta$) is defined to be the composite map $\quad \alpha<f>\beta = \Omega f$.

Let Y is a compact metric space and $\alpha\times\beta : (X,A) \to (Y\times Y, (y,y))$ can be uniformly normalized. Let $f : (X,A) \to (Y\vee Y, (y,y))$ be a u-normalization of $\alpha\times\beta$. The u-sum with respect to f (denote by $\alpha<f>_u\beta$) is defined to be the composite map $\alpha<f>_u\beta = \Omega f$.

6. The cohomotopy and uniform cohomotopy groups. LEMMA 6.1 and 6.2 in [10] are easily modified to following forms.

LEMMA 6.1. *Let (X, A) be a pair with* $\dim F < 2n$ *for any closed set $F \subset X\text{-}A$. For any map $f : (X,A) \to (S^n\times S^n, (p,p))$, there exists a u-normalization g of f such that $f \overset{u}{\simeq} g$ rel. $f^{-1}(S^n\vee S^n)$.*

LEMMA 6.2. *Let (X,A) be a paracompact pair with* $\dim(X-A) < 2n-1$.

If α, β, α', β' : $(X,A) \to (S^n,p)$ with $\alpha \overset{u}{\simeq} \alpha'$ and $\beta \overset{u}{\simeq} \beta'$ and if $g : (X,A) \to (S^n\vee S^n, (p,p))$ is a u-normalization of $\alpha\times\beta$ and $g' : (X,A) \to (S^n\vee S^n, (p,p))$ is a u-normalization of $\alpha'\times\beta'$, then $\Omega g \overset{u}{\simeq} \Omega g'$. If $\alpha\simeq\alpha'$, $\beta\simeq\beta'$ then for any normalizations g, g' of $\alpha\times\beta$, and $\alpha'\times\beta'$ we have $\Omega g \simeq \Omega g'$.

THEOREM 6.3. *If (X,A) is a paracompact pair with* $\dim(X-A) < 2n-1$, *the homotopy classes $\{\alpha\}$ of maps of (X,A) into (S^n,p) form an abelian group with the law of composition $\{\alpha\} + \{\beta\} = \{\alpha<f>\beta\}$, where f is an arbitrary normalization of $\alpha\times\beta$.*

A proof of this theorem is quite similar with that of Theorem 6.3 in [10].

THEOREM 6.4 *If (X,A) is a paracompact pair with* $\dim(X-A) < 2n-1$, *the uniform homotopy classes $\{\alpha\}_u$ of maps α of (X,A) into (S^n,p) form an abelian group with the law of compcosition* $\{\alpha\}_u+\{\beta\}_u = \{\alpha <f>_u \beta\}_u$, *where f is an arbitrary u-normalization of $\alpha\times\beta$.*

This theorem will be proved, if, each lemma used in the proof of Theorom 6.3 of [10] is replaced by the corresponding lemma in the paper.

The groups whose existences were proved in Theorem 6.3 and 6.4 are called the n-th cohomotopy group and uniform cohomotopy group and these are denoted by $\pi^n(X, A)$ and $\pi^n_u(X,A)$ respectively.

REMARK 1. In the following, we shall use $\pi^n(X,A)$ and $\pi^n_u(X, A)$ to denote the sets of homotopy classes and uniform homotopy classes of maps of (X,A) into (S^n,p) respectively, even if (X,A) doesn't satisfy the hypothesis of Theorem 6.3. Hence $\pi^n(X,A)$ and $\pi^n_u(X,A)$ are sets of homotopy classes and uniform homotopy classes and are defined for any pair (X,A).

If (Y,A) is a compact pair, then $\pi^n(X,A)$ and $\pi^n_u(X,A)$ are the same set. It is obvious that $\pi^n(X,A)$ and $\pi^n_u(X,A)$ for a paracompact pair (X,A) with $\dim(X-A) < 2n-1$ are both generalizations of Borsuk's cohomotopy group for a compact pair.

NOTATIONS. $\pi^n(X)$ and $\pi^n_u(X)$ will denote $\pi^n(X, 0)$ and $\pi^n_u(X, 0)$ respectively.

THEOREM 6.5. *If P is a space consisting of a single point, then* $\pi^n(P) = 0$, $\pi^n_u(P) = 0$ *for* $n \geqq 1$.

7. The induced homomorphisms. Let (X,A) and (Y,B) be two pairs and let $f: (X,A)\to(Y,B)$ be continuous. If $\alpha: (Y,B)\to(S^n,p)$, then $\alpha f: (X,A)\to(S^n,p)$, and if $\alpha\simeq\beta$ (or $\alpha \overset{u}{\simeq} \beta$), then $\alpha f\simeq\beta f$ (or $\alpha f \overset{u}{\simeq} \beta f$). Hence, f induces two mappings

$$f^{\#}: \pi^n(Y,B)\to\pi^n(X,A),$$
$$f^{\#}_u: \pi^n_u(Y,B)\to\pi^n_u(X,A)$$

defined by $f^{\#}\{\alpha\} = \{\alpha f\}$ for $\{\alpha\}\in\pi^n(Y,B)$ and by $f^{\#}_u\{\alpha\}_u = \{\alpha f\}_u$ for $\{\alpha\}_u \in \pi^n_u(Y,B)$ respectively.

$f^{\#}$ and $f^{\#}_u$ are to be considered here as merely set transformations as no dimension restrictions have been imposed on either pair (X, A) or (Y, B).

REMARK 2. Let K be a CW-complex [12] consisting of at most $(2n-2)$-dimensional cells. Then, by [7, Lemma 5], $\dim K < 2n-1$ and also K is paracompact [9]. Therefore, for any closed subset L of K, $\pi^n(K, L)$ and $\pi^n_u(K, L)$ are abelian groups.

LEMMA 7.1. *If (X,A) and (Y,B) are paracompact pairs with* $\dim(X-A)$

$< 2n-1$ *and* $\dim(Y-B) < 2n-1$, *then* $f^{\#}: \pi^n(Y, B) \to \pi^n(X, A)$ *and* $f^{\#}_u$: $\pi^n_u(Y, B) \to \pi^n_u(X, A)$ *are both homomorphisms.*

THEOREM 7.2. *If* $f: (X, A) \to (X, A)$ *is the identity, then* $f^{\#}$ *and* $f^{\#}_u$ *are the identity transformations on* $\pi^n(X, A)$ *and* $\pi^n_u(X, A)$ *respectively.*

THEOREM 7.3. *If* $f: (X, A) \to (Y, B)$ *and* $g: (Y, B) \to (Z, C)$, *then* $(fg)^{\#} = f^{\#}g^{\#}$, *and* $(gf)^{\#}_u = f^{\#}_u g^{\#}_u$.

THEOREM 7.4. *If* $f, g: (X, A) \to (Y, B)$ *are homotopic, then* $f^{\#} = g^{\#}$.

THEOREM 7.4′. *If* (Y, B) *a compact pair and* $f, g: (X, A) \to (Y, B)$ *are u-homotopic,*[3] *then* $f^{\#}_u = g^{\#}_u$.

If (X, A) is a pair, let (X_A, q_A) be the pair consisting of a space X_A obtained from by identifing A to a point q_A, and the point q_A.
Let $f: (X, A) \to (X_A, q_A)$ be the natural map which maps $Y-A$ homeomorphically onto $X_A - q_A$.

THEOREM 7.5. *With the above notation,* $f^{\#}$ *is a* 1-1 *map of* $\pi^n(X_A, q_A)$ *onto* $\pi^n(X, A)$. *Also* $f^{\#}_u$ *is a* 1-1 *map of* $\pi^n_u(X_A, q_A)$ *onto* $\pi^n_u(X, A)$.

