ABSOLUTE CESÀRO SUMMABILITY OF ORTHOGONAL SERIES II (CORRECTION AND REMARK TO THE PREVIOUS PAPER)

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Introduction. In the previous paper [8] we proved a theorem on absolute Cesàro summability at a point for the Fourier series of functions in $L^p(0, 2\pi)$ and stated that the summability $|C, r|$ $(r > 1/p)$ is of local property for the class $L^p(p > 1)$; but this result is incorrect for $p > 2$.[1] Indeed the summability $\left|C, \frac{1}{2}\right|$ is not of local property even for the class of continuous functions; this fact will be shown here in §2. In §1 we shall give an improvement of the summability theorem.

On the other hand, as Bosanquet-Kestelman and Yano have shown [3; 9], the summability $|C, 1/p|$ $(1 \leqq p < 2)$ is not of local property for the class L^p; we shall give a proof of this by a counter example in §3.

1. The summability theorem of the author [8; Theorem 5] will be improved as follows:

THEOREM 1. *Let $1 < p \leqq 2$ and let $f(t) \in L^p(0, 2\pi)$. If the integral*

$$(1) \qquad \int_0^\pi \frac{|\varphi_x(t)|^p}{t} \left|\log \frac{1}{t}\right|^\alpha dt$$

is convergent for some $\alpha > p - 1$, where

$$\varphi_x(t) = f(x + t) + f(x - t) - 2f(x),$$

then the Fourier series of $f(t)$ is summable $|C, \delta|$ for every $\delta > 1/p$ at $t = x$.

We obtain this result replacing the condition (1) by the stronger condition:

$$(2) \qquad \frac{1}{t}\int_0^t |\varphi_x(u)|^p du = O\left(\left|\log \frac{1}{t}\right|^{-p-\varepsilon}\right) \quad (\varepsilon > 0) \text{ as } t \to 0,$$

from which it follows easily the convergence of the integral (1).

LEMMA 1. (i) *The Cesàro kernel of order r, $-1 < r < 1$, is written as*

$$K_n^r(t) = G_n^r(t) + H_n^r(t),$$

where

$$G_n^r(t) = \frac{\sin\left[\left(n + \frac{1}{2} + \frac{r}{2}\right)t - \frac{r}{2}\pi\right]}{A_n^r\left(2 \sin \frac{t}{2}\right)^{1+r}},$$

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1) Throughout the Part II of the previous paper [8], the condition "$p>1$" should be replaced by the condition "$1<p\leqq 2$."

$$|K_n^r(t)| \leqq Cn \textit{ and } |H_n^r(t)| \leqq Cn^{-1}t^{-2},$$

C being a positive constant.[2)]

(ii) *If* $0 < r < 1$, *we have*

$$|K_n^r(t)| \leqq Cn^{-r}t^{-1-r} \quad \textit{for} \quad \pi/n \leqq t \leqq \pi.$$

The first part of the lemma is due to Kogbetlianz (Cf. Hardy-Littlewood [4] or Zygmund [10], p. 212), and the second is well known (Cf. Zygmund [10], p. 48).

PROOF OF THEOREM 1. Denote by $\sigma_n^\delta(t)$ the n-th (C, δ) mean of the Fourier series of $f(t)$. Since we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\sigma_n^\delta(x) - \sigma_{n-1}^\delta(x)| &= \delta \sum_{n=1}^{\infty} \frac{|\sigma_n^{\delta-1}(x) - \sigma_n^\delta(x)|}{n} \\
&\leqq \sum_{n=1}^{\infty} \frac{|\sigma_n^{\delta-1}(x) - f(x)|}{n} + \sum_{n=1}^{\infty} \frac{|\sigma_n^\delta(x) - f(x)|}{n} \\
&= S_1 + S_2 \text{ say,}
\end{aligned}$$

it is sufficient to prove the finiteness of the sums S_1 and S_2 for δ such as $1/p < \delta < 1$.

