INTEGRABILITY OF TRIGONOMETRICAL SERIES I

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1. R. P. Boas [1], G. Sunouchi [2] and B. Sz-Nagy [3] have proved the following theorems.

THEOREM I. *If* $a_n \downarrow 0$ *and* $g(x) = \Sigma a_n \cos nx$, *then a necessary and sufficient condition that* $\Sigma a_n/n^\gamma$ *converges, is that* $x^{\gamma-1} g(x) \in L$ *for* $0 < \gamma < 1$.
The same holds for sine series.

THEOREM II. *If* $g(x)$ *is positive and even in* $|x| < \pi$ *and is decreasing in* $(0, \pi)$ *and* (a_n) *is cosine coefficients of* $g(x)$, *then a necessary and sufficient condition that* $\Sigma|a_n|/n^\gamma$ $(0 < \gamma < 1)$ *converges, is that* $x^{\gamma-1} g(x) \in L$.
The same holds for sine series.

These theorems give the condition that absolute convergence of $\Sigma a_n/n^\gamma$ is equivalent to absolute integrability of $g(x)/x^{1-\gamma}$.

We prove theorems, replaced absolute convergence and absolute integrability by conditional convergence and Cauchy integrability respectively, wholy or partially. Our theorems are closely related to those due to R. P. Boas [4] and S. Izumi [5].

2. THEOREM 1. *Let* $0 < \alpha < 1$ *and*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

If $x^{\alpha-1} f(x)$ *is absolutely integrable, then the series* $\Sigma a_n/n^\alpha$ *converges*[1].
The same holds for sine series.

This theorem contains the counter part of Theorem I.

For the proof we use a lemma, due to R. Salem [6], (cf. Zygmund [7]).

LEMMA 1. *Let* $0 < M < N$, $0 < \alpha < 1$ *and* $0 \leqq t \leqq \pi$. *Then there is an absolute constant* C *such that*

$$\left| \int_M^N \frac{\cos ut}{u^\alpha} du - \sum_{n=M}^{N} \frac{\cos nt}{n^\alpha} \right| \leqq \frac{C}{M^\alpha}. \tag{1}$$

PROOF. It is sufficient to prove (1) for non-integral M and N. Let the integral and sum of (1) be I and S, respectively. If we put

$$\xi(u) = [u] + 1/2 \qquad (u \neq 1, 2, \ldots.),$$

then

$$S = \int_M^N \frac{\cos ut}{u} d\xi(u).$$

1) R. P. Boas [1] has in fact proved the theorem for the case $a_n \geqq 0$.

Further writing $X(u) = u - [u] - 1/2$, we get

$$S - I = \int_M^N \frac{\cos ut}{u^\alpha} dX(u)$$

$$= \left[\frac{\cos ut}{u^\alpha} X(u) \right]_M^N - \int_M^N \left(\frac{\cos ut}{u^\alpha} \right)' X(u) du$$

$$= \left[\frac{\cos ut}{u^\alpha} X(u) \right]_M^N - \int_M^N \left(\frac{1}{u^\alpha} \right)' \cos ut X(u) du$$

$$- \int_M^N \frac{(\cos ut)'}{u^\alpha} X(u) du$$

$$\equiv T_1 - T_2 - T_3,$$

say. Now

$$|T_1| \leqq C/M^\alpha,$$

$$|T_2| \leqq \alpha \int_M^N \frac{|\cos ut|}{u^{\alpha+1}} |X(u)| du \leqq C/M^\alpha.$$

Finally, since $X(u) \sim -\dfrac{1}{\pi} \sum_{n=1}^\infty \dfrac{\sin 2\pi nu}{n}$, we have

$$T_3 = t \int_M^N \frac{\sin ut}{u^\alpha} X(u) du$$

$$= -\frac{t}{\pi} \sum_{n=1}^\infty \frac{1}{n} \int_M^N \frac{\sin ut \cdot \sin 2\pi nu}{u^\alpha} du,$$

where the inner integral is less than $C/M^\alpha n$ in absolute value, and then $|T_3| \leqq C/M^\alpha$. Thus we get the required inequality (1).