THEOREM 7.6. *Let* V *be an open set contained in* A *and* $j: (X-V, A-V) \to (X, A)$ *be the identity map. Then* $j^{\#}$ *is a* 1-1 *map of* $\pi^n(X, A)$ *onto* $\pi^n(X-V, A-V)$, *and also* $j^{\#}_u$ *is a* 1-1 *map of* $\pi^n_u(X, A)$ *onto* $\pi^n_u(X-V, A-V)$.

The above results are easily proved and so we shall omit the proofs (cf. [10, § 7]).

8. The coboundary operators. In this section set transformations from $\pi^n(A)$ into $\pi^{n+1}(X, A)$ and from $\pi^n_u(A)$ into $\pi^{n+1}_u(X, A)$ will be defined.

Let (X, A) be a pair and let $\alpha: A \to S^n$. Then there exists an extension $\tilde{\alpha}$ of α which maps (X, A) into (E^{n+1}_+, S^n). Let $\psi: (E^{n+1}_+ \times I, S^n \times I) \to (S^{n+1}, E^{n+1}_-)$ be defined so that if $\psi_t = \psi | (E^{n+1}_+ \times t)$, then

$\psi_0 =$ identity map of (E^{n+1}_+, S^n) into (S^{n+1}, E^{n+1}_-)

$\psi_1: (E^{n+1}_+, S^n) \to (S^{n+1}, p)$ is a homeomorphism of $E^{n+1}_+ - S^n$ onto $S^{n+1} - p$.

LEMMA 8.1 *If* $\alpha_1, \alpha_2: A \to S^n$ *are homotopic and if* $\tilde{\alpha}_1, \tilde{\alpha}_2: (X, A) \to (E^{n+1}_+, S^n)$ *are extensions of* α_1 *and* α_2 *respectively, then* $\psi_1 \tilde{\alpha}_1 \simeq \psi_1 \tilde{\alpha}_2$. *Moreover, if* α_1, α_2 *are uniformly homotopic, then* $\psi_1 \tilde{\alpha}_1 \overset{u}{\simeq} \psi_1 \tilde{\alpha}_2$.

PROOF. Let $F: ([(X \times 0) \cup (A \times I) \cup (X \times I)], A \times I) \to (E^{n+1}_+, S^n)$ be defined by

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3) See [8; p. 84].

$$\left.\begin{array}{l} F(x, 0) = \widetilde{\alpha_1}(x) \\ F(x, 1) = \widetilde{\alpha_2}(x) \end{array}\right\} \quad \text{for all } x \in X,$$

$F | A \times I$ = a homotopy between α_1 and α_2 in S^n.

Then, F has an extension $F' : (X \times I, A \times I) \to (E_+^{n+1}, S^n)$ and $\psi_1 F'$ is a homotopy between $\psi_1\widetilde{\alpha_1}$ and $\psi_1\widetilde{\alpha_2}$.

Let $\alpha_1 \overset{u}{\simeq} \alpha_2$. We define a map $F: ([(X \times 0) \cup (A \times I) \cup (X \times I)], A \times I) \to (E_+^{n+1}, S^n)$ by taking

$$F(x, 0) = \widetilde{\alpha_1}(x)$$
$$F(x, 1) = \widetilde{\alpha_2}(x)$$

$F | A \times I$ = a u-homotopy between α_1 and α_2 in S^n.

Let $\phi : Z \to \beta(X)$ be the Čech compactification of X ([2]), and set $\overline{\phi(A)} = \beta(A)$. Then, by [3, Proof of Theorem 9.3], there exists a map G: $([(\beta(X) \times 0) \cup (\beta(A) \times I) \cup (\beta(X) \times 1)], \beta(A) \times I) \to (E_+^{n+1}, S^n)$ such that $G(\phi(x), t) = F(x, t)$ $(x \in (X \times 0) \cup (A \times I) \cup (X \times 1))$. By the homotopy extension theorem G has an extension $G' : (\beta(X) \times I, \beta(A) \times I) \to (E_+^{n+1}, S^n)$. We put $F'(x, t) = G'(\phi(x), t)$. Then F' is an extension of F. Since $\beta(X)$ is compact, G' is a uniform homotopy, hence F' is also uniform homotopy. Since ψ_1 is uniformly continuous, $\psi_1 F'$ is a uniform homotopy between $\psi_1\widetilde{\alpha_1}$ and $\psi_1\widetilde{\alpha_2}$, i. e. $\psi_1\widetilde{\alpha_1} \overset{u}{\simeq} \psi_1\widetilde{\alpha_2}$.

Definition 8. 2 The coboundary operator Δ is a set transformation of $\pi^n(A)$ into $\pi^{n+1}(X, A)$ defined by

$$\Delta\{\alpha\} = \{\psi_1\widetilde{\alpha}\} \text{ for } \{\alpha\} \in \pi^n(A),$$

where $\widetilde{\alpha}: X \to E_+^{n+1}$ is an extension of α.

The uniform coboundary operator Δ_u is a set transformation of $\pi_u^n(A)$ into $\pi_u^{n+1}(X, A)$ defined by

$$\Delta_u\{\alpha\}_u = \{\psi_1\widetilde{\alpha}\}_u \quad \text{for } \{\alpha\}_u \in \pi_u^n(A),$$

where $\widetilde{\alpha}: X \to E_+^{n+1}$ is an extension of α.

Lemma 8. 1 shows that $\{\psi_1\widetilde{\alpha_1}\}$ and $\{\psi_1\widetilde{\alpha}\}_u$ are indepedent of the choice of $\alpha \in \{\alpha\}$ and $\alpha \in \{\alpha\}_u$ and of the extension $\widetilde{\alpha}$, hence are uniquely determined by $\{\alpha\}$ and $\{\alpha\}_u$ respectively.

We have following results and proofs of these are quite parallel with that of Lemma 8. 3 and Theorem 8. 4 [10].

Lemma 8. 3. *If (X, A) is a paracompact pair with* $\dim A < 2n-1$ *and* $\dim(X-A) < 2n-1$, *then* $\Delta: \pi^n(A) \to \pi^{n+1}(X, A)$ *and* $\Delta_u : \pi_u^n(A) \to \pi_u^n(X, A)$ *are both homomorphisms.*

Theorem 8. 4. *For any map $f: (X, A) \to (Y, B)$ commutativities hold in the diagrams*

$$\begin{array}{ccc} \pi^n(B) & \xrightarrow{\Delta} & \pi^{n+1}(Y,B) \\ {\scriptstyle (f\mid A)^\#}\downarrow & & \downarrow{\scriptstyle f^\#} \\ \pi^n(A) & \xrightarrow{\Delta} & \pi^{n+1}(X,A), \end{array} \qquad \begin{array}{ccc} \pi^n_u(B) & \xrightarrow{\Delta_u} & \pi^{n+1}_u(Y,B) \\ {\scriptstyle (f\mid A)^\#_u}\downarrow & & \downarrow{\scriptstyle f^\#_u} \\ \pi^n_u(A) & \xrightarrow{\Delta_u} & \pi^{n+1}_u(X,A). \end{array}$$

9. The limiting process. Let (X,A) be a pair, and (M,m) be a simplicial pair consisting of a simplicial complex M and its vertex m. Let $\mathfrak{U}_\nu$ be a locally finite covering of X, and denote by (K_ν, L_ν) the nerve of $\mathfrak{U}_\nu$.

