UNIFORM CONVERGENCE OF SOME TRIGONOMETRICAL SERIES

HIROSHI HIROKAWA

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1. Introduction. On the uniform convergence of some trigonometrical series, G. Sunouchi [6] proved the following theorem.

THEOREM A. *Let* $0 < \alpha < 1$. *If*

$$\sum_{n}^{\infty} |\Delta a_\nu| = O(n^{-\alpha}), \tag{1.1}$$

where $\Delta a_\nu = a_\nu - a_{\nu+1}$, *and*

$$\sum_{1}^{n} \nu a_\nu = o(n^{\alpha}), \tag{1.2}$$

then the series

$$\sum_{1}^{\infty} a_\nu \sin \nu x \tag{1.3}$$

converges uniformly in $0 \leqq x \leqq \pi$.

Concerning this theorem, we shall prove the following

THEOREM 1. *Let* $0 < \alpha < 1$. *If* (1.1) *holds and*

$$t_n^\beta = o(n^{\beta\alpha}), \quad (\beta > 0), \tag{1.4}$$

where t_n^β *is* (C, β)*-sum of the sequence* $\{\nu a_\nu\}$, *then the sine series* (1.3) *converges uniformly in* $0 \leqq x \leqq \pi$.

Recently M. Satô [5] considered the cosine analogue of Theorem A. Concerning the cosine series we shall prove the following

THEOREM 2. *Under the assumptions of Theorem* 1, *the series*

$$\sum_{1}^{\infty} a_\nu \cos \nu x \tag{1.5}$$

converges uniformly in $0 \leqq x \leqq \pi$.

In this Theorem, if we put $\beta = 1$, we get Theorem of Satô [5]. Now, the following theorems are known.

THEOREM B. (I. Ôyama [4]) *Let* $0 < \alpha < 1$, *and* Σa_ν *be convergent. Then, if* (1.1) *holds* and

$$r_n \equiv \sum_{n}^{\infty} a_\nu = o(n^{\alpha-1})$$

the series (1.3) *and* (1.5) *converge uniformly in* $0 \leqq x \leqq \pi$.

THEOREM C. (S. Izumi and N. Matsuyama [3], I. Ôyama[4])
Let $0 < \alpha < 1$ *and* Σa_ν *be convergent. Then, if* (1.1) *holds and*

$$\sum_1^n r_\nu = o(n^{2\alpha-1}),$$

where $r_n = \sum_n^\infty a_\nu$, *then the series* (1.3) *and* (1.5) *converge uniformly in* $0 \leqq x \leqq \pi$.

Concerning these Theorems, we have

THEOREM 3. *Let* $0 < \alpha < 1$ *and* Σa_ν *be convergent. Then, if* (1.1) *holds and*

$$\tau_n^{\beta-1} = o(n^{\beta\alpha-1}), \tag{1.6}$$

where τ_n^β *is* (C, β)*-sum of the sequence* $\{r_\nu\}$ *and* β *is a positive number, then the series* (1.3) *and* (1.5) *converge uniformly in* $0 \leqq x \leqq \pi$.

In this Theorem, if we put $\beta = 1$, then we get Theorem B, and if we put $\beta = 2$, then we get Theorem C. This Theorem was suggested by Prof. G. Sunouchi.

Furthermore we have following

THEOREM 4. *Let* $0 < \alpha < 1$. *If* (1.1) *holds and*

$$s_n^\beta = o(n^{\beta\alpha-1}), \tag{1.7}$$

where s_n^β *is* (C, β)*-sum of the sequence* $\{a_\nu\}$ *and* β *is a positive number, then the series* (1.3) *and* (1.5) *converge uniformly in* $0 \leqq x \leqq \pi$.

In this paper, the main theorems are Theorems 1 and 2. These Theorems are proved in §2 and §3, respectively, Theorems 3 and 4 are corollaries of Theorems 1 and 2. The proof of these are in §4.

I. Ôyama [4] proved that, under the assumption (1.1), (1.4) and (1.6) are equivalent for $\beta = 1$. Also, we can easily see that (1.7) implies (1.6) for $\beta = 1$. But these facts are not valid for general $\beta > 0$. Finally, in §5, we apply these Theorems to summability methods of Riemann and Zygmund.