For the sum S_2 we have

$$\begin{aligned}
\pi S_2 = \pi \sum_{n=1}^{\infty} \frac{|\sigma_n^\delta(x) - f(x)|}{n} &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^\pi \varphi_x(t) K_n^\delta(t)\, dt \right| \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi/n} |\varphi_x(t) K_n^\delta(t)|\, dt + \sum_{n=1}^{\infty} \frac{1}{n} \int_{\pi/n}^{\pi} |\varphi_x(t) K_n^\delta(t)|\, dt \\
&= A + B \text{ say.}
\end{aligned}$$

By Lemma 1 (ii) we get

$$\begin{aligned}
A &\leqq C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi/n} n |\varphi_x(t)|\, dt = C \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt \\
&= C \sum_{k=1}^{\infty} k \int_{\pi/(k+1)}^{\pi/k} |\varphi_x(t)|\, dt \\
&\leqq C_1 \sum_{k=1}^{\infty} \int_{\pi/(k+1)}^{\pi/k} \frac{|\varphi_x(t)|}{t}\, dt \\
&= C_1 \int_0^\pi \frac{|\varphi_x(t)|}{t}\, dt\ ;
\end{aligned}$$

2) In what follows $C, C_1, C_2, \ldots$ are positive constants independent of the variables.

$$B \leqq C\sum_{n=1}^{\infty}\frac{1}{n}\int_{\pi/n}^{\pi}|\varphi_x(t)|\frac{dt}{n^{\delta}t^{1+\delta}}$$
$$= C\sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}}\sum_{k=1}^{n-1}\int_{\pi/(k+1)}^{\pi/k}\frac{|\varphi_x(t)|}{t^{1+\delta}}dt$$
$$\leqq C_2\sum_{k=1}^{\infty}\frac{1}{k}\int_{\pi/(k+1)}^{\pi/k}\frac{|\varphi_x(t)|}{t^{1+\delta}}dt < C_3\sum_{k=1}^{\infty}\int_{\pi/(k+1)}^{\pi/k}\frac{|\varphi_x(t)|}{t}dt$$
$$= C_3\int_0^{\pi}\frac{|\varphi_x(t)|}{t}dt.$$

Applying the Hölder inequality we have easily
$$\int_0^{\pi}\frac{|\varphi_x(t)|}{t}dt \leqq \left(\int_0^{\pi}\frac{dt}{t\left|\log\frac{1}{t}\right|^{\alpha/(p-1)}}\right)^{1-1/p}\left(\int_0^{\pi}\frac{|\varphi_x(t)|^p}{t}\left|\log\frac{1}{t}\right|^{\alpha}dt\right)^{1/p};$$
the first factor on the right-hand side is finite as $\alpha > p-1$ and so is the second by the assumption. Thus we get at once: $S_2 < \infty$.

Now we estimate the sum S_1. We divide it into two sums:
$$\pi S_1 = \pi\sum_{n=1}^{\infty}\frac{|\sigma_n^{\delta-1}(x) - f(x)|}{n} = \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^{\pi}\varphi_x(t)K_n^{\delta-1}(t)dt\right|$$
$$\leqq \sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\left|\int_0^{\pi}\varphi_x(t)K_k^{\delta-1}(t)dt\right|$$
$$\leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\left|\int_0^{\pi/2^n}\right| + \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\right| \quad = P + Q \text{ say.}$$

By Lemma 1 (i) we have
$$P \leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=2n}^{2^{n+1}-1}Ck\int_0^{\pi/2^n}|\varphi_x(t)|dt$$
$$\leqq 2C\sum_{n=0}^{\infty}2^n\int_0^{\pi/2^n}|\varphi_x(t)|dt$$
$$= 2C\sum_{n=0}^{\infty}2^n\sum_{m=n}^{\infty}\int_{\pi/2^{m+1}}^{\pi/2^m}|\varphi_x(t)|dt$$
$$\leqq 4C\sum_{m=0}^{\infty}2^m\int_{\pi/2^{m+1}}^{\pi/2^m}|\varphi_x(t)|\,dt \leqq C_3\sum_{m=0}^{\infty}\int_{\pi/2^{m+1}}^{\pi/2^m}\frac{|\varphi_x(t)|}{t}dt$$
$$= C_3\int_0^{\pi}\frac{|\varphi_x(t)|}{t}dt$$

which is finite as in the former case. The sum Q will be further divided into two sums :

$$
\begin{aligned}
Q &\leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\varphi_x(t)G^{\delta-1}_k(t)dt\right| \\
&\quad + \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\varphi_x(t)H^{\delta-1}_k(t)dt\right| \\
&= Q_1 + Q_2 \text{ say.}
\end{aligned}
$$