LEMMA 9.1. *If $\phi, \phi' : (X,A) \to (K_\nu, L_\nu)$ are canonical, then $\phi \simeq \phi'$. In particular, $\mathfrak{U}_\nu$ is finite, then $\phi \overset{u}{\simeq} \phi'$.*

This lemma is easily proved (cf. [10, Lemma 13.1])

If $\mathfrak{U}_\mu$ is a locally finite refinement of $\mathfrak{U}_\nu$, (denote by $\mathfrak{U}_\mu > \mathfrak{U}_\nu$), let

$$T_{\mu\nu} : (K_\mu, L_\mu) \to (K_\nu, L_\nu)$$

be a projection from (K_μ, L_μ) to (K_ν, L_ν) (by a projection is meant a simplicial map $T_{\mu\nu}$ with the property that if $T_{\mu\nu}(u^i_\mu) = u^j_\nu$ then $U^i_\mu \subset U^j_\nu$). Such a map exists because $\mathfrak{U}_\mu$ is a refinement of $\mathfrak{U}_\nu$. If $\overline{T}_{\mu\nu}$ is another such projection, then it is easily verified that $T_{\mu\nu} \simeq \overline{T}_{\mu\nu}$ and if $\mathfrak{U}_\nu$ is finite, $T_{\mu\nu} \overset{u}{\simeq} \overline{T}_{\mu\nu}$.

Let $D(K_\mu, L_\mu; M, m)$ denote the homotopy classes of maps of (K_μ, L_μ) into (M, m). Then $T_{\mu\nu}$ and $\overline{T}_{\mu\nu}$ induce a unique transformation

$$T^\#_{\mu\nu} = \overline{T}^\#_{\mu\nu} : D(K_\nu, L_\nu; M, m) \to D(K_\mu, L_\mu; M, m).$$

If $\mathfrak{U}_\lambda > \mathfrak{U}_\mu > \mathfrak{U}_\nu$ and let $T_{\lambda\mu}, T_{\mu\nu}$ be projections, then $T_{\mu\nu} T_{\lambda\mu}$ is a projection of (K_λ, L_λ) into (K_ν, L_ν). Thus the system $\{D(K_\mu, L_\mu; M, m), T^\#_{\mu\nu}\}$ forms a direct spectrum, where $\{\mathfrak{U}_\mu\}$ is the totality of all locally finite coverings of X. Denote by $\overline{D}(X, A; M, m)$ the limit set of this spectrum. Let $\{\mathfrak{U}_\mu\}_F$ be the totality of all finite coverings of X. Let $\overline{D}_F(X, A; M, m)$ be the limit set of the spectrum $\{D(K_\mu, L_\mu, M, m); T^\#_{\mu\nu}\}$ based on $\{\mathfrak{U}_\mu\}_F$.

For each element $\mathfrak{U}_\mu \in \{\mathfrak{U}_\mu\}$, choose a canonical map

$$h_\mu : (X, A) \to (K_\mu, L_\mu).$$

Such a map exists by Lemma 3.3.

LEMMA 9.2. *If $h_\mu : (X,A) \to (K_\mu, L_\mu)$ is canonical and $T_{\mu\nu} : (K_\mu, L_\mu) \to (K_\nu, L_\nu)$ is a projection, then $T_{\mu\nu} h_\mu : (X, A) \to (K_\nu, L_\nu)$ is canonical, where $\mathfrak{U}_\mu > \mathfrak{U}_\nu$ are locally finite coverings of X.*

PROOF. See [10, Lemma 13.2].

COROLLARY 9.3. *h_ν is homotopic to $T_{\mu\nu} h_\mu$. If $\mathfrak{U}_\nu$ is finite, then h_ν is uniformly homotopic to $T_{\mu\nu} h_\mu$.*

This corollary immediately follows from Lemma 9.1 and 9.2.

If $u_\nu \in D(K_\nu, L_\nu;\ M, m)$, $[u_\nu]$ and $[u_\nu]_F$[4] denote the elements of $\overline{D}(X, A;\ M, m)$ and $\overline{D}_F(X, A;\ M, m)$ determined by u_ν.

If $u_\mu \in [u_\nu]$, then there is $\lambda > \mu, \nu$, such that $T^{\#}_{\lambda\nu}\ u_\nu = T^{\#}_{\lambda\mu}\ u_\mu$. By Corollary 13. 3,

$$h^{\#}_\nu\ u_\nu = (T_{\lambda\mu} h_\lambda)^{\#} u_\nu = h^{\#}_\lambda\ T^{\#}_{\lambda\mu}\ u_\nu.$$

Similarly $h^{\#}_\mu\ u_\mu = h^{\#}_\lambda\ T^{\#}_{\lambda\mu}\ u_\mu$. Hence $h^{\#}_\nu\ u_\nu = h^{\#}_\mu\ u_\mu$. Also, if $u_\mu \in [u_\nu]_F$, then we have $h^{\#}_{\nu,u}\ u_\nu = h^{\#}_{\mu,u}\ u_\mu$. Therefore, we can define

$$\overline{h}: \overline{D}(X, A;\ M, m) \to D(X, A;\ M, m).$$

and
$$\overline{h}_u: \overline{D}_F(X, A;\ M, m) \to D_u(X, A;\ M, m)$$

by $\overline{h}[u_\nu] = h^{\#}_\nu\ u_\nu$, and $\overline{h}_u[u_\nu]_F = h^{\#}_{\nu u}\ u_\nu$, where $D_u(Y, A;\ M, m)$ denotes the uniform homotopy classes of maps of (X, A) into (M, m).

THEOREM 9. 4. *If (X, A) is a paracompact pair, then $\overline{h}$ maps $\overline{D}(X, A;$ $M, m)$ 1-1 onto $D(X, A;\ M, m)$. $\overline{h}_u$ is a 1-1 transformation of $\overline{D}_F(X, A; M, m)$ onto $D_u(X, A;\ M, m)$, where M is finite complex, but the paracompactness of X is not assumed.*

PROOF. By Lemma 3. 4 it is obvious that $\overline{h}$ and $\overline{h}_u$ are onto. First, we prove that $\overline{h}$ is one-to-one.

Let $\overline{h}[u_\nu] = \overline{h}[v_\mu]$ so that $h^{\#}_\nu u_\nu = h^{\#}_\mu v_\mu$. Let $\{\alpha\} = u_\nu$, $\{\beta\} = v_\mu$. Then $\alpha\, h_\nu \simeq \beta\, h_\mu$. We have to prove that there exists $\lambda > \mu, \nu$ such that $\alpha\, T_{\lambda\mu} \simeq \beta\, T_{\lambda\nu}$. Therefore we can assume that α and β are simplicial maps. If α, β are not simplicial, then by [11; Theorem 36] there are simplicial approximations α', β' of α, β with respect to suitable subdivisions K_μ', K_λ' of K_μ, K_ν respectively. Let $\mathfrak{U}_\mu'$ be the covering of X consisting of the sets h_μ^{-1} (star u_μ), where star u_μ is the open star of a vertex $u_\mu \in K_\mu'$ with respect to K_μ'. Let $\mathfrak{U}_\nu'$ be the corresponding covering for K_ν' and h_ν. Since X is paracompact, there exist locally finite refinements $\mathfrak{U}_\mu''$ and $\mathfrak{U}_\nu''$ of $\mathfrak{U}_\mu'$ and $\mathfrak{U}_\nu'$ respectively. Let $\tau_\mu: (K_\mu'', L_\mu'') \to (K_\mu', L_\mu')$ and $\tau_\nu : (K_\nu'', L_\nu'') \to (K_\nu', L_\nu')$ be mappings corresponding to τ which is used in Lemma 3. 4. Then, by Lemma 3. 4, $\tau_\nu\, h_\nu'' \simeq h_\nu$, $\tau_\mu\, h_\mu'' \simeq h_\mu$, where (K_ν'', L_ν''), (K_μ'', L_μ'') are nerves of $\mathfrak{U}_\nu''$, $\mathfrak{U}_\mu''$ and h_μ'', h_ν'' are canonical maps. In this case, τ_ν and τ_μ are projections, and $\alpha'\, \tau_\nu\ h_\nu'' \simeq \beta'\, \tau_\mu\, h_\mu''$, and $\alpha'\, \tau_\nu$, $\beta'\, \tau_\mu$ are simplicial maps. Therfore we may assume that α and β are simplicial maps with respect to K_ν and K_μ respectively.