2. **Proof of Theorem 1.** We can easily see that the series (1.3) converges uniformly in $0 < \varepsilon \leqq x \leqq \pi$ by (1.1) and Abel's lemma,*) where ε is a positive number. Therefore, for the proof it is sufficient to show the uniform convergence of (1.3) at $x = 0$.

Let us put

$$\begin{aligned} \sum_1^\infty a_\nu \sin \nu x &= \sum_{\nu=1}^M a_\nu \sin \nu x + \sum_{\nu=M+1}^\infty a_\nu \sin \nu x \\ &= U(x) + V(x), \end{aligned} \tag{2.1}$$

say, where M will be determined later. Using Abel's transformation and (1.1), we get

$$\begin{aligned} V(x) &= \sum_{\nu=M+1}^\infty a_\nu \sin \nu x \\ &= \sum_{\nu=M+1}^\infty \Delta a_\nu \cdot \bar{D}_\nu(x) + \bar{D}_M(x) a_{M+1}, \end{aligned}$$

where $\bar{D}_\nu(x)$ is conjugate Dirichlet kernel.

*) We remark that (1.1) and (1.4) implies $a_\nu = o(1)$.

We can easily see that $\bar{D}_\nu(x) = O(x^{-1})$ uniformly. Further, since $a_\nu = o(1)$, we have

$$a_n = \sum_{\nu=n}^{\infty} \Delta a_\nu = O\left(\sum_{\nu=n}^{\infty} |\Delta a_\nu|\right) = O(n^{-\alpha}) \tag{2.2}$$

by (1.1). Thus. from (1.1) and (2.2), we get

$$\begin{aligned} V(x) &= O\left(\sum_{\nu=M+1}^{\infty} |\Delta a_\nu| x^{-1}\right) + O(M^{-\alpha}x^{-1}) \\ &= O(M^{-\alpha}x^{-1}). \end{aligned} \tag{2.3}$$

Putting $[\beta] = \gamma$, by repeated use of Abel's transformation γ-times, we have

$$\begin{aligned} U(x) &= \sum_{\nu=1}^{M-\gamma} t_\nu^\gamma \Delta_\nu^\gamma(x) + t_{M-\gamma+1}^\gamma \Delta_{M-\gamma+1}^{\gamma-1}(x) + \dots \\ &\qquad \dots + t_{M-1}^2 \Delta_{M-1}^1(x) + t_M^1 \Delta_M^0(x) \\ &= W(x) + \sum_{\nu=1}^{\gamma} U_\nu(x), \end{aligned} \tag{2.4}$$

say, where

$$\Delta_n^0(x) = \sin nx/n, \quad \Delta_n^k(x) = \Delta_n^{k-1}(x) - \Delta_{n+1}^{k-1}(x)$$

and

$$U_\nu(x) = t_{M-\nu+1}^\nu \Delta_{M-\nu+1}^{\nu-1}(x).$$

Since

$$\Delta_n^{2k}(x) = (-1)^{k+1}2^{2k} \int_0^x \left(\sin\frac{t}{2}\right)^{2k} \cos(n+k)t\, dt, \tag{2.5a}$$

$$\Delta_n^{2k+1}(x) = (-1)^{k+1}2^{2k+1} \int_0^x \left(\sin\frac{t}{2}\right)^{2k+1} \sin\left(n+k+\frac{1}{2}\right)t\, dt \tag{2.5b}$$

for $k = 0, 1, 2, \dots,$ we have

$$\Delta_n^k(x) = O(n^{-1}x^k) \tag{2.6}$$

by the second mean value theorem. From (1.4) and $t_n^0 = na_n = O(n^{1-\alpha})$ (by (2.2)), using Dixson-Ferrar's convexity theorem [1], we have

$$t_n^\nu = O\left\{(n^{1-\alpha})^{1-\frac{\nu}{\beta}} (n^{\beta\alpha})^{\frac{\nu}{\beta}}\right\} = O(n^{\{(1-\alpha)(\beta-\nu)+\alpha\beta\nu\}/\beta}), \tag{2.7}$$

$$(\nu = 1, 2, 3, \dots, \gamma).$$

Hence, by (2.6), (2.7)

$$\begin{aligned} U_\nu(x) &= O(M^{\{(1-\alpha)(\beta-\nu)+\alpha\beta\nu\}/\beta} x^{\nu-1}/M) \\ &= O(x^{\nu-1}M^{(\alpha\beta\nu+\alpha\nu-\alpha\beta-\nu)/\beta}). \end{aligned} \tag{2.8}$$