By the Hölder inequality, if we put $q = p/(p-1)$,

$$
\begin{aligned}
Q_1 &\leqq \sum_{n=0}^{\infty}\frac{2^{n/p}}{2^n}\left(\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\varphi_x(t)\frac{\cos\left\{\left(k+\frac{\delta}{2}\right)t-\frac{\delta}{2}\pi\right\}}{A^{\delta-1}_k\left(2\sin\frac{t}{2}\right)^{\delta}}dt\right|^q\right)^{1/q} \\
&\leqq C_4\sum_{n=0}^{\infty}\frac{1}{2^{n(\delta-1/p)}}\left(\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\frac{\varphi_x(t)\cos\left(\frac{\delta}{2}t-\frac{\delta}{2}\pi\right)}{\left(2\sin\frac{t}{2}\right)^{\delta}}\cos kt\,dt\right|^q\right)^{1/q} \\
&\quad + C_4\sum_{n=0}^{\infty}\frac{1}{2^{n(\delta-1/p)}}\left(\sum_{k=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\frac{\varphi_x(t)\sin\left(\frac{\delta}{2}t-\frac{\delta}{2}\pi\right)}{\left(2\sin\frac{t}{2}\right)^{\delta}}\sin kt\,dt\right|^q\right)^{1/q}
\end{aligned}
$$

Applying the Hausdorff-Young inequality (Cf. [10], p. 190) we get easily

$$
\begin{aligned}
Q_1 &\leqq 2C_4\sum_{n=0}^{\infty}\frac{1}{2^{n(\delta-1/p)}}\left(\int_{\pi/2^n}^{\pi}\left|\frac{\varphi_x(t)}{\left(2\sin\frac{t}{2}\right)^{\delta}}\right|^p dt\right)^{1/p} \\
&\leqq C_5\sum_{n=0}^{\infty}\frac{1}{2^{n(\delta-1/p)}}\left(\int_{\pi/2^n}^{\pi}\frac{|\varphi_x(t)|^p}{t^{\delta p}}dt\right)^{1/p}
\end{aligned}
$$

By the Hölder inequality, if $\alpha > p-1$ we have

$$
\begin{aligned}
Q_1 &\leqq C_5\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\alpha/(p-1)}}\right)^{1-1/p}\left(\sum_{n=0}^{\infty}\frac{(n+1)^{\alpha}}{2^{n(\delta p-1)}}\int_{\pi/2^n}^{\pi}\frac{|\varphi_x(t)|^p}{t^{\delta p}}dt\right)^{1/p} \\
&\leqq C_6\left(\sum_{n=0}^{\infty}\frac{(n+1)^{\alpha}}{2^{n(\delta p-1)}}\sum_{m=0}^{n-1}\int_{\pi/2^{m+1}}^{\pi/2^m}\frac{|\varphi_x(t)|^p}{t^{\delta p}}dt\right)^{1/p} \\
&\leqq C_7\left(\sum_{m=0}^{\infty}\frac{(m+1)^{\alpha}}{2^{m(\delta p-1)}}\int_{\pi/2^{m+1}}^{\pi/2^m}\frac{|\varphi_x(t)|^p}{t^{\delta p}}dt\right)^{1/p} \\
&\leqq C_8\left(\sum_{m=0}^{\infty}\int_{\pi/2^{m+1}}^{\pi/2^m}\frac{|\varphi_x(t)|^p}{t^{\delta p}}t^{\delta p-1}\left|\log\frac{1}{t}\right|^{\alpha}dt\right)^{1/p}
\end{aligned}
$$

$$\leqq C_8\left(\int_0^\pi \frac{|\varphi_x(t)|^p}{t}\left|\log\frac{1}{t}\right|^\alpha dt\right)^{1/p}$$

which is finite by the assumption. Finally, by Lemma 1 (i) we get

$$Q_2 \leqq \sum_{n=0}^\infty \frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1}\int_{\pi/2^n}^\pi |\varphi_x(t)|\,\frac{C}{kt^2}\,dt$$