Now, let $\mathfrak{U}$ be the covering of X consisting of the open stars of vertices of K. Let E: $(X \times I,\ A \times I) \to (M, m)$ be a homotopy between $\alpha\, h_\nu$ and $\alpha\, h_\mu$. Then F^{-1} $(\mathfrak{U})$ covers X. Chose a locally finite covering $\mathfrak{U}_\lambda = \{U^\alpha_\lambda\}$ and finite coverings $\mathfrak{W}_\alpha = \{W^i_\alpha,\ i = 1, \cdots, \mathrm{h}(\alpha)\}$ corresponding to each suffix α such that

4). In this case it is always assumed that all coverings $\mathfrak{U}_\nu$ are finite and M is finite.

(a) $\mathfrak{U}_\lambda$ is a common refinement of $\mathfrak{U}_\mu$ and $\mathfrak{U}_\nu$.

(b) totality of products $U^a_\lambda \times W^i_a$ $(i = 1, \cdots, n(\alpha))$ forms a renfiement of $F^{-1}(\mathfrak{U})$. Since X is paracompact and I is compact, it is not difficult to show that such coverings $\mathfrak{U}_\lambda, \mathfrak{W}_a$ exist (see [7, p. 91]).

Let $\varepsilon(\alpha)$ be the Lebesgue's number ([6, p. 39]) of the covering $\mathfrak{W}_a$. Put $\Omega(n) = \{\alpha \mid \varepsilon(\alpha) > 1/2^n\}$ and let K^n_λ denote the subcomplex of the nerve (K_λ, L_λ) of $\mathfrak{U}_\lambda$ consisting of vertices u^a_λ such that $\alpha \in \Omega(n)$.

Then
$$K^1_\lambda \subset K^2_\lambda \subset \cdots \subset K^n_\lambda \subset \cdots,$$
and
$$\sum_{i=1}^{\infty} K^i_\lambda = K_\lambda.$$

Let $\alpha^* : (K_\lambda \times 0, L_\lambda \times 0) \to (M, m)$ and $\beta^* : (K_\lambda \times 1, L_\lambda \times 1) \to (M, m)$ be the maps defined by
$$\alpha^*(p, 0) = \alpha T_{\lambda\mu}(p), \quad \beta^*(p, 1) = \beta T_{\lambda\nu}(p) \quad (p \in K_\lambda).$$
We shall prove $\alpha^* \simeq \beta^*$. This proof is the induction on n.

Let n be a fixed integer, and we divide I up 2^n minor intervals $I^i_n = [i-1/2^n, i/2^n]$ $(i = 1, \cdots, 2^n)$. Put $\beta^0_n = \alpha^* | K^n_\lambda$, $\beta^{2^n+1}_n = \beta^* | K^n_\lambda$. For each set U^a_λ $(\alpha \in \Omega(n))$, each I^i_n is contained in a some element of $\mathfrak{W}_a$. Hence, by (b), each product $U^i_\lambda \times I^i_n$ is contained in a set F^{-1} (star m^{ai}), where m^{ai} is a vertex of M. Let $\beta^i_n(u^a) = m^{ai}$, then it is obvious that β^i_n $(i = 1, \cdots, 2^n)$ are simplicial maps of K^n_λ into M and $\beta^i_n(L^n_\lambda) = m$, where $L^n_\lambda = K^n_\lambda \cap L_\lambda$. Thus $\beta^i_n : (K^n_\lambda, L^n_\lambda) \to (M, m)$.

If $u^0, \cdots, u^p$ are vertices of a simplex of K, then $U^0 \cap \cdots \cap U^p \neq \phi$. It is obvious that
$$F^{-1}\left(\bigcap_{j=0}^{p} \mathrm{star}\,\beta^i_n(u^j) \cap \bigcap_{j=0}^{p} \mathrm{star}\,\beta^{i+1}_n(u^j)\right) \supset \left(\bigcap_{j=0}^{p} U^j \times I^i_n \bigcap_{j=0}^{p} U^j \times I^{i+1}_n\right), \quad \text{where}$$
$0 \leqslant i \leqslant 2^n$ and I^0_n, $I^{2^n+1}_n$ are to mean the point $t = 0$, $t = 1$ respectively. Since the set on right equals
$$\bigcap_{j=0}^{p} (U^j \times t_i) = (\bigcap_j U^j) \times t_i \neq \phi,$$
where $t_i = I^i_n \cap I^{i+1}_n = i/2^n$. Hence, for each $0 \leqslant i \leqslant 2^n$, $\beta^i_n(u^0), \cdots, \beta^i_n(u^p)$, $\beta^{i+1}_n(u^0)$, , $\beta^{i+1}_n(u^p)$ are vertices of a simplex of M. Thus $\beta^i_n(x)$ and $\beta^{i+1}_n(x)$ both belong to a closed simplex of M for any $x \in K$ and for each $0 \leqslant 1 \leqslant 2^n$. Moreover, if $x \in L^n_\lambda$, $\beta^i_n(x) = m$. Thus we can define a map $G^i_n : (K^n_\lambda \times I, L^n_\lambda \times L) \to (M, m)$ which maps $x \times I$ linearly onto the segment of M joining $\beta^i_n(x)$ to $\beta^{i+1}_n(x)$ $(x \in K^n_\lambda, 0 \leqslant i \leqslant 2^n)$. Define a map $G_n : (K^n_\lambda \times I, L^n_\lambda \times I) \to (M, m)$ by taking

$$G_n(x,t) = \begin{cases} G_n^i(x, 2t-i), \ \frac{i}{2^n} \leqslant t \leqslant \frac{i+1}{2^n}, \quad i = 0, 1, \cdots, 2^n\text{-}2, \\ G^{2^n-1}(x, 2^{n+1}(t-1)+1), \ \frac{2^n-1}{2^n} \leqslant t \leqslant \frac{2^n-1}{2^n} + \frac{1}{2^{n+1}} \\ G_n^{2^n}(x, 2^{n+1}(t-1)+1), \ \frac{2^n-1}{2^n} + \frac{1}{2^{n+1}} \leqslant t \leqslant 1. \end{cases}$$

Then G_n is a homotopy between $\alpha^*|K_\lambda^n$ and $\beta^*|K_\lambda^n$.

Next, in the same way we define $\beta_{n+1}^j: (K_\lambda^{n+1}, L_\lambda^{n+1}) \to (M, m)$ $(j=1, \cdots, 2^{n+1})$ under the following additional condition: if $j = 2i$, $j = 2i-1$ and if $\alpha \in \Omega$, then $\beta_{n+1}^j(u^a) = \beta_n^i(u^a)$. If $j = 2i$ or $j = 2i-1$, then $I_{n+1}^j \subset I_n^i$, therfore this restriction is consistent with the above definitions of β's. Thus β_{n+1}^j have been defined, hence, by the same way as the preceding, we have a homotopy $G_{n+1} : (K_\lambda^{n+1} \times I, L_\lambda^{n+1} \times I) \to (M, m)$ between $\alpha^*|K_\lambda^{n+1}$ and $\beta^*|K_\lambda^{n+1}$.

By the additional condition for β_{n+1}^j it is easily seen that $G_n \simeq G_{n+1}|K_\lambda^n \times I$. Hence, by [12, (J), p. 228], there exists a homotopy H_{n+1} between $\alpha^*|K_\lambda^{n+1}$ and $\beta^*|K_\lambda^{n+1}$ such that $H^{n+1}|K_\lambda^n \times I = H^n (\equiv G_n)$.

Starting with $H_1 = G_1$ it follows by induction on n that there exists a sequence of homotopies H_n: $(K_\lambda^n \times I, L_\lambda^n \times I) \to (M, m)$ between $\alpha^*|K_\lambda^n$ and $\beta^*|K_\lambda^n$ such that $H_{n+1}|K_\lambda^{n+1} = H^n (n = 1, 2, \cdots)$. Define a homotopy H: $(K_\lambda \times I, L_\lambda \times I) \to (M, m)$ by

$$H|K_\lambda^n \times I = H_n.$$

Then, by [12, (I), p. 228], H is continuous. Therefore H is a homotopy between α^* and β^*, i.e. $\alpha^* \simeq \beta^*$. Thus our assertion has been proved.