By the well-known formula

$$t_\nu^\gamma = \sum_{n=0}^{\nu} (-1)^{\nu-n} \binom{\beta-\gamma}{\nu-n} t_n, \qquad (t_0 = 0), \tag{2.9}$$

where $\binom{m}{n} = \dfrac{m(m-1)\ldots.(m-n+1)}{n!}$ and $\binom{0}{0} = 1$, we have

$$\begin{aligned} W(x) &= \sum_{\nu=1}^{M-\gamma} t_\nu^\gamma \Delta_\nu^\gamma(x) \\ &= \sum_{\nu=1}^{M-\gamma} \left\{ \sum_{n=0}^{\nu} (-1)^{\nu-n} \binom{\beta-\gamma}{\nu-n} t_n^\beta \right\} \Delta_\nu^\gamma(x) \\ &= \sum_{n=0}^{M-\gamma} t_n^\beta \sum_{\nu=n}^{M-\gamma} (-1)^{\nu-n} \binom{\beta-\gamma}{\nu-n} \Delta_\nu^\gamma(x). \end{aligned}$$

Here, we consider the two cases, the first is, γ is even and the second, is odd. For the first, from (2. 5a), we have

$$\begin{aligned} W(x) &= \sum_{n=0}^{M-\gamma} t_n^\beta \sum_{\nu=n}^{M-\gamma} (-1)^{\nu-n} \binom{\beta-\gamma}{\nu-n} \int_0^x (-1)^{\frac{\gamma}{2}+1} 2^\gamma \left(\sin\frac{\gamma}{2}\right)^\gamma \cos\left(\nu+\frac{t}{2}\right) t\, dt \\ &= \sum_{n=0}^{M-\gamma} t_n^\beta (-1)^{\frac{\gamma}{2}+1} 2\gamma \int_0^x \sum_{\nu=n}^{M-\gamma} (-1)^{\nu-n} \binom{\beta-\gamma}{\nu-n} \cos\left(\nu+\frac{\gamma}{2}\right) t \left(\sin\frac{t}{2}\right)^\gamma dt \qquad (2.10) \\ &= \sum_{n=0}^{M-\gamma} (-1)^{\frac{\gamma}{2}+1} 2^\gamma t_n^\beta \int_0^x \sum_{\nu=0}^{M-\gamma-n} (-1)^\nu \binom{\beta-\gamma}{\nu} \cos\left(\nu+n+\frac{\gamma}{2}\right) t \left(\sin\frac{t}{2}\right)^\gamma dt. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} \cos\left(\nu+n+\frac{\gamma}{2}\right) t \\ &= R\left\{ \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} \exp(i\nu x) \exp i\left(n+\frac{\gamma}{2}\right) t \right\} \\ &= 2^{\beta-\gamma} \left(\sin\frac{t}{2}\right)^{\beta-\gamma} \cos\left\{ \left(\frac{\beta}{2}+n\right) t + \frac{\beta-\gamma}{2}\pi \right\}, \end{aligned}$$

we write $W(x)$ in the form

$$\begin{aligned} W(x) &= \sum_{n=0}^{\infty} (-1)^{\frac{\gamma}{2}} 2^\gamma t_n^\beta \left[\int_0^x \left(\sin\frac{t}{2}\right)^\beta \cos\left\{ \left(\frac{\beta}{2}+n\right) t + \frac{\beta-\gamma}{2}\pi \right\} dt \right. \\ &\quad \left. - \int_0^x \sum_{\nu=M-\gamma-n+1}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} \cos\left(\nu+n+\frac{\gamma}{2}\right) t \left(\sin\frac{t}{2}\right)^\gamma dt \right] \qquad (2.11) \\ &= W_1(x) - W_2(x), \end{aligned}$$

say. By the second mean value theorem

$$\int_0^x \left(\sin\frac{t}{2}\right)^\beta \cos\left\{ \left(\frac{\beta}{2}+n\right) t - \frac{\beta-\gamma}{2}\pi \right\} dt = O(x^\beta n^{-1}),$$

and then

$$\begin{aligned} W_1(x) &= o\left(\sum_{n=1}^{M-\gamma} n^{\beta\alpha} x^\beta / n \right) \qquad (2.12) \\ &= o(M^{\beta\alpha} x^\beta). \end{aligned}$$