$$\leqq C_9\sum_{n=0}^\infty \frac{1}{2^n}\int_{\pi/2^n}^\pi \frac{|\varphi_x(t)|}{t^2}\,dt$$

$$= C_9\sum_{n=0}^\infty \frac{1}{2^n}\sum_{m=0}^{n-1}\int_{\pi/2^{m+1}}^{\pi/2^m} \frac{|\varphi_x(t)|}{t^2}\,dt$$

$$\leqq C_9\sum_{m=0}^\infty \frac{1}{2^m}\int_{\pi/2^{m+1}}^{\pi/2^m} \frac{|\varphi_x(t)|}{t^2}\,dt$$

$$\leqq C_{10}\int_0^\pi \frac{|\varphi_x(t)|}{t}\,dt,$$

which is also finite.

Combining the above estimations we complete the proof.

2. The theorem just proved is not true for $p > 2$, and indeed the summability $|C, \delta|$ $\left(\delta \leqq \frac{1}{2}\right)$ of the Fourier series is not decided by the local behaviour of the function in the neighbourhood of an assigned point, More precisely we shall prove the following theorem.

THEOREM 2. *There exists a continuous function of period 2π, which vanishes in the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and whose Fourier series is not summable $\left|C, \frac{1}{2}\right|$ at the origin.*

For the proof we shall use the following lemmas.

LEMMA 2. *Let $\{r_i(u)\}$ be the Rademacher system. For a given set $E\subset(0,1)$, $|E| > 0$, there correspond a positive integer $N = N(E)$ and a positive constant $A = A(E)$ such that for any sequence of real numbers $\{a_i\}$ and for any integer $P > N$ we have*

$$\int_E \left|\sum_{i=N}^P a_i r_i(u)\right| du \geqq A\left(\sum_{i=N}^P a_i^2\right)^{1/2};$$

further, if $\Sigma a_i^2 < \infty$, then P may be infinite.

This lemma is essentially known (Cf. [8], Lemma 1).

LEMMA 3. *If $\Sigma a_n^2 (\log n)^{1+\epsilon} < \infty$ $(\epsilon > 0)$, then for almost all u the series*

$$\sum_{n=1}^{\infty} r_n(u) a_n \cos nt \tag{3}$$

is a Fourier series of a continuous function.

This is due to Paley and Zygmund [7]. (Cf. [10]. p. 127)

PROOF OF THEOREM 2. Let us put

$$\begin{aligned} f_u(t) &= \sum_{n=2}^{\infty} \frac{r_n(u)}{n^{1/2}(\log n)^{3/2}} \cos 2nt \qquad && \text{for } \frac{\pi}{2} < x \leqq \pi, \\ &= 0 && \text{for } 0 \leqq x \leqq \frac{\pi}{2}, \end{aligned} \tag{4}$$

$f_u(t) = f_u(-t)$ and $f_u(t+2\pi) = f_u(t)$ for all t, $-\infty < t < \infty$. We shall first prove that the Fourier series of the function $f_u(t)$ is non-summable $\left|C, \frac{1}{2}\right|$ at $t = 0$ for almost all u. To show this we consider the series

$$\sum_{n=1}^{\infty} \left| \sigma_{n+1}^{1/2}(0, u) - \sigma_n^{1/2}(0, u) \right| \tag{5}$$

where $\sigma_n^{1/2}(t, u)$ is the n-th $\left(C, \frac{1}{2}\right)$ mean of the Fourier series of $f_u(t)$.

Suppose, on the contrary, that the Fourier series of $f_u(t)$ is summable $\left|C, \frac{1}{2}\right|$ at $t = 0$ for all u in a set $E \subset (0, 1)$ of positive measure, that is, the series (5) is convergent for $u \in E$. We can suppose that the sum (5) is majorated uniformly by a constant M for every $u \in E$. Then using the well known formula we have