Next, we shall prove that, if M is finite, then $\overline{h_u}$ is one-to-one. Let $\overline{h_u}[u_\mu]_F = \overline{h_u}[v_\nu]_F$ so that $h_{\mu,u}^{\#} u_\mu = h_{\nu,u}^{\#} v_\nu$. Let $\{\alpha\} = u_\mu$, $\{\beta\} = v_\nu$. Then $\alpha\, h_\mu \overset{\mathfrak{U}}{\simeq} \beta\, h_\nu$. We have to prove that there exists a finite covering $\mathfrak{U}_\lambda$ such that $\mathfrak{U}_\lambda$ is a common refiniement of $\mathfrak{U}_\mu, \mathfrak{U}_\nu$ and $\alpha\, T_{\lambda\mu} \simeq \beta\, T_{\lambda\nu}$. To prove this we may assume without any loss of generality that $\mu = \nu$.

Let $\phi : X \to \beta(X)$ be the Čech compactification of X, and let $\beta(A) = \phi(A)$. Since A is closed, $\phi|A : A \to \beta(A)$ is also the compactification of A. For each open set u_μ^i of $\mathfrak{U}_\mu$ we put $V_\mu^i = \beta(X) - \phi(X - U_\mu^i)$. Then $\{V_\mu^i\}$ forms a covering $\mathfrak{U}_\mu$ of $\beta(X)$. (cf. [3, b), p. 229]).

Since K_μ is compact there exists the unique map h_μ^*: $(\beta(X), \beta(A)) \to (K_\mu, L_\mu)$ such that

$$h_\mu = h_\mu^* \phi.$$

Let $\mathfrak{W}$ be the covering consisting of open stars of vertices of K_μ and let us put $\mathfrak{V}_{\mu'} = h_\mu^{*-1}(\mathfrak{W})$. Let $\mathfrak{V}_\nu$ be a common finite refinement of $\mathfrak{V}_\mu$ and $\mathfrak{V}_{\mu'}$ and let $\mathfrak{U}_\nu = \phi^{-1}(\mathfrak{V}_\nu)$. Since $\phi^{-1}(\mathfrak{V}_\mu)$ is a refinement of $\mathfrak{U}_\mu$, $\mathfrak{U}_\nu$ is a refinement of $\mathfrak{U}_\mu$ and $(h^*\phi)^{-1}(\mathfrak{W}) = h^{-1}(\mathfrak{W})$. The nerve (K_ν, L_ν) of $\mathfrak{U}_\nu$ may be regarded as the nerve of $\mathfrak{V}_\nu$. Let $h: (X, A) \to (K_\nu, L_\nu)$ and $h_\nu^*: (\beta(X), \beta(A)) \to (K_\nu, L_\nu)$ be canonical maps. Then it is obvious that $h_\nu^*\phi$ is a canonical map of (X, A) into (K_ν, L_ν). Hence, by Corollary 9.1, we have

$$h_\nu \overset{u}{\simeq} h_\nu^*\phi.$$

Let $T_{\nu\mu}$ be a projection of the nerve (K_ν, L_ν) of $\mathfrak{U}_\nu$ into the nerve (K_μ, L_μ) of $\mathfrak{U}_\mu$ Then, by Corollary 9.3, we have

$$T_{\nu\mu}\, h_\nu \overset{u}{\simeq} h_\mu.$$

Since α, β and $T_{\nu\mu}$ are uniformly continuous, we have

$$\alpha\, h_\mu \overset{u}{\simeq} \alpha\, T_{\nu\mu} h_\nu^*\phi \quad \text{and} \quad \beta\, h_\mu \overset{u}{\simeq} \beta\, T_{\nu\mu} h_\nu^*\phi.$$

But $\alpha\, h_\mu \overset{u}{\simeq} \beta\, h_\mu$, hence

$$T_{\nu\mu} h_\nu^*\phi \overset{u}{\simeq} \beta\, T_{\nu\mu} h_\nu^*\phi.$$

Therefore we have

$$\alpha\, T_{\nu\mu} h_\nu^* \simeq \beta\, T_{\nu\mu} h_\nu^*.$$

Since $\beta(X)$ is compact, by the first part of Theorem 9.4, there exists a finite refinement $\mathfrak{V}_\lambda$ of $\mathfrak{V}_\nu$ such that

$$\alpha\, T_{\nu\mu} T_{\lambda\nu} \simeq \beta\, T_{\nu\mu} T_{\lambda\nu},$$

where $T_{\lambda\nu}$ is a projection of the nerve (K_λ, L_λ) of $\mathfrak{V}_\lambda$ into the nerve (K_ν, L_ν) of $\mathfrak{V}_\nu$. Let $\mathfrak{U}_\lambda = \phi^{-1}(\mathfrak{V}_\nu)$. Then (K_λ, L_λ) may be regarded as the nerve of $\mathfrak{U}_\lambda$ and $T_{\lambda\nu}$ may be regarded as a projection of the nerve of $\mathfrak{U}_\lambda$ into the nerve of $\mathfrak{U}_\nu$. Hence $T_{\nu\mu} T_{\lambda\nu} \simeq T_{\lambda\nu}$, where $T_{\lambda\nu}$ is a projection of the nerve (K_λ, L_λ) of $\mathfrak{U}_\lambda$ into the nerve (K_μ, L_μ) of $\mathfrak{U}_\mu$. Hence we have $\alpha\, T_{\lambda\mu} \simeq \beta\, T_{\lambda\mu}$. Therefore our assertion is proved. Thus the theorem has been completely established.

REMARK 3. Let $\{\mathfrak{U}_\mu\}'$ and $\{\mathfrak{U}_\mu\}'_F$ be cofinal parts of $\{\mathfrak{U}_\mu\}$ and $\{\mathfrak{U}_\mu\}_F$ respectively. We remark that the limit sets of $\{D(K_\mu, L_\mu;\ M, m)\}$ based on $\{\mathfrak{U}_\mu\}'$ and $\{\mathfrak{U}_\mu\}_F'$ are same with $\overline{D}(X, A; M, m))\}$ and $\overline{D}_F(X, A;\ M, m)$ respectively.

We shall now apply Theorem 9.4 to cases of cohomotopy and u-cohomotopy groups.

Let (X, A) be a paracompact pair with $\dim X < 2n-1$. We consider the totality of all locally finite coverings of X with order $<2n-1$. Let (K_λ, L_λ) be the nerve of $\mathfrak{U}_\lambda$. Then dim $K_\lambda < 2n-1$([7, Lemma 5]), and K_λ is paracompact ([7, Lemma 4]), therefore $\pi_n(K_\lambda, L_\lambda)$ is an abelian group. If $\mathfrak{U}_\lambda > \mathfrak{U}_\mu$, then $T_{\lambda\mu}^{\#}$ is a homomorphism, thus we have a direct spectrum of homomorphisms. Let $\bar{\pi}^n(K, A)$ denote the limit group of this spectrum.

Also, let $\bar{\pi}_F^n(X, A)$ denote the limit group of the direct spectrum of homomorphisms based on all finite coverings with order $<2n-1$. From Theorem 9. 4, Remark 2 and [3, Theorem 3. 5], we have immediately the following theorem.

THEOREM 9.5. *If* (X, A) *is a paracompact pair with* $\dim X < 2n-1$, *then* $\bar{h}$ *maps* $\bar{\pi}^n(X, A)$ *isomorphically onto* $\pi^n(X, A)$, *and* $\bar{h}_u$ *maps* $\bar{\pi}^n_F(X, A)$ *isomorphically onto* $\pi^n_u(X, A)$.