Now we have

$$
\begin{aligned}
W_2(x) &= o\Big(\sum_{n=0}^{M-\gamma} n^{\beta\alpha} \sum_{\nu=M-\gamma+n+1}^{\infty} \nu^{-(\beta-\gamma+1)} x^{\gamma}/(\nu+n)\Big) \\
&= o\Big(\frac{(M-\gamma)^{\beta\alpha}}{M-\gamma+1} \sum_{n=0}^{M-\gamma} (M-\gamma-n+1)^{-\beta+\gamma} x^{\gamma}\Big) \qquad (2.13) \\
&= o\Big(x^{\gamma} M^{\beta\alpha-1} \sum_{n=1}^{M-\gamma+1} n^{-\beta+\gamma}\Big) \\
&= o(x^{\gamma} M^{\beta\alpha-\beta+\gamma}).
\end{aligned}
$$

Thus, from (2.3), (2.8), (2.12) and (2.13)

$$
\begin{aligned}
\sum_{\nu=1}^{\infty} a_\nu \sin \nu x &= O(1/xM^{\alpha}) + o(x^{\beta} M^{\beta\alpha}) \qquad (2.14) \\
&\quad + o(x^{\gamma} M^{\beta\alpha-\beta+\gamma}) + \sum_{\nu=1}^{\gamma} o(x^{\nu-1} M^{(\alpha\beta\nu+\alpha\nu-\alpha\beta-\nu)/\beta}),
\end{aligned}
$$

We note that (2.14) holds also when the summation is extended on $1 \leqq \nu \leqq N$, N being a function of x such that $N \to \infty$ as $x \to 0$.

We can now prove the uniform convergence of (1.3) at $x = 0$. For this purpose, it is sufficient to prove the convergence of

$$\sum_{\nu=1}^{N} a_\nu \sin \nu x_N$$

as $N \to \infty$ for any sequence $\{x_N\}$ tending to zero. Now we have, by (2.14) and its remark,

$$
\begin{aligned}
\sum_{\nu=1}^{N} a_\nu \sin \nu x_N &= O(1/x_N M^{\alpha}) + o(x_N^{\beta} M^{\beta\alpha}) + o(x_N^{\gamma} M^{\beta\alpha-\beta+\gamma}) \\
&\quad + \sum_{\nu=1}^{\gamma} o(x_N^{\nu-1} M^{(\alpha\beta\nu+\alpha\nu-\alpha\beta\nu-\nu)/\beta}).
\end{aligned}
$$

When we put $M = \left[(\varepsilon x_N)^{-\frac{1}{\alpha}}\right]$, where ε is an arbitrary positive number, we have

$$O(1/x_N M^{\alpha}) = O(\varepsilon) \leqq \varepsilon, \qquad o(x_N^{\beta} M^{\beta\alpha}) = o(1),$$

$$o(x_N^{\gamma} M^{\beta\alpha-\beta+\gamma}) = o(x^{\gamma-\beta+\frac{\beta}{\alpha}-\frac{\gamma}{\alpha}}) = o(x_N^{(\beta-\gamma)(\frac{1}{\alpha}-1)}) = o(1),$$

and

$$o(x_N^{\nu-1} M^{(\alpha\beta\nu+\alpha\nu-\alpha\beta-\nu)/3} = o(x_N^{\nu-1+(\alpha\beta\nu+\alpha\nu-\alpha\beta-\nu)/\alpha\beta}) = o(x_N^{\nu(1-\alpha)}) = o(1)$$

for $\nu = 1, 2, \ldots, \gamma$.