We shall now prove Theorem 1. Since

$$a_n = \frac{2}{\pi} \int_0^\pi f(t) \cos nt \, dt,$$

We have

$$\sum_{n=M}^N \frac{a_n}{n^\alpha} = \frac{2}{\pi} \sum_{n=M}^N \frac{1}{n^\alpha} \int_0^\pi f(t) \cos nt \, dt$$

$$= \frac{2}{\pi} \int_0^\pi f(t) \left(\sum_{n=M}^N \frac{\cos nt}{n^\alpha} \right) dt$$

$$= \frac{2}{\pi} \int_0^\pi f(t) dt \int_M^N \frac{\cos ut}{u^\alpha} du$$

$$- \frac{2}{\pi} \int_0^\pi f(t) dt \left[\int_M^N \frac{\cos ut}{u^\alpha} du - \sum_{n=M}^N \frac{\cos nt}{n^\alpha} \right]$$

$$= \frac{2}{\pi}(I_1 + I_2),$$

say. By Lemma 1, we get

$$|I_2| \leqq \frac{C}{M^\alpha}\int_0^\pi |f(t)|dt = o(1) \qquad (M \to \infty).$$

Writing

$$I_1 = \int_0^\pi = \int_0^{\pi/L} + \int_{\pi/L}^\pi = I_{1,1} + I_{1,2},$$

we have

$$|I_{1,1}| \leqq \int_0^{\pi/L} |f(t)|t^{\alpha-1}\,dt \left|\int_{Mt}^{Nt} \frac{\cos u}{u^\alpha}\,du\right|$$

$$\leqq C\int_0^{\pi/L} |f(t)|t^{\alpha-1}\,dt,$$

which is $o(1)$ for sufficiently large L, and

$$I_{1,2} = \int_{\pi/L}^\pi \frac{|f(t)|t^{\alpha-1}}{M^\alpha t^\alpha}\,dt \int_{Mt}^{\xi} \cos u\,du$$

$$= O\left(\frac{1}{M^\alpha}\int_{\pi/L}^\pi \frac{|f(t)|}{t}\,dt\right) \qquad (Mt < \xi < Nt),$$

which is $o(1)$ for sufficiently large $M = M(L)$.

Accordingly, $I_1 + I_2 = o(1)$ as $M \to \infty$, and then $\Sigma\, a_n/n^\alpha$ converges.

Proof for the sine series is quitely similar.

3. THEOREM 2. *Let* $0 < \alpha < 1$ *and*

$$f(x) \sim \sum_{n=1}^\infty a_n \cos nx.$$

If $\Sigma a_n/n^\alpha$ *converges absolutely, then* $f(t)t^{\alpha-1}$ *is integrable in the Cauchy sense.*

The same holds for sine series.

This theorem contains direct part of Theorem 2.

Let us prove Theorem 2. We have

$$\int_{\pi/N}^{\pi/M} f(t)t^{\alpha-1}\,dt = \int_{\pi/N}^{\pi/M} \frac{dt}{t^{1-\alpha}}\sum_{n=1}^\infty a_n \cos nt$$

$$= \sum_{n=1}^\infty \frac{a_n}{n^\alpha}\int_{\pi/N}^{\pi/M} \frac{\cos nt}{t^{1-\alpha}}\,dt = \sum_{n=1}^\infty \frac{a_n}{n^\alpha}\int_{n\pi/N}^{n\pi/M} \frac{\cos u}{u^{1-\alpha}}\,du \tag{2}$$

$$= \sum_{n=1}^\infty \frac{a_n}{n^\alpha}\int_{n\pi/M}^\infty \frac{\cos u}{u^{1-\alpha}}\,du - \sum_{n=1}^\infty \frac{a_n}{n^\alpha}\int_{n\pi/N}^\infty \frac{\cos u}{u^{1-\alpha}}\,du$$