$$\begin{aligned} M|E| &> \int_E \sum_{n=1}^{\infty} |\sigma_{n+1}^{1/2}(0, u) - \sigma_n^{1/2}(0, u)| du \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \int_E \left| \sum_{k=1}^{n} \frac{A_{n-k}^{1/2}}{A_{n+1}^{1/2}} \frac{k}{(n+1)(n-k+1)} \right. \\ &\qquad \left. \times \int_{\pi/2}^{\pi} \cos kt \sum_{m=2}^{\infty} \frac{r_m(u)}{m^{1/2}(\log m)^{3/2}} \cos 2mt \, dt \right| du \\ &\geqq C_{11} \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \int_E \left| \sum_{m=2}^{\infty} \frac{r_m(u)}{m^{1/2}(\log m)^{3/2}} \right. \\ &\qquad \left. \times \sum_{k=1}^{n} \frac{k}{(n-k+1)^{1/2}} \int_{\pi/2}^{\pi} \cos kt \cos 2mt \, dt \right| du \\ &\geqq C_{11} \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \int_E \left| \sum_{m=N}^{\infty} \right| du - C_{11} \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \int_E \left| \sum_{m=2}^{N-1} \right| du \end{aligned} \tag{6}$$

$$= C_{11}S - C_{11}T$$

say, where N is an integer determined in Lemma 2. From the Khintchine inequality the second sum T in the last expression is majorated by

$$\sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \left\{ \sum_{m=2}^{N-1} \frac{1}{m(\log m)^3} \left(\sum_{k=1}^{n} \frac{k}{(n-k+1)^{1/2}} \int_{\pi/2}^{\pi} \cos kt \cos 2mt\, dt \right)^2 \right\}^{1/2}$$

$$\leqq 2 \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \left\{ \sum_{m=2}^{N-1} \frac{1}{m(\log m)^3} \left(\sum_{j=1}^{[n/2]} \frac{2j-1}{(n-2j+2)^{1/2}} (-1)^j \right.\right.$$

$$\left.\left. \times \frac{2j-1}{(2j-1)^2 - (2m)^2} \right)^2 \right\}^{1/2} \tag{7}$$

$$+ \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \left\{ \sum_{m=2}^{N-1} \frac{1}{m(\log m)^3} \left(\frac{2m\pi}{4(n-2m+1)^{1/2}} \right)^2 \right\}^{1/2}$$

$$= T_1 + T_2 \quad \text{say.}$$

The third sum in the expression T_1 is, in absolute value, not greater than

$$\left| \frac{(2m-1)^2}{(2m-1)^2 - (2m)^2} \sum_{j=\nu}^{m} \frac{(-1)^j}{(n-2j+2)^{1/2}} \right| + \left| \frac{(2m+1)^2}{(2m+1)^2 - (2m)^2} \sum_{j=m+1}^{\mu} \frac{(-1)^j}{(n-2j+2)^{1/2}} \right|$$

where $1 \leqq \nu \leqq m$, $m+1 \leqq \mu \leqq [n/2]$ in virtue of the mean value theorem. As $1 \leqq m < N$ this expression is again majorated by a constant depending only on N. Hence, as we see easily, the sum T_1 is not greater than a constant $C(N)$. For the sum T_2 we obtain

$$T_2 \leqq \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \left\{ \sum_{m=2}^{N-1} \frac{1}{m(\log m)^3} \left(\frac{m\pi}{2\sqrt{3}} \right)^2 \right\}^{1/2}$$

which is clearly less than a constant $C_1(N)$. Therefore we have from (7)

$$T \leqq T_1 + T_2 \leqq C(N) + C_1(N) < \infty. \tag{8}$$

On the other hand for the sum S, using Lemma 2, we get easily

$$S \geqq A \sum_{n=2N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{\infty} \frac{1}{m(\log m)^3} \left\{ \sum_{k=1}^{n} \frac{k}{(n-k+1)^{1/2}} \int_{\pi/2}^{\pi} \cos kt \cos 2mt\, dt \right\}^2 \right]^{1/2}$$

$$\geqq A \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3} \left\{ \frac{2m}{(n-2m+1)} \frac{\pi}{4} \right.\right. \tag{9}$$

$$+ \sum_{j=1}^{m} \frac{(-1)^{m-j}}{(n-2j+2)^{1/2}} \frac{(2j-1)^2}{(2m)^2 - (2j-1)^2}$$

$$\left.\left. + \sum_{j=m+1}^{[n/2]} \frac{(-1)^{j-m-1}}{(n-2j+2)^{1/2}} \frac{(2j-1)^2}{(2j-1)^2 - (2m)^2} \right\}^2 \right]^{1/2}.$$