Let (X, A) be a paracompact pair with $\dim X < 2n-1$, and let $i: A \to X$, $j: X \to (X, A)$ be identity maps. The cohomotopy and u-cohomotopy sequences of the pair (X, A) are sequences

$$\pi^n(X, A) \xrightarrow{j^{\#}} \cdots \xrightarrow{j^{\#}} \pi^m(X) \xrightarrow{i^{\#}} \pi^m(A) \xrightarrow{\Delta} \pi^{m+1}(X, A) \xrightarrow{j^{\#}} \cdots,$$

and

$$\pi^n_u(X, A) \xrightarrow{j^{\#}_u} \cdots \xrightarrow{j^{\#}_u} \pi^m_u(X) \xrightarrow{i^{\#}_u} \pi^m_u(A) \xrightarrow{\Delta_u} \pi^{m+1}_u(X, A) \xrightarrow{j^{\#}_u} \cdots .$$

Let $\mathfrak{U}_\nu$ be as before and let $i_\nu: L_\nu \to K_\nu$ and $j_\nu: K_\nu \to (K_\nu, L_\nu)$ be the identity. Then the cohomotopy sequence of (K_ν, L_ν) is the sequence

$$\pi^n(K_\nu, L_\nu) \xrightarrow{j^{\#}_\nu} \cdots \xrightarrow{j^{\#}_\nu} \pi^m(K_\nu) \xrightarrow{i^{\#}_\nu} \pi^m(L_\nu) \xrightarrow{\Delta_\nu} \pi^{m+1}(K_\nu, L_\nu) \xrightarrow{j^{\#}_\nu} \cdots .$$

If $\mu < \nu$, then $T^{\#}_{\nu\mu}$ commutes with each map of the cohomotopy sequences of (K_ν, L_ν) and (K_μ, L_μ), and therefore, can be regarded as a homomorphism of the cohomotopy sequence of (K_ν, L_ν) into that of (K_μ, L_μ). Hence, we have a direct limit of sequences whose limit sequence is denoted by

$$(9.1) \qquad \bar{\pi}^n(X, A) \xrightarrow{\bar{j}} \cdots \xrightarrow{\bar{j}} \bar{\pi}^m(X, A) \xrightarrow{\bar{i}} \bar{\pi}^m(A) \xrightarrow{\bar{\Delta}} \bar{\pi}^{m+1}(X, A) \xrightarrow{\bar{j}} \cdots .$$

Since $\bar{h}$ commutes with each map in this sequence and the cohomotopy sequence of (X, A), it can be regarded as a homomorphism of the sequenece (9.1) into the cohomotopy sequence of (X, A). Also we have a direct limit sequence

$$(9.2) \qquad \bar{\pi}^n_F(X, A) \xrightarrow{\bar{j}} \cdots \xrightarrow{\bar{j}} \bar{\pi}^n_F(X, A) \xrightarrow{\bar{i}} \bar{\pi}^m_F(A) \xrightarrow{\bar{\Delta}} \bar{\pi}^m_F(X, A) \xrightarrow{\bar{i}} \cdots ,$$

and $\bar{h}_u$ may be regarded as a homomorphism of the sequence (9.2) into the u-cohomotopy sequence of (X, A). Thus Theorem 9.5 can be given the following forms.

THEOREM 9.6. *If* (X, A) *is a paracompact pair with* $\dim X < 2n-1$, *then* $\bar{h}$ *maps the sequence* (9.1) *isomorphically onto the cohmotopy sequence of* (X, A), *and* $\bar{h}_u$ *maps the sequence* (9.2) *isomorphically onto the u-cohomotopy sequence of* (X, A).

10. The exactness axiom. DEFINITION 10.1. A sequence of groups and homomorphisms

$$G_1 \xrightarrow{g_1} \quad \to G_n \xrightarrow{g_n} G_{n+1} \xrightarrow{g_{n+1}} \cdots$$

is said to be exact sequence if the image of g_i equals the kernel of g_{i+1} for all i.

THEOREM 10.2.[t] *The cohomotopy and u-cohomotopy sequences of a paracompact pair* (X, A) *with* $\dim X < 2n-1$ *are both exact.*

Theorem 10.2. is called the exactness axiom. In virtue of Theorem 9.6 and [5, p. 689], it is sufficient to prove the exactness axiom for simplicial pair (K, L).

Spanier's proof of the exactness axiom for finite simplicial pair (K, L) is based on Lemma 15.1 [10]. We shall prove that this lemma is also true for arbitrary not necessary locally finite simplicial pair (K, L). To this end we must prove the following lemmas.

LEMMA 10.3. *Let* K *be an infinite simplicial complex with the weak topology,* L *be a closed subcomplex and* V *be a neighborhood of* L. *Put* $\dot{V} = \overline{V} - V$. *Then there exists a real continuous function* $0 \leqslant \varepsilon(x) \leqslant 1$ *defined on* $\overline{V}$ *such that* $\varepsilon(x) = 1$ *if* $x \in \dot{V}$ *and* $\varepsilon(x) = 0$ *if and only if* $x \in L$.

PROOF. Let $N(K)$ be the complex K topologized by the natural metric, and $i: N(K) \to K$ be the identity transformation. It is obvious that i is an open (not necessary continuous) transformation. Since $N(K)$ is a metric space, there exists a sequence $\{U_n'\}$ of open neighborhoods of $N(L)$ such that $\bigcap U_n' = N(K)$. Therefore $\{U_n\}$ $(U_n = iU_n')$ is a sequence of open neighborhoods of L such that $\bigcap U_n = L$. Let $V_n = \overline{V} \cap U_n$. Then $\bigcap V_n = L$ and V_n are open in the normal subspace $\overline{V}$. By Urysohn's lemma there exist real continuous functions $0 \leqslant \varepsilon_n(x) \leqslant 1$ defined on $\overline{V}$ such that $\varepsilon_n(x) = 0$ if $x \in L$, $\varepsilon_n(x) = 1$ if $x \in \overline{V} - V_n$. We define

$$\varepsilon(x) = \sum \varepsilon(x)/2^n \qquad (x \in \overline{V}),$$

then $\varepsilon(x)$ is a real continuous function defined on $\overline{V}$ such that $0 \leqslant \varepsilon(x) \leqslant 1$. If $x \in \overline{V}-V$, then obviously $\varepsilon(x) = 1$. If $\varepsilon(x) = 0$ then, for each n $\varepsilon_n(x) = 0$. Hence $x \notin \overline{V} - V_n$, i.e. $x \in V_n$. Therefore $x \in \bigcap V_n = L$. Thus the function $\varepsilon(x)$ is required.

LEMMA 10.4. *Let* X *be a paracompact space and let* $\hat{X}$ *denote the joint of* X *with a point* P. *For any neighborhood* U *of* X *containing* P, *there exists a real continuous function* $0 \leqslant t(x) < 1$ *defined on* X *such that* $\{(x, t(x)) \mid x \in X\} \subset U$.

PROOF. By the topology of $\hat{X}$, for each point $x \in X$ there exists a neighborhood $x \in V_x$ in X and a real number $0 \leqslant \varepsilon(x) < 1$ such that $(y, t) \in U$ if $y \in V_x$, $\varepsilon(x) \leqslant t \leqslant 1$. Such V_x's form a covering of X. Since X is paracompact, there exists a locally finite refinement $\{U_\alpha\}$ of this covering. For each set $U_\alpha \in \{U_\alpha\}$ we choose a set $V_x \subset V_\alpha$, and we associate to U_α a real number $\varepsilon_\alpha = \varepsilon(x)$. Let N be the nerve of the covering $\{U_\alpha\}$ with the natural metric (cf, [3, 1]), and $\phi: X \to N$ denote a canonical map. Denote by $\{\phi_a(x)\}$ $(\Sigma\phi_a(x) = 1)$ the barycentric coordinates of $\phi(x)$ $(x \in X)$, then, for each α, ϕ_a is a continuous function defined on X and $\phi_a(x)$ are zero except a fininite set of them. Thus

$$t(x) = \Sigma_a \phi_a(x)\, \varepsilon_a$$

has the meaning and is continuous. Since $\Sigma_a\phi_a(x) = 1$ and $0 \leqq \varepsilon_a < 1$, Min $(\varepsilon_a) \leqq t(x) < 1$ for all $x \in X$. Thus the function $t(x)$ is desired.