Therefore, we get

$$\sum_{\nu=1}^{N} a_\nu \sin \nu x_N = o(1).$$

For the second case, that is, γ is odd, we can prove similarly so that we omitt its proof. Thus, the Theorem is proved.*)

3. Proof of Theorem 2. Firstly we prove the following lemma. This lemma was proved by M. Satô[5] for $\beta = 1$.

LEMMA. *Under the assumptions of Theorem* 1, *the series* Σa_ν *is convergent.*

PROOF. We shall consider the case that $0<\beta<1$. Let s_n be the n-th partial sum of Σa_ν. Then, by the well-known formula (2.9), we have

$$\begin{aligned} s_p - s_q &= \sum_{\nu=q+1}^{p} a_\nu \\ &= \sum_{\nu=q+1}^{p} \nu a_\nu \frac{1}{\nu} \\ &= \sum_{\nu=q+1}^{p} \frac{1}{\nu} \sum_{n=0}^{\nu} (-1)^{\nu-n} \binom{\beta}{\nu-n} t_n^\beta \qquad (3.1) \\ &= \left(\sum_{\nu=1}^{p} - \sum_{\nu=1}^{q}\right) \frac{1}{\nu} \sum_{n=0}^{\nu} (-1)^{\nu-n} \binom{\beta}{\nu-n} t_n^\beta \\ &= P - Q, \end{aligned}$$

say. Then

$$\begin{aligned} P &= \sum_{\nu=0}^{p} \frac{1}{\nu} \sum_{n=0}^{\nu} (-1)^{\nu-n} \binom{\beta}{\nu-n} t_n^\beta \\ &= \sum_{n=0}^{p} t_n^\beta \sum_{\nu=n}^{p} (-1)^{\nu-n} \binom{\beta}{\nu-n} \frac{1}{\nu} \\ &= \sum_{n=0}^{p} t_n^\beta \sum_{\nu=0}^{p-n} (-1)^{\nu} \binom{\beta}{\nu} \frac{1}{\nu+n}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{\beta}{\nu} \frac{1}{\nu+n} &= \int_0^1 x^{n-1}(1-x)^\beta \, dx \\ &= \Gamma(n)\Gamma(\beta+1)/\Gamma(n+\beta+1) \qquad (3.2) \\ &= O(n^{-\beta-1}), \qquad \text{(See Titchmarsh [9, p. 56])} \end{aligned}$$

we write P and Q in the form

$$\begin{aligned} P &= \sum_{n=0}^{p} t_n^\beta \Gamma(n)\Gamma(\beta+1)/\Gamma(n+\beta+1) - \sum_{n=0}^{p} t_n^\beta \sum_{\nu=p-n+1}^{\infty} (-1)^{\nu} \binom{\beta}{\nu} \frac{1}{\nu+n} \\ &= P_1 - P_2, \end{aligned}$$

say, and

$$\begin{aligned} Q &= \sum_{n=0}^{q} t_n^\beta \Gamma(n)\Gamma(\beta+1)/\Gamma(n+\beta+1) - \sum_{n=0}^{q} t_n^\beta \sum_{\nu=p-n+1}^{\infty} (-1)^{\nu} \binom{\beta}{\nu} \frac{1}{\nu+n} \\ &= Q_1 - Q_2, \end{aligned}$$

*) The method of the proof was used in Hirokawa and Sunouchi [7].

say. Then, from (1.4) and (3.2)

$$P_1 - Q_1 = \sum_{n=q+1}^{p} t_n^\beta \Gamma(n)\Gamma(\beta+1)/\Gamma(n+\beta+1)$$
$$= o\Big(\sum_{n=q+1}^{p} n^{\beta\alpha}/n^{\beta+1}\Big)$$
$$= o(1).$$