Among the three terms in the last curly bracket, the second, say, S' is positive because its summand is an increasing function of j in absolute value and its m-th term is positive. We consider the third term:

$$\tag{10} \begin{aligned} &\sum_{j=m+1}^{[n/2]} \frac{(-1)^{j-m-1}}{(n-2j+2)^{1/2}} \frac{(2j-1)^2}{(2j-1)^2-(2m)^2} \\ &\quad = \sum_{j=m+1}^{[n/2]} (-1)^{j-m-1} a_j(m,n) \end{aligned}$$

say. By elementary consideration we shall see that there exists an integer $\nu, m < \nu \leqq [n/2]$ such that, when j varies from $m+1$ to $[n/2], a_j(n,m)$ decreases for $m < j \leqq \nu$ and increases for $\nu < j \leqq [n/2]$. Hence (10) can be written as

$$\begin{aligned} &\sum_{j=m+1}^{\nu} (-1)^{j-m-1} a_j(m,n) \\ &\qquad + \sum_{j=\nu+1}^{[n/2]} (-1)^{j-m-1} a_j(n,m) = S'' + S''' \end{aligned}$$

say; the first sum S'' is positive since its first term is positive; and we get easily

$$\tag{11} |S'''| \leqq \frac{(n-1)^2}{(n-1)^2-(2m)^2}$$

Therefore we deduce from (9) that

$$\tag{12} \begin{aligned} S &\geqq A \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3} \left\{ \frac{m\pi}{2(n-2m+1)^{1/2}} + S' + S'' + S''' \right\}^2 \right]^{1/2} \\ &\geqq \frac{\pi}{2} A \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3} \left\{ \frac{m}{(n-2m+1)^{1/2}} + S' + S'' \right\}^2 \right]^{1/2} \\ &\qquad - A \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3} S'''^2 \right]^{1/2} \\ &= \frac{\pi}{2} AS_1 - AS_2 \end{aligned}$$

say. From (11) we have

$$\tag{13} \begin{aligned} S_2 &\leqq \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3} \left(\frac{(n-1)^2}{(n-1)^2-(2m)^2} \right)^2 \right]^{1/2} \\ &\leqq \sum_{n=4N}^{\infty} \frac{1}{n^{1/2}} \left[\sum_{m=N}^{[n/2]-1} \frac{1}{m(\log m)^3 (n-2m-1)^2} \right]^{1/2} \\ &\leqq \sum_{n=4N}^{\infty} \frac{1}{n^{1/2}} \left[\frac{1}{(n-n/2-1)^2} \sum_{m=N}^{[n/4]} \frac{1}{m(\log m)^3} \right. \\ &\qquad \left. + \frac{1}{n(\log n/4)^3} \sum_{m=[n/4]+1}^{[n/2]-1} \frac{1}{(n-2m-1)^2} \right]^{1/2} \end{aligned}$$

$$\leqq C_{12} \sum_{n=4N}^{\infty} \frac{1}{n(\log n)^{3/2}} < \infty.$$

Since S' and S'' are positive we get

$$S_1 \geqq \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\sum_{m=[n/4]}^{[n/2]-1} \frac{m}{(\log m)^3(n-2m+1)} \right]^{1/2} \tag{14}$$

$$\geqq C_{13} \sum_{n=4N}^{\infty} \frac{1}{n^{3/2}} \left[\frac{n}{(\log n)^3} \sum_{m=[n/4]}^{[n/2]-1} \frac{1}{(n-2m+1)} \right]^{1/2}$$

$$\geqq C_{14} \sum_{n=4N}^{\infty} \frac{1}{n \log n} = \infty.$$

Hence from (12), (13) and (14) we obtain $S = \infty$, and from the estimation (8) we get easily a contradiction with the inequality (6). Therefore we conclude that the series (5) diverges for almost all u, that is the Fourier series of $f_u(t)$ is non-summable $\left| C, \frac{1}{2} \right|$ at $t = 0$ for almost all u.