Let X_1 denote the set of points which are represented by $(x, t(x))$. Then it is easily verified that the correspondence $x \to (x, t(x))$ gives a homeomorphism of X onto X_1.

LEMMA 10.5. *Let K be a (not necessary locally finite) simplicial complex with the weak topology and let $\hat{K}$ denote the joint of K with a point P. If* dim $K < 2n-1$, *then Δ maps $\pi^n(K)$ onto $\pi^{n+1}(\hat{K}, K)$.*

PROOF. The proof of this lemma is parallel with Spanier's proof of Lemma 15.1 [10]. Let $\{\alpha\} \in \pi^{n+1}(\hat{K}, K)$ so that $\alpha: (\hat{K}, K)$ (S^{n+1}, p). Let q be the north pole S^{n+1} (q is the center of E_+^{n+1}). By the simplicial approximation theorem [11, Theorem 36], we may assume that α has been chosen in $\{\alpha\}$ so that $\alpha^{-1}(q)$ has dimension less than n. Let $L = \alpha^{-1}(q)$ and $\hat{L}$ = the joint of L with P. Then dim $\hat{L} < n+1$.

Let σ be a closed simplex on S^{n+1} containing p in its interior and not containing q. Let $\dot{\sigma}$ be the boundary of σ and let $M = \alpha^{-1}(\sigma), N = \alpha^{-1}(\dot{\sigma})$. Define $M' = \hat{L} \cap M$, and $N' = \hat{L} \cap N$. Then dim $M' < n+1$ and $\alpha | M'$ maps (M', N') into $(\sigma, \dot{\sigma})$. Hence, by Lemma 3.5 there is a map $F : (M' \times I, N' \times I) \to (\sigma, \dot{\sigma})$ such that

$$\left.\begin{aligned} F(x, 0) &= \alpha(x) \\ F(x, 1) &\in \dot{\sigma} \end{aligned}\right\} \quad \text{for all } x \in M'$$

$$F(x, t) = \alpha(x) \qquad \text{for all } x \in N',\ t \in I.$$

Define $F': ((M \times 0 \cup [M' \cup K \cup N] \times I) \to (\sigma, \dot{\sigma})$ by

$$F'(x, t) = \begin{cases} F(x, t) & \text{if } x \in M' \\ \alpha(x) & \text{if } x \in K \cup N \\ \alpha(x) & \text{if } x \in M \text{ and } t = 0. \end{cases}$$

Since σ is a contractible space, there is an extension G of F' which maps $(M \times I, N \times I)$ into $(\sigma, \dot{\sigma})$. Define α': $(\hat{K}, K) \to (S^n, p)$ by

$$\alpha'(x) = \begin{cases} G(x, 1) & \text{if } x \in M, \\ \alpha(x) & \text{if } x \in \hat{K}-M. \end{cases}$$

Then α' is continous, $\alpha' \in \{\alpha\}$, $\alpha'^{-1}(q) = L$ and $\alpha'(\hat{L})$ does not meet p.

Hence there is a neighborhood V of $\hat{L}$ such that the closure $\overline{V}$ of V does not map onto p under α'. Put $\dot{V} = \overline{V} - V$. Since V and $\hat{L}$ are disjoint closed sets in $\overline{V}$, by Lemma 10.4, there is a real continuous function $0 \leqq \varepsilon(x) \leqq 1$ such that $\varepsilon(X) = 1$ if $x \in \dot{V}$ and $\varepsilon(x) = 0$ if and only if $x \in \hat{L}$. For $x \in \overline{V}$, let $a(x)$ be the point of $S^{n+1}-p$ which divides the segment from $\alpha'(x)$ to q in the ratio $1-\varepsilon(x):\varepsilon(x)$. Define a homotopy H of α' as follows. If $x\in\overline{V}$, H maps $x\times I$ linearly onto the segment of $S^{n+1}-p$ joining $\alpha'(x)$ to $a(x)$. If $x\notin\overline{V}$, H maps $x\times I$ onto $\alpha'(x)$. Let $\beta(x)=H(x,1)$. Then $\alpha'\simeq\beta$ rel. K and $\beta^{-1}(q)=\hat{L}$.

For $x\in\overline{V}$, let $b(x)$ be the point on the segment from x to P which divides this segment in the ratio $1-\varepsilon(x):\varepsilon(x)$. Define a deformation H' of $\hat{K}$ on itself as follows. If $x\in\overline{V}$, H' maps $x\times I$ linearly onto the segment joining x to $b(x)$. If $x\in\hat{K}-\overline{V}$, H' maps $x\times I$ onto x. Then H' is continuous. Let $f:(\hat{K},K)\rightarrow(\hat{K},K)$ be defined by $f(x)=H'(x,1)$. Then f is homotopic rel. K to the identity map of $\hat{K}$ onto itself, and f maps $\hat{K}-\hat{L}$ homeomorphically onto $\hat{K}-P$. Let $\gamma:(\hat{K},K)\rightarrow(S^{n+1},p)$ be defined by

$$\gamma(x)=\begin{cases}\beta(f^{-1}(x)) & \text{if } x\neq p,\\ q & \text{if } x=p.\end{cases}$$

Then it is obvious that γ is continuous and $\gamma H'$ is a homotopy rel. K between γ and $\gamma f=\beta$. Also $\gamma^{-1}(q)=P$.

Let T be a small closed $(n+1)$-cell on S^{n+1} with center at q and boundary S_1^n. Let U be a neighborhood of the point P which is mapped into T by γ. Such a neighborhood U exists certainly. Since $\hat{K}$ is paracompact ([7, Lemma 4]), by Lemma 10.4, there is a real continuous function $t(x)$ defined on K so that $0\leqq t(x)<1$. Let K_1 be the set of points $(x,t(x))$ $(x\in K)$. Let D define to be the set $\{(x,t)\mid t\leqq t(x)\}$. Since $K_1\not\ni P$ and $\gamma^{-1}(q)=P$, $\gamma(K_1)\not\ni q$. Push $\gamma(K_1)$ along geodesic arc from q until K_1 is mapped into S_1^n and follow this by a deformation of γ (keeping the inverse image $S^{n+1}-\operatorname{int}T$ pointwise fixed) to get a new map $\gamma'\simeq\gamma$ rel. K with $\gamma'^{-1}(q)=P$. Deform γ' to a map γ'' : $(\hat{K},K)\rightarrow(S^{n+1},p)$ such that γ'' agrees with on $K-D$, $\gamma''(D)\subset S^{n+1}-\operatorname{int}T$ and $\gamma''\simeq\gamma$ rel. K.

Deform S^{n+1} into itself along arcs from q so that at the end of the deformation (T,S_1^n) is mapped homeomorphically onto (E_+^{n+1},S^n) and $S^{n+1}-\operatorname{int}T$ is mapped into E_-^{n+1} with p and q kept fixed during the homotopy. Let the final map of (S^{n+1},p) into (S^{n+1},p) be g. Then $g\gamma''$ maps $(\hat{K},K)$ into (S^{n+1},p), maps K_1 into S^n and D into E^{n+1} and is homotopic to α rel. K. Let h be a correspondence $x\rightarrow(x,t(x))$. Then, as we have already noticed, h is a homeomorphism of K onto K. Define $\zeta=g\gamma''h$. Then ζ maps K into S^n, and it is obvious that $\Delta\{\zeta\}=\{g\gamma''\}=\{\alpha\}$. Thus the lemma has been proved.