On the other hand

$$P_2 = \sum_{n=0}^{p} t_n^\beta \sum_{\nu=p-n+1}^{\infty} (-1)^\nu \binom{\beta}{\nu} \frac{1}{\nu+n}$$
$$= \Big(\sum_{n=0}^{p/2} + \sum_{n=p/2+1}^{p}\Big) t_n^\beta \sum_{\nu=p-n+1}^{\infty} (-1)^\nu \binom{\beta}{\nu} \frac{1}{\nu+n}$$
$$= P_3 + P_4,$$

say. Then

$$P_3 = O\Big(\sum_{n=0}^{p/2} |t_n^\beta| \sum_{\nu=p-n+1}^{\infty} 1/\nu^{\beta+1}(\nu+n)\Big)$$
$$= O\Big(\sum_{n=0}^{p/2} |t_n^\beta|/(p+1)(p-n+1)^\beta\Big)$$
$$= O\Big(\frac{1}{p^{\beta+1}} \sum_{n=0}^{p/2} |t_n^\beta|\Big)$$
$$= o(p^{\beta\alpha+1}/p^{\beta+1})$$
$$= o(p^{\beta\alpha-\beta})$$
$$= o(1)$$

and

$$P_4 = o\Big(\sum_{n=p/2+1}^{p} n^{\beta\alpha} \sum_{\nu=p-n+1}^{\infty} 1/\nu^{\beta+1}(\nu+n)\Big)$$
$$= o\Big(p^{\beta\alpha-1} \sum_{n=p/2+1}^{p} 1/(p-n+1)^\beta\Big)$$
$$= o\Big(p^{\beta\alpha-1} \sum_{n=1}^{p/2} 1/n^\beta\Big)$$
$$= o(p^{\beta\alpha-\beta})$$
$$= o(1).$$

Similar method shows that $Q_2 = o(1)$. Thus we get

$$s_p - s_q = (P_1 + P_2) - (Q_1 + Q_2)$$
$$= (P_1 - Q_1) + (P_2 - Q_2)$$
$$= o(1).$$

Therefore Σa_ν converges for $0 < \beta < 1$. *)

Next, we shall consider the case that $\beta \geqq 1$. Putting $[\beta] = \gamma$, by repeated

*) The method of the proof was suggested by Prof. G. Sunouchi.

use of Abel's transformations γ-times, we have

$$\begin{aligned} s_p - s_{q-1} &= \sum_{\nu=q}^{p} a_\nu \\ &= \sum_{\nu=q}^{p} \nu\, a_\nu \cdot \frac{1}{\nu} \\ &= \sum_{\nu=q}^{p-\gamma} t_n^\gamma \Delta_n^\gamma + \sum_{\nu=1}^{\gamma} t_{p-\nu-1}^{\nu} \Delta_{p-\nu-1}^{\nu-1} - \sum_{\nu=1}^{\gamma} t_{q-\nu-2}^{\nu} \Delta_{q-\nu-1}^{\nu-1} \\ &= R_0 - \sum_{\nu=1}^{\gamma} R_\nu + \sum_{\nu=1}^{\gamma} R'_\nu, \end{aligned}$$

say, where $\Delta_n^0 = 1/n$, and $\Delta_n^k = \Delta_n^{k-1} - \Delta_{n+1}^{k-1}$.

Since $\Delta_n^\nu = O(1/n^{\nu+1})$, from (2.7)

$$\begin{aligned} R_0 &= o\Big(\sum_{\nu=q}^{p-\gamma} \nu^{\{(1-\alpha)(\beta-\gamma)+\alpha\beta\nu\}/\beta} \nu^{-\gamma-1}\Big) \\ &= o(q^{\{(1-\alpha)(\beta-\gamma)+\alpha\beta\gamma\}/\beta-\alpha}) \\ &= o(q^{(1-\alpha)(\beta-\gamma-\beta\gamma)/\beta}) \\ &= o(1), \end{aligned}$$

and

$$\begin{aligned} R_\nu &= o(q^{\alpha\nu-\nu+(1-\alpha)(\beta-\nu)/\beta}) \\ &= o(q^{(\alpha-1)(\beta\nu-\beta+\nu)/\beta}) \\ &= o(1) \end{aligned}$$

for $\nu = 1, 2, \dots, \gamma$. Hence $\sum_{\nu=1}^{\gamma} R_\nu = o(1)$. Similarly $\sum_{\nu=1}^{\gamma} R'_\nu = o(1)$.

Therefore, we have

$$s_p - s_{q-1} = o(1).$$

Thus, the proof of Lemma is complete.

PROOF OF THEOREM. The method is similar as in former section.