$$= I_1 - I_2,$$

say. We write

$$I_1 = \sum_{n=1}^{N-1} + \sum_{n=N}^{\infty} = I_{1,1} + I_{1,2}.$$

$$|I_{1,1}| \leqq \sum_{n=1}^{N} |s_n| \left| \int_{n\pi/N}^{(n+1)\pi/N} \frac{\cos u}{u^{1-\alpha}} du \right| + |s_N| \left| \int_{(N-1)\pi/N}^{(N-1)\pi/M} \frac{\cos u}{u^{1-\alpha}} du \right|$$

where $s_n = \sum_{\nu=1}^{n} \frac{a_\nu}{\nu^\alpha}$. Since we can suppose that $s_n \to 0$ $(n \to \infty)$, the second term of the right side is $o(1)$, and the first term is

$$\sum_{n=1}^{N-2} |s_n| \left| \int_{n\pi/N}^{(n+1)\pi/N} \frac{du}{u^{1-\alpha}} \right| \leqq \frac{1}{N} \sum_{n=1}^{N-2} |s_n| \left(\frac{N}{n}\right)^{1-\alpha}$$

$$= \frac{1}{N^\alpha} \sum_{n=1}^{N-2} \frac{|s_n|}{n^{1-\alpha}} = o(1).$$

Hence $I_{1,2} = o(1)$[1]. By the absolute convergence of $\sum a_n/n^\alpha$ and boundedness of the integral $\int_v^w \frac{\cos u}{u^{1-\alpha}} du$, we get $I_1 = o(1)$. Since we can similarly prove that $I_2 = o(1)$, $f(t)t^{\alpha-1}$ is integrable in the Cauchy sense, which is the required.

4. THEOREM 3. *Let* $0 < \alpha < 1$ *and*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

If $\sum a_n/n^\alpha$ *is convergent and*

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-\alpha}} \max_{kM \leqq m < (k+1)M} |s_m - s_{kM}| = o(1) \qquad (M \to \infty) \tag{2}$$

where $s_n = \sum_{k=1}^{n} a_k/k^\alpha$, *then* $f(t)t^{\alpha-1}$ *is integrable in the Cauchy sense.*

The condition (2) is satisfied when $a_n/n^\alpha \downarrow 0$ or $a_n = o(1/n^{1-\alpha})$, more generally when

$$\max_{kM \leqq m < (k+1)M} \left| \sum_{n=kM}^{m} \frac{a_n}{n^\alpha} \right| = o(1/k^\alpha (\log k)^2) \quad (M \to \infty).$$

For the proof of Theorem 3, it is sufficient to prove $I_{1,1} = o(1)$ and $I_{1,2} = o(1)$ in (2) in the proof of Theorem 2. $I_{1,1} = o(1)$ is already proved.

Now, in the sum

$$I_{1,2} = \sum_{n=N}^{\infty} \frac{a_n}{n^\alpha} \int_{n\pi/N}^{\infty} \frac{\cos u}{u^{1-\alpha}} du,$$

1) In the proof of $I_1 = o(1)$, ordinary convergence of $\sum a_n/n^\alpha$ is used. If its absolute convergence is used, the proof becomes simpler. But this proof is used in the proof of Theorem 3.

$\int_v^\infty \frac{\cos u}{u^{1-\alpha}}\, du$ is a simusoidal function of v, and takes extremum value at $(k+1/2)$ $(k=1,2,\ldots.)$. If we put

$$I_{1,2} = \sum_{n=N}^{\infty} = \sum_{n=N}^{3N/2-1} + \sum_{k=1}^{\infty} \sum_{n=(k+1/2)N}^{(k+3/2)N-1} = J_0 + \sum_{k=1}^{\infty} J_k ,$$

then

$$|J_k| \leqq \left| \sum_{n=(k+1/2)N}^{\mu} \frac{a_n}{n^\alpha} \right| \cdot \left| \int_{(k+1/2)N}^{\infty} \frac{\cos u}{u^{1-\alpha}}\, du \right|$$

$$\leqq \frac{1}{k^{1-\alpha}} \max_{(k+1/2)N \leqq \mu \leqq (k+3/2)N} \left| \sum_{n=(k+1/2)N}^{\mu} \frac{a_n}{n^\alpha} \right|$$

for $k \geqq 1$ and J_0 may be similarly estimated. By (2) we get $I_{1,2} = o(1)$. Thus we get the theorem.