PROOF OF THEOREM 10.2. Spanier's proof [10, § 16] of exactness axiom for a finite simplicial pair is applied to an infinite simplicial pair without any modification, if we use Lemma 10.5 instead of Lemma 15.1 [10]. Therefore, the cohomotopy sequence of any simplicial pair satisfies the exactness axiom.

Hence, by Theorem 9.4 and [5, p 689], the cohomotopy and u-cohomotopy sequences of a paracompact pair are both exact.

If $X \supset A \supset B$ and $i: A \to (A, B)$ denotes the identity map, then the compositions $\varDelta i^{\#}: \pi^n(A, B) \to \pi^{n+1}(X, A)$ and $\varDelta_u i^{\#}: \pi^n_u(A, B) \to \pi^{n+1}_u(X, A)$ will also be denoted by $\varDelta$ and $\varDelta_u$ respectively. If (X, A, B) is a triple consisting of a paracompact space X with $\dim X < 2n-1$ and closed subset A, B with $B \subset A$, let $i: (A, B) \to (X, B)$ and $j: (X, B) \to (X, A)$ denote the identity maps. Then the cohomotopy and u-cohomotopy sequences of the triple (X, A, B) are the sequences of groups and homomorphisms

$$\pi^n(X, A) \xrightarrow{j^{\#}} \cdots \xrightarrow{j^{\#}} \pi^m(X, B) \xrightarrow{i^{\#}} \pi^m(A, B) \xrightarrow{\varDelta} \pi^{m+1}(X, A) \xrightarrow{j^{\#}} \cdots ,$$

and

$$\pi^n_u(X, A) \xrightarrow{j^{\#}_u} \cdots \xrightarrow{j^{\#}_u} \pi^m_u(X, B) \xrightarrow{i^{\#}_u} \pi^m_u(A, B) \xrightarrow{\varDelta_u} \pi^{m+1}_u(X, A) \xrightarrow{j^{\#}_u} \cdots .$$

THEOREM 10.5. *The cohomotopy and u-cohomotopy sequences of a paracompact triple (X, A, B) with $\dim X < 2n-1$ are both exact.*

As is well-known, this theorem may be derived in a purely algebraic fashion from Theorem 6.5, 7.2, 7.3, 7.5 and 10.2.

REMARK 4. Let (X, A) be any arbitrary pair with $\dim X < 2n-1$ and let $\phi: X \to \beta(X)$ the Čech compactification of X and put $\beta(A) = \overline{\phi(A)}$. Then $\dim \beta(X) < 2n-1$ and $\phi \mid A: A \to \beta(A)$ is the Čech compactification of A. Thus ϕ induces a 1-1 transformation ϕ_u of the u-cohomotopy sequence of the compact pair $(\beta(X), \beta(A))$ onto the sequence of sets and set transfromations

$$\pi^n_u(X, A) \xrightarrow{j^{\#}_u} \cdots \xrightarrow{j^{\#}_u} \pi^m_u(X, A) \xrightarrow{i^{\#}_u} \pi^m_u(A) \xrightarrow{\varDelta_u} \pi^{m+1}_u(X, A) \to \cdots . \tag{10.1}$$

Therfore we can define the group operation for each set of (10.1) so that ϕ_u becomes the isomorphism, then we have an exact homorphism sequence of (X, A), which will be also called the u-cohomotopy sequence of (X, A). If X is paracompact, then (10.1) is a homomorphism sequence of groups, and ϕ_u is an isomorphism. Therefore the above definition ϕ is consistence with that of the u-cohomotopy group of a pair. Also we can define the group operation for each set of (10.1) so that $\bar{h}_u$ in Theorem 9.5 becomes the isomorphism. It is easily seen that two definitions of group operations are the same. Moreover we notice here that Part II of [10] is extended to any (not necessary paracompact) pair (X, A), by using of u-cohomotopy sequence and ordinary Čech cohomology sequence of (X, A).

11. Comparision with cohomology groups. Let (X, A) be a pair. Let $H^n(X, A; G)$ and $H^n_F(X, A; G)$ denote the n-th Čech cohomology groups of (X, A) with coefficients in an abelian group G, based on infinite coverings, and finite coverings respectively.

If $\{\alpha\} \in \pi^n(X, A)$, then $\alpha^* s^n \ni \check{H}^n(X, A; n^n)$ and if $\alpha \in \{\alpha\}$, then $\beta^* s^n = \alpha^* s^n$ by the homotopy axiom for Čech theory (cf. [4, E 4]), where α^*, β^* are induced homomorphism of Čech cohomology groups, n^n is the n-th homotopy group of the sphere S^n, and s^n is a generator of $H^n(S^n, p; n^n) = H^n_F(S^n, p; n^n)$ (for the convension of the orientation of s see [10, § 17]).

Hence, there is induced a transformation

$$\overline{\phi} : \pi^n(X, A) \to H^n(X, A; n^n)$$

defined by $\overline{\phi}\{\alpha\} = \alpha^* s^n$.

If $\{\alpha\}_u \in \pi^n_u(X, A)$, then $\alpha^* s \in H^n_F(X, A; n^n)$ and if $\beta \in \{\alpha\}_u$, then it is easily verified that $\beta^* s^n = \alpha^* s^n$, where $\alpha^*, \beta^* : H^n_F(S^n, p; n^n) \to H^n_F(X, A; n^n)$ are induced homomorphisms of α, β. Hence, there is induced a transformation

$$\overline{\phi}_u : \pi^n_u(X, A) \to H^n_F(X, A; n^n)$$

defined by $\overline{\phi}_u\{\alpha\}_u = \alpha^* s^n$.

THEOREM 11.1. *If* $\dim (X-A) < 2n-1$ *then* $\overline{\phi}_u$ *is a homomorphism, and furthermore, if* X *is paracompact, then* $\overline{\phi}$ *is a homomorphism.*

THEOREM 11.2. *If* f *is a map of any pair* (X, A) *into another* (Y, B), *then commutativities hold in the diagrams*:

$$\begin{array}{ccc} \pi^n(Y,B) & \xrightarrow{f^\#} & \pi^n(X,A) \\ \overline{\phi}\downarrow & & \downarrow\overline{\phi} \\ H^n(Y,B;n^n) & \xrightarrow{f^*} & H^n(X,A;n^n) \end{array} \qquad \begin{array}{ccc} \pi^n_u(Y,B) & \xrightarrow{f^\#_u} & \pi^n_u(X,A) \\ \overline{\phi}_u\downarrow & & \overline{\phi}_u\downarrow \\ H^n(Y,B;n^n) & \xrightarrow{f^*} & H^n_F(X,A;n^n). \end{array}$$

It is not difficult to prove these two theorems and so we shall omitt the proofs (cf. [10, § 17]).

Combining Theorem 9.5, 11.1 and 11.2 with the Hopf classification theorem for simplicial pair ([12]), we have easily the following generalized Hopf classification theorems.

THEOREM 11.3. *If* (X, A) *is a paracompact pair with* $\dim (X-A) < n$ $(n > 1)$, *then* $\overline{\phi}$ *maps* $\pi^n(X, A)$ *isomorphically onto* $H^n(X, A; n^n)$.

THEOREM 11.4. *For any pair* (X, A) *with* $\dim X < n$ $(n > 1)$, $\overline{\phi}_u$ *maps* $\pi^n_u(X, A)$ *isomorphically onto* $H^n(X, A; n^n)$, *wheve* n^n *denotes the group* $\pi_n(s^n)$.

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+) ADDED IN PROOF. Prof. K. Morita has obtained the same result independently. Cf. K. Morita, Cohomotopy groups for fully normal spaces, Science Report of the Tokyo Bunrika Daigaku, 4, No. 98 (1952).

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