We shall prove that the uniform convergence of (1.5) at $x = 0$. Let us write

$$\begin{aligned} \sum_{\nu=1}^{\infty} a_\nu \cos \nu x &= \Big(\sum_{\nu=1}^{M} + \sum_{\nu=M+1}^{\infty}\Big) a_\nu \cos \nu x \\ &= U(x) + V(x), \end{aligned}$$

say, where M will be determined later. Then we have

$$V(x) = O(1/xM^\alpha) \tag{3.3}$$

by the analogous method to the one which we obtain (2.3). As in § 2, putting $[\beta] = \gamma$, by repeated use of Abel's transformation γ-times, we get

$$U(x) = \sum_{\nu=1}^{M} a_\nu \cos \nu x$$

$$= -\sum_{\nu=1}^{M} \nu\, a_\nu \int_0^x \sin \nu\, x\, dx + \sum_{\nu=1}^{M} a_\nu$$

$$= -\sum_{\nu=1}^{M-\gamma} t_n^\gamma\, \Delta_n^\gamma(x) - \sum_{\nu=1}^{\gamma} t_{n-\nu-1}^\nu\, \Delta_{n-\nu-1}^{\nu-1}(x) + \sum_{\nu=1}^{M} a_\nu$$

$$= -W(x) - \sum_{\nu=1}^{\gamma} t_{n-\nu-1}^\nu\, \Delta_{x-\nu-2}^{\nu-1}(x) + \sum_{\nu=1}^{M} a_\nu,$$

say, where $\Delta_n^0(x) = \int_0^x \sin nx\, dx$, and $\Delta_n^k(x) = \Delta_n^{k-1}(x) - \Delta_{n+1}^{k-1}(x)$.

Since

$$\Delta_n^{2k}(x) = 2^{2k} \int_0^x \left(\sin \frac{t}{2} \right)^{2k} \sin (n+k)t\, dt,$$

$$\Delta_n^{2k+1}(x) = 2^{2k+1} \int_0^x \left(\sin \frac{t}{2} \right)^{2k+1} \cos \left(n + k + \frac{1}{2} \right) t\, dt$$

for $k = 0, 1, 2, \ldots,$ we can proceed the proof as in § 2. Since Σa_ν convergent by Lemma, we have

$$\sum_{\nu=1}^{\infty} a_\nu \cos \nu x - \sum_{\nu=1}^{\infty} a_\nu = O(1/xM^\alpha) + o(x^\beta M^{\beta\alpha}) + o(x^\gamma M^{\beta\alpha-\beta+\gamma})$$

$$+ \sum_{\nu=1}^{\gamma} o(x^{\nu-1} M^{(\alpha\beta\nu-\alpha\nu-\alpha\beta-\nu)/\beta}).$$

Hence we can prove Theorem 2 as in § 2.

4. Proof of Theorems 3–4. For our purpose, it is sufficient to prove that each of the conditions (1. 6) and (1. 7) implies the condition (1. 4). First, we shall prove the former.

By definition of r_n,

$$\sum_{\nu=1}^{n} \nu\, a_\nu = \sum_{\nu=1}^{n} \nu\, (r_\nu - r_{\nu+1})$$

$$= \sum_{\nu=1}^{n} r_\nu - n r_{n+1},$$

that is,

$$t_n^1 = \tau_n^1 - n\, \tau_{n+1}^0.$$

Further, for a positive integer β, we have, using Abel's lemma,

$$t_n^\beta = \beta\, \tau_n^\beta - n\, \tau_{n+1}^{\beta-1}.$$

But, an easy calculation shows that this expression holds for any positive number β. Then, using (1. 6),

$$t_n^\beta = \beta \sum_{n=1}^{n} \tau_n^{\beta-1} - n\, \tau_{n+1}^{\beta-1}$$

$$= o\left(\sum_{n=1}^{n} n^{\beta\alpha-1}\right) + o(n^{\beta\alpha})$$
$$= o(n^{\beta\alpha}).$$

Thus, it was prove that the condition (1.6) implies the condition (1.4). Next we consider the latter case. we have Using Abel's lemma and putting

$$s_n = \sum_{1}^{n} a_\nu \,(= s_n^1),$$

we have

$$\sum_{1}^{n} \nu\, a_\nu = -\sum_{1}^{n=1} s_\nu + ns_n,$$

that is,

$$t_n^1 = -\sum_{1}^{n-1} s_\nu^1 + ns_n^1.$$

Further we have

$$t_n^\beta = -\beta\sum_{1}^{n-1} s_\nu^\beta + ns_n^\beta.$$

Therefore, from (1.7), we have

$$t_n^\beta = o\left(\sum_{1}^{n} \nu^{\beta\alpha-1}\right) + o(n^{\beta\alpha})$$
$$= o(n^{\beta\alpha}).$$

Thus, using Theorems 1-2, Theorems 3-4 follow.

