ON THE DIRECT-PRODUCT OF OPERATOR ALGEBRAS, III

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Y. Misonou, in his preceding paper [3], has studied the direct-product of arbitrary W^*-algebras generalizing the notion due to F. J. Murray-J. von Neumann [4] for factors of type I. It is defined as follows.

Let A_1 and A_2 be W^*-algebras on Hilbert spaces H_1 and H_2 respectively. Then the direct-product $A_1 \otimes A_2$ of A_1 and A_2 is defined as the weak closure (in the algebra of bounded operators on $H_1 \times H_2$) of the algebraical direct-product $A_1 \odot A_2$.

The main purpose of the present note is to clarify the relationship between A_{1*}, A_{2*} and $(A_1 \otimes A_2)_*$, where A_* denotes the set of all σ-weak continuous (ultra-faiblement continue) linear functions on a W^*-algebra A [1], and as its application we shall prove Y. Misonou's Theorem 1.

Let A_1 and A_2 be W^*-algebras on Hilbert spaces H_1 and H_2 respectively, and let $A_1 \otimes A_2$ be the direct-product of A_1 and A_2 on $H_1 \times H_2$. Moreover, let A_{1*}, A_{2*} and $(A_1 \otimes A_2)_*$ be the Banach space of all σ-weakly continuous linear functionals on A_1, A_2 and $A_1 \otimes A_2$ respectively, then

$$(A_{1*})^* = A_1, (A_{2*})^* = A_2 \text{ and } ((A_1 \otimes A_2)_*)^* = A_1 \otimes A_2$$

as Banach spaces (cf. J. Dixmier [1]).

As in [6], we shall denote the operator bound $\left\| \sum_{i=1}^{n} x_i \times y_i \right\|$ $(x_i \in A_1, y_i \in A_2)$ by $\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)$, therefore its associate norm α' is defined as follows (cf. R. Schatten [5]):

$$\alpha'\left(\sum_{j=1}^{m} \varphi_j \times \psi_j\right) = \sup \frac{\left|\left(\sum_{j=1}^{m} \varphi_j \times \psi_j\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)\right|}{\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)}$$

where sup is taken over all expressions $\sum_{i=1}^{n} x_i \times y_i \in A_1 \odot A_2$; that is, α'-norm is a functional norm.

Then our aimed theorem is the following

THEOREM 1. $\quad A_{1*} \times_{\alpha'} A_{2*} = (A_1 \otimes A_2)_*.$

We shall devide the proof into two steps.

LEMMA 1. $\quad A_{1*} \odot A_{2*} \subseteqq (A_1 \otimes A_2)_*.$

PROOF. $\varphi \in A_{1*}$, $\psi \in A_{2*}$ imply the following representations [1]

$$\varphi(x) = \sum_{i=1}^{\infty} < \xi_i x, \bar{\xi_i} >, \psi(y) = \sum_{j=1}^{\infty} < \eta_j y, \bar{\eta_j} >$$

where $\sum_{i=1}^{\infty} \|\xi_i\|^2$, $\sum_{i=1}^{\infty} \|\bar{\xi}_i\|^2$, $\sum_{j=1}^{\infty} \|\eta_j\|^2$, and $\sum_{j=1}^{\infty} \|\bar{\eta}_j\|^2$ are all finite. Therefore

$$\varphi \times \psi(x \times y) = \sum_{i,j} < \xi_i x, \bar{\xi}_i > \cdot <\eta_j y, \bar{\eta}_j >$$

$$= \sum_{i,j} < (\xi_i \times \eta_i)(x \times y), \bar{\xi}_i \times \bar{\eta}_j >$$

and

$$\sum_{i,j} \|\xi_i \times \eta_j\|^2 = \sum_{i,j} \|\xi_i\|^2 \cdot \|\eta_j\|^2 = \sum_i \|\xi_i\|^2 \cdot \sum_j \|\eta_j\|^2$$

and similarly $\sum_{i,j} \|\bar{\xi}_i \times \bar{\eta}_j\|^2$ are both finite, and, since $A_1 \odot A_2$ is weakly dense in $A_1 \otimes A_2$, $\varphi \times \psi$ can be extended onto $A_1 \otimes A_2$ preserving its representation; then $\varphi \times \psi \in (A_1 \otimes A_2)_*$.

Since an arbitrary element of $A_{1*} \odot A_{2*}$ is a linear combination of the elements of the form $\varphi \times \psi$, we complete the proof of Lemma 1.

LEMMA 2. $A_{1*} \odot A_{2*}$ is α'-dense in $(A_1 \otimes A_2)_*$.

PROOF. Let ϕ be an arbitrary element of $(A_1 \otimes A_2)_*$, and let its representation be $\phi(.) = \sum_{i=1}^{\infty} < u_i \cdot, v_i >$, where $\sum_{i=1}^{\infty} \|u_i\|^2$ and $\sum_{i=1}^{\infty} \|v_i\|^2$ are finite.

