COMPLETELY CONTINUOUS OPERATORS WITH PROPERTY F

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1. The purpose of the present note is to generalize a matrix theorem of McCoy [5] for completely continuous operators on a Hilbert space. Our method is an infinite dimensional version of that of Goldhaber [1]. As an application, an analogue of a theorem of Motzkin-Taussky [6] type follows, which is closely related to a recent theorem of I. Kaplansky [4].

2. If a and b are commutable $n \times n$ matrices, it is well known that they satisfy the following

PROPERTY F. For any polynomial p of two variables, the proper values of $p(a, b)$ are expressible as $p(\lambda_i, \mu_i)$, where $\{\lambda_i\}$ and $\{\mu_i\}$ are proper values (counting their multiplicities and being arranged in suitable order) of a and b.

However, it is also known that the commutativity is not necessary for property F. N. McCoy [5] showed that it is equivalent to the following

PROPERTY M. The commutator

$$[a, b] = ab - ba$$

of a and b belongs to the radical N of the algebra A which is generated by a and b.

To extend the theorem of McCoy for operators on a Hilbert space, the terms of the above properties will be rearranged. It is expected in the below, A of Property M is generated by a and b as a Banach algebra, i. e., the generation includes the process of limitation. Also, it is expected that Property M is introduced into a pair of elements of a Banach algebra by its purely algebraical character.

As a generalization of McCoy's theorem, we shall state as follows:

THEOREM 1. *For a pair of completely continuous operators on a Hilbert space, Property M and Property F are equivalent.*

As an analogue of Motzkin-Taussky-Kaplansky's theorem, we shall state as follows:

THEOREM 2. *For a pair of completely continuous hermitean operators, the following three statements are eqvivalent*;

(i) *They commute,*
(ii) *They have Property F,*
(iii) *They have Property M.*

Clearly, (i) implies (ii), and (ii) implies (iii) by Theorem 1, whence it is sufficent to prove Theorem 2 that (iii) implies (i).

The PROOF of THEOREM 1 will be divided into two steps.

LEMMA 1. *For a pair of completely continuous operators, Property F implies Property M.*

PROOF. Let D be a dense subalgebra of A which is generated algebraically by a and b. Each element of D can be expressed as $p(a, b)$ where p is a polynomial of two variables. If $c = [a, b]$, then $p(a, b)c$ has 0 spectrum, whence c is properly quasi-nilpotent in the sense of Hille [2; p. 458, 480] and so c belongs to the radical of D. This implies that the principal left ideal L generated by c consists of elements with 0 spectra.

Since J. D. Newburgh [7] shows that the spectra of completely continuous operators are continuous, the closure K of L in A consists of elements with 0-spectra. Clearly K is a left ideal of A, whence K is a nil-ideal, and so K is contained in N by a theorem of N. Jacobson [3; Theorem 16]. Therefore c is contained in N.

LEMMA 2. *For completely continuous operators, Property M implies Property F.*

PROOF. Since the proper values of completely continuous operator a coincide with the spectrum of a in the Banach algebra generated by a after a theorem of K. Yosida [8], it is also true in A by the spectre invariance of completely continuous operators.

Since A/N is commutative by the hypothesis, the spectrum of a coincides also with the value $a(M)$ in the Gelfand representation by functions. It is also true for b, and the spectrum corresponds to $b(M)$. Theorefore, arranging their order we can correspond the proper values of a and b by

$$\lambda_i = a(M_i), \quad \mu_i = b(M_i)$$

for some maximal ideal M_i.

By the Gelfand representation theorem, it is obvious that $p(\lambda_i, \mu_i)$ is a proper value of $p(a, b)$, and it enumerates all of them. This is Property F.

Concerning the multiplicity, no difficulty can occur, since it is not hard to see that the proper space belonging to same indicesed proper values is the intersection of the proper space belonging to same proper values of two operators. That is, the multiplicities corresponding to M is same for two operators.

To prove the Theorem 2, we shall begin with the following

LEMMA 3. *If B is a B^*-algebra, and if a and b are hermitean elements of B having Property M, then a and b commute.*

PROOF. Let A be the B^*-subalgebra generated by a and b. A is clearly the closure of the set of all $p(a, b)$ where p runs through all polynomials of two variables. Since a B^*-algebra is semi-simple, the radical N of A vanishes, whence the hypothesis implies $ab - ba = 0$. This completes the proof of Lemma.

If a and b are completely continuous and hermitean, then Property F and Property M are equivalent and also equivalent to the commutativity by Lemma 3, which completes the Theorem.

3. Before to conclude the note, we wish to list a few remarks.

(1) In Theorem 1, no special property of the unitarity of the underlying Hilbert space is used. Hence Theorem 1 can be generalized for a pair of completely continuous operators on a complex Banach space. Moreover, by a verbal change in the proof, Theorem 1 can be extended for a Banach algebra with a suitable restriction. However, by its special restriction, the authors can not find an application of this generalization.

(2) Recently, I. Kaplansky [4] generalized the Motzkin-Taussky Theorem for completely continuous operators of Hilbert spaces in its original form. Since Property L is equivalent to Property F for matrices, it is also expected for completely continuous operators. However, the author can not show this. If it is done, then our Theorem 2 becomes an equivalent formulation of Kaplansky's theorem.

(3) In Theorem 2, we have shown (iii) implies (i). But, it is not also difficult to show (ii) implies (i), since the commutator of hermitean operators is skew symmetric, and since the Property F gives the commutator has 0-spectrum.

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