From the definition of $f_u(t)$, it is continuous in the closed interval $\left[\frac{\pi}{2}, \pi \right]$ for almost all u as we see by Lemma 3. Consequently we can choose a u, say u_0, such that the function $f_{u_0}(t)$ is continuous everywhere except perhaps at $t = \frac{\pi}{2}$ (mod π) and its Fourier series is not summable $\left| C, \frac{1}{2} \right|$ at $t = 0$. Since the function $f_{u_0}(t)$ may have a discontinuity of finite jump at $t = \frac{\pi}{2}$ (mod π), we add it a suitable even function $l(t)$ of period 2π which vanishes for $0 \leqq t \leqq \frac{\pi}{4}$ and for $\frac{\pi}{2} < t \leqq \pi$ and is linear for $\frac{\pi}{4} < t < \frac{\pi}{2}$ with a resulting continuous function:

$$g(t) = f_{u_0}(t) + l(t).$$

The function $l(t)$ being of bounded variation its Fourier series is summable $|C, \delta|$ for every $\delta > 0$ at $t = 0$ in virtue of the Bosanquet theorem [1]. Therefore by the above result for $f_{u_0}(t)$ the Fourier series of $g(t)$ is not summable $\left| C, \frac{1}{2} \right|$ at $t = 0$. Obviously $g(t) = 0$ for $-\frac{\pi}{4} \leqq t \leqq \frac{\pi}{4}$ and we complete the proof.

3. The summability $|C, \delta|$ is of local property for the class L^p if $\delta > 1/p$ and $1 \leqq p \leqq 2$. The case $p = 1$ is due to Basanquet [2] and the case $1 < p \leqq 2$ is a consequence of the Theorem 1. But the summability $|C, 1/p|$ is not of local property for the class L^p $(1 \leqq p < 2)$. This fact was already proved by Basanquet and Kestelman [3] for $p = 1$ and by Yano [9] for $p > 1$. We shall show this by a simple counter examples.

THEOREM 3. *For $1 \leqq p < 2$, there exists a function $\in L^p$ of period 2π which vanishes in the interval $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$ and whose Fourier series is not summable $|C, 1/p|$ at the origin.*

LEMMA 3. *If a series Σc_n is summable $|C, \delta|$ $(\delta > 0)$, then the series $\Sigma |c_n|/n^\delta$ is convergent.*

This is due to Kogbetliantz [6].

PROOF OF THEOREM 3. We suppose first that $1 < p \leqq 2$. Let us consider the series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{n^{1-1/p} \log n} \tag{8}$$

which is the Fourier series of a function $\in L^p$ (Zygmund [10], 9, 501, p. 212).

Let us define:

$$\begin{aligned} f(x) &= \sum_{n=2}^{\infty} \frac{\cos 2nx}{n^{1-1/p} \log n} \qquad && \text{for } \frac{\pi}{2} < x \leqq \pi, \\ &= 0 && \text{for } 0 \leqq x \leqq \frac{\pi}{2}, \end{aligned}$$

$f(x) = f(-x)$ and $f(x + 2\pi) = f(x)$. Obviously $f(x) \in L^p$ $(0, 2\pi)$. We shall prove that its Fourier series is not summable $|C, 1/p|$ at $x = 0$. Denote by a_k the k-th Fourier coefficient of $f(x)$. We have

$$\begin{aligned} a_{2k} &= \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos 2kx \sum_{n=2}^{\infty} \frac{\cos 2nx}{n^{1-1/p} \log n} dx \\ &= \frac{1}{2k^{1-1/p} \log k}, \qquad (k = 2, 3, \ldots.) \end{aligned}$$

since the termwise integration is permitted by the well known theorem (Cf. [10], p. 91). Therefore we get

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^{1/p}} \geqq \sum_{k=2}^{\infty} \frac{|a_{2k}|}{(2k)^{1/p}} = \frac{1}{2^{1+1/p}} \sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty,$$

and it fails the necessary condition for the summability $|C, 1/p|$ at the origin by Lemma 3.

In the case $p = 1$ we consider instead of (8) the following series:

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}, \tag{9}$$

which is a Fourier series of an integrable function ([10], p. 109). Constructing an analogous function using (9), we can easily obtain a required example.

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