Now, if an arbitrary small positive number ε is given, then there exists an integer n_0 such that

$$\Big(\sum_{i=n_0+1}^{\infty} \|u_i\|^2\Big)^{1/2} \Big(\sum_{i=n_0+1}^{\infty} \|v_i\|\Big)^{1/2} < \varepsilon/3.$$

Put $\phi'(.) = \sum_{i=1}^{n_0} < u_i \cdot, v_i >$ on $A_1 \otimes A_2$, then

$$\alpha'(\phi - \phi') = \|\phi - \phi'\| \leqq \Big(\sum_{n_0+1}^{\infty} \|u_i\|^2\Big)^{1/2}\Big(\sum_{n_0+1}^{\infty} \|v_i\|^2\Big)^{1/2} < \varepsilon/3.$$

Since $\{u_i\}, \{v_i\}$ are elements of $H^1 \times H_2$ and $H_1 \odot H_2$ is dense in $H_1 \times H_2$, there exist $\sum_{j=1}^{p_i} \xi_j^i \times \xi_j^i$, $\sum_{k=1}^{q_i} \eta_k^i \times \bar{\eta}_k^i$ such that

$$\Big\|u_i - \sum_{j=1}^{p_i} \xi_j^i \times \xi_j^i\Big\| < \varepsilon/3n_0\lambda \qquad (i = 1, 2, \ldots, n_0)$$

$$\Big\|v_i - \sum_{k=1}^{q_i} \eta_k^i \times \bar{\eta}_k^i\Big\| < \varepsilon/3n_0\lambda$$

where $\lambda = \max (\|u_i\|, \|v_i\|)$.

Now define the functionals

$$\phi''(.) = \sum_{i=1}^{n_0} < \Big(\sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i\Big)\cdot, v_i >$$

and
$$\phi_0(.) = \sum_{i=1}^{n_0} < \Big(\sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i\Big)\cdot, \sum_{k=1}^{q_i} \eta_k^i \times \bar{\eta}_k^i >,$$

then clearly $\phi_0(.) \in A_{1*} \odot A_{2*}$, and

$$\alpha'(\phi - \phi_0) \leqq \alpha'(\phi - \phi') + \alpha'(\phi' - \phi'') + \alpha'(\phi'' - \phi_0)$$

$$\leqq \varepsilon/3 + \alpha'\Big(\sum_{i=1}^{n_0} < \Big(u_i - \sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i\Big)\cdot, v_i > \Big) + \alpha'\Big(\sum_{i=1}^{n_0} < \Big(\sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i\Big)\cdot,$$

$$\Big(v_i - \sum_{k=1}^{q_i} \eta_k^i \times \bar{\eta}_k^i\Big) > \Big) \leqq \varepsilon/3 + \sum_{i=1}^{n_0} \| u_i - \sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i \| \, \| v_i \| + \sum_{i=1}^{n_0} \| \sum_{j=1}^{p_i} \xi_j^i \times \bar{\xi}_j^i \|$$

$$\| v_i - \sum_{k=1}^{q_i} \eta_k^i \times \bar{\eta}_k^i \| \leqq \varepsilon/3 + \sum_1^{n_0} \lambda\varepsilon/3n_0\lambda + \sum_1^{n_0} \lambda\varepsilon/3n_0\lambda = \varepsilon. \qquad \text{Q.E.D.}$$

COROLLARY 1. *Let $C(H)$ be the set of all completely continuous operators on a Hilbert space H, then*

$$(C(H_1) \times {}_{\alpha}C(H_2))^* = C(H_1)^* \times {}_{\alpha'}C(H_2)^*.$$

PROOF. Let $F(H_i)$ be the full operator algebras on H_i, then $F(H_i)_*$ are trace classes of operators on H_i; and moreover $F(H_i)_*$ are conjugate spaces of $C(H_i)$ $(i = 1, 2)$. And finaly by [6]

$$C(H_1) \times {}_{\alpha'}C(H_2) = C(H_1 \times H_2),$$

therefore by our Theorem 1,

$$\begin{aligned}[C(H_1) \times {}^{\alpha}C(H_2)]^* &= C(H_1 \times H_2)^* \\ &= F(H_1 \times H_2)_* \\ &= F(H_1)_* \times {}_{\alpha'}F(H_2)_* \\ &= C(H_1)^* \times {}_{\alpha'}C(H_2)^*. \end{aligned} \qquad \text{q.e.d.}$$

THEOREM 2 (Y. Misonou [3; Thm. 1]). *Let A_1 be a W^*-algebra on Hilbert spaces H_1 and K_1; and let A_2 be a W^*-algebra on a Hilbert space H_2 and K_2. Then the direct product of A_1 and A_2 on $H_1 \times H_2$ is algebraically $*$-isomorphic to the one of A_1 and A_2 on $K_1 \times K_2$.*

PROOF. A_{1*}, A_{2*} and α-norm are determined by algebraic characters of A_1 and A_2, and therefore α'-norm is determined algebraically. Now let $A_1 \otimes A_2 = A$ on $H_1 \times H_2$ and $A_1 \otimes A_2 = B$ on $K_1 \times K_2$. Then by the above mentioned facts and by our theorem A_* and B_* are isometric as Banach spaces.

Now, by our Theorem

$$A = (A_*)^* = (A_{1*} \times {}_{\alpha'}A_{2*})^*$$
$$B = (B_*)^* = (A_{1*} \times {}_{\alpha'}A_{2*})^*.$$

Therefore, by identifying the elements of A and B as functionals on $A_1{}^* \times {}_{\alpha'}A_{2*}$, we have the isometry between A and B as Banach spaces.

While by a Theorem due to R. V. Kadison [2; Thm. 14], the Banach space isometry between two W^*-algebras, is a direct sum of $*$-isomorphism and anti-$*$-isomorphism.

Now, the above-described isometry between A and B contains clearly $A_1 \odot A_2$ in the $*$-isomorphic part. Since $A_1 \odot A_2$ is dense in A, the proof of Theorem 2 is completed.

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