Concluding this section, we note that (1.4) does not imply (1.7) in general. For an example, we put $\beta = 1$. Then, since $na_n = t_n^1 - t_{n-1}^1$,

$$s_n^1 = \sum_{1}^{n} a_\nu = \sum_{1}^{n}(t_\nu^1 - t_{\nu-1}^1)/\nu$$
$$= \sum_{1}^{n-1} t_\nu^1/\nu(\nu+1) + t_n^1/n$$

Further, putting $t_n^1/n^\alpha = \eta_n$,

$$s_n^1 = \sum_{\nu=1}^{n-1} \frac{\nu^\alpha}{\nu(\nu+1)}\eta_\nu + \frac{n^\alpha}{n}\eta_n.$$

Hence

$$\frac{s_n^1}{n^{\alpha-1}} = \frac{1}{n^{\alpha-1}}\sum_{\nu=1}^{n-1}\frac{\nu^\alpha}{\nu(\nu+1)}\eta_\nu + \eta_n$$
$$= \sum_{\nu=1}^{\infty} c_{n,\nu}\,\eta_\nu,$$

say, where

$$\begin{aligned} c_{n,\nu} &= \nu^\alpha/\nu(\nu+1)n^{\alpha-1} && (\nu \leqq n-1),\\ &= 1 && (\nu = n),\\ &= 0 && (\nu > n). \end{aligned}$$

Since $\alpha - 1 < 0$,

$$\lim_{n\to\infty} c_{n,\nu} = \infty,$$

for an arbitrarily fixed ν. Thus $\| c_{n,\nu} \|$ is not Toeplitz Matrix. Therefore, (1. 4) does not implies (1. 7).

5. The series $\sum a_\nu$ is said (R_1)-summable to zero when

$$\sum_{\nu=0}^{\infty} \frac{s_\nu}{\nu} \sin \nu x,$$

where $s_\nu = \sum_{1}^{\nu} \frac{a_\nu}{\nu}$ converges for $0 < x < x_0$ and tends to zero as $x \to 0$.

The series Σa_ν is said $(R, 1)$-summable (or Lebesgue summable) to zero when

$$\sum_{\nu=1}^{\infty} a_\nu (\sin \nu x)/\nu x$$

converges for $0 < x < x_0$ and tends to zero as $x \to 0$. Further, the series Σa_ν is said $(K, 1)$-summable to zero when

$$\sum_{\nu=1}^{\infty} a_\nu \int_x^\pi \frac{\sin \nu t}{tg \dfrac{1}{2} t} dt$$

converges for $0 < x < x_0$ and tends to zero as $x \to 0$.

THEOREM 5. *Let* $0 < \alpha < 1$. *Suppose that* (1. 1) *and one of the conditions* (1. 4), (1. 6) *and* (1. 7) *are satisfied. Then, the series* $\sum a_\nu$ *is* (R_1)-, $(R, 1)$-, *and* $(K, 1)$-*summable to zero, respectively.*

PROOF. Under the assumptions of Theorem, the series (1. 3) and (1. 5) are uniformly convergent in $0 \leqq x \leqq \pi$. Hence each series is a Fourier series of some continuous function. For (R_1)-method, O. Szász [8] proved that Fourier series is summable (R_1) at continuity point of function.

This fact holds for $(R, 1)$-method. Thus, Σa_ν is summable (R_1) and $(R, 1)$. On the other hand, S. Izumi [2] proved that (R_1)-method and $(K, 1)$-method are equivalent for Fourier series. Thus we have our Theorem for $(K, 1)$-method.

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DEPARTMENT OF MATHEMATICS, CHIBA INSTITUTE OF TECHNOLOGY, CHIBA.