5. THEOREM 4. *Let $0 < \alpha < 1$ and*

$$f(x) \sim' \sum_{n=1}^{\infty} a_n \cos nx.$$

If $f(t)/t^{\alpha-1}$ is integrable in the Cauchy sense and

$$\sum_{k=1}^{M} \frac{1}{k^\alpha} \max_{k\pi/M \leqq \xi \leqq (k+1)\pi/M} \left| \int_{k\pi/M}^{\xi} \frac{f(t)}{t^{1-\alpha}}\, dt \right| = o(1) \qquad (M \to \infty), \tag{3}$$

then the series $\sum a_n/n^\alpha$ converges.

(3) is satisfied when $f(t) = o(1/t^\alpha)$, or more generally when

$$\sum_{k=1}^{M} \frac{1}{k} \max_{k\pi/M \leqq \xi \leqq (k+1)\pi/M} \left| \int_{k\pi/M}^{\xi} f(t) dt \right| = o(1/M^{1-\alpha}) \qquad (M \to \infty).$$

Let us now prove Theorem 4.

$$\sum_{n=M}^{N} \frac{a_n}{n^\alpha} = \frac{2}{\pi} \sum_{n=M}^{N} \frac{1}{n^\alpha} \int_0^\pi f(t) \cos nt\, dt$$

$$= \frac{2}{\pi} \int_0^\pi f(t) \left(\sum_{n=M}^{N} \frac{\cos nt}{n^\alpha} \right) dt$$

$$= \frac{2}{\pi} \int_0^\pi f(t)\, dt \int_M^N \frac{\cos ut}{u^\alpha}\, du$$

$$+ \frac{2}{\pi} \int_0^\pi f(t) \left(\sum_{n=M}^{N} \frac{\cos nt}{n^\alpha} - \int_M^N \frac{\cos ut}{u^\alpha}\, du \right) dt$$

$$= I_1 + I_2 ,$$

say. By Lemma 1, $I_2 = o(1)$.

$$I_1 = \int_0^\pi f(t)\, dt \int_M^\infty \frac{\cos ut}{u^\alpha}\, du - \int_0^\pi f(t) dt \int_N^\infty \frac{\cos ut}{u^\alpha}\, du$$

$$= I_{1,1} - I_{1,2},$$

say, and

$$I_{1,1} = \int_0^{\pi/M} + \int_{\pi/M}^{\pi} = I_{1,1,1} + I_{1,1,2},$$

say. We have

$$I_{1,1,1} = \int_0^{\pi/M} \frac{f(t)}{t^{1-\alpha}} dt \int_{Mt}^{\infty} \frac{\cos v}{v^{\alpha}} dv = o(1).$$

$$I_{1,1,2} = \int_{\pi/M}^{\pi} \frac{f(t)}{t^{1-\alpha}} dt \int_{Mt}^{\infty} \frac{\cos v}{v^{\alpha}} dv$$

$$= \int_{\pi/M}^{3\pi/2M} + \sum_{k=1}^{M-2} \int_{(k+1/2)\pi/M}^{(k+3/2)\pi/M} + \int_{(M-1/2)\pi/M}^{\pi}$$

$$= J_0 + \sum_{k=1}^{M-1} J_k + J_M,$$

say, where

$$|J_k| \leqq \frac{A}{k^{\alpha}} \max_{(k+1/2)\pi/M \leqq \xi \leqq (k+3/2)\pi/M} \left| \int_{(k+1/2)\pi/M}^{\xi} \frac{f(t)}{t^{1-\alpha}} dt \right|$$

for $1 \leqq k \leqq M-1$. We get similar estimations for J_0 and J_M. Thus we get $I_{1,1,2} = o(1)$ by (3).

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