SOME PARTITION FORMULAS

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1. In a recent paper [1], Bailey showed that Ramanujan's formula

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{(1-x^n)^6} \tag{1.1}$$

is a consequence of

$$\wp(u)-\wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}, \tag{1.2}$$

when the $\wp$-functions are replaced by their Fourier series and the σ-functions are replaced by their product formulas. For some similar results obtained from

$$\wp'(u) = -\sigma(2u)/\sigma^4(u) \tag{1.3}$$

see [2]. In the present note we derive, to begin with, some additional consequences of (1.2) associated with the modulus 3.

2. As Bailey shows, (1.2) implies the identity

$$\begin{aligned}&\sum_{-\infty}^{\infty}\left\{\frac{aq^n}{(1-aq^n)^2}-\frac{bq^n}{(1-bq^n)^2}\right\}\\ &= a\prod\begin{bmatrix} ab, & q/ab, & b/a, & qa/b, & q, & q, & q, & q \\ a, & a, & q/a, & q/a, & b, & b, & q/b, & q/b\end{bmatrix},\end{aligned} \tag{2.1}$$

where

$$\prod\begin{bmatrix} a_1a_2\ldots a_r \\ b_1b_2\ldots b_r\end{bmatrix} = \prod_{n=0}^{\infty}\frac{(1-a_1q^n)\ldots(1-a_rq^n)}{(1-b_1q^n)\ldots(1-b_rq^n)}.$$

We also note the elementary formula

$$\begin{aligned}&\sum_{-\infty}^{\infty}\left\{\frac{aq^n}{(1-aq^n)^2}-\frac{bq^n}{(1-bq^n)^2}\right\}\\ &= \frac{a}{(1-a)^2}-\frac{b}{(1-b)^2}+\sum_{1}^{\infty}\frac{mq^m}{1-q^m}(a^m+a^{-m}-b^m-b^{-m}).\end{aligned} \tag{2.2}$$

In (2.1) take $q=x^6$, $a=x$, $b=x^2$. The right member becomes

$$\begin{aligned}& x\sum_{1}^{\infty}\frac{(1-x^{6n-3})^2(1-x^{6n-1})(1-x^{6n-5})(1-x^{6n})^4}{(1-x^{6n-1})^2(1-x^{6n-5})^2(1-x^{6n-2})^2(1-x^{6n-4})^2}\\ &= x\sum_{1}^{\infty}\frac{(1-x^{6n})^6(1-x^{6n-3})^3}{(1-x^{6n})^2(1-x^{6n-1})(1-x^{6n-2})^2(1-x^{6n-3})(1-x^{6n-4})^2(1-x^{6n-5})}\\ &= x\prod_{1}^{\infty}\frac{(1-x^{3n})^3(1-x^{6n})^3}{(1-x^n)(1-x^{2n})}.\end{aligned}$$

Making the same substitution in (2.2) we get the identity

$$\tag{2.3} \frac{x}{(1-x)^2} - \frac{x^2}{(1-x^2)^2} + \sum_1^\infty \frac{mx^{6m}}{1-x^{6m}}(x^m + x^{-m} - x^{2m} - x^{-2m}) = x\prod_1^\infty \frac{(1-x^{3n})^3(1-x^{6n})^3}{(1-x^n)(1-x^{2n})}.$$

In the same way if we take $q = x^6$, $a = x$, $b = x^3$, we get

$$\tag{2.4} \frac{x}{(1-x)^2} - \frac{x^3}{(1-x^3)^3} + \sum_1^\infty \frac{mx^{6m}}{1-x^{6m}}(x^m + x^{-m} - x^{3m} - x^{-3m}) = x\prod_1^\infty \frac{(1-x^{2n})^4(1-x^{6n})^4}{(1-x^n)^2(1-x^{3n})^2},$$

while the substitution $q = x^6$, $a = x^2$, $b = x^3$ leads to

$$\tag{2.5} \frac{x^2}{(1-x^2)^2} - \frac{x^3}{(1-x^3)^3} + \sum_1^\infty \frac{mx^{6m}}{1-x^{6m}}(x^{2m} + x^{-2m} - x^{3m} - x^{-3m}) = x^2\prod_1^\infty \frac{(1-x^n)^2(1-x^{6n})^{12}}{(1-x^{2n})^4(1-x^{3n})^6}.$$

Note that comparison of (2.3), (2.4), (2.5) yields

$$\tag{2.6} \prod_1^\infty \frac{(1-x^{3n})^3(1-x^{6n})^3}{(1-x^n)(1-x^{2n})} = \prod_1^\infty \frac{(1-x^{2n})^4(1-x^{6n})^4}{(1-x^n)^2(1-x^{3n})^2} - x\prod_1^\infty \frac{(1-x^n)^2(1-x^{6n})^{12}}{(1-x^{2n})^4(1-x^{3n})^6}.$$

3. Now consider the terms in the left member of (2.3) with exponents divisible by 3. Clearly these contribute

$$\frac{3x^3}{(1-x^3)^2} - \frac{3x^6}{(1-x^6)^2} + 3\sum_1^\infty \frac{mx^{18m}}{1-x^{18m}}(x^{3m} + x^{-3m} - x^{6m} - x^{-6m}).$$

Hence if we put

$$\tag{3.1} \prod_1^\infty (1-x^n)^{-1}(1-x^{2n})^{-1} = \sum_0^\infty a(m)x^m,$$

it is evident that

$$\sum_1^\infty a(3n-1)x^{3n}\prod_1^\infty (1-x^{3n})^3(1-x^{6n})^3 = 3x^3\prod_1^\infty \frac{(1-x^{9n})^3(1-x^{18n})^3}{(1-x^{3n})(1-x^{6n})}$$

and therefore

$$\tag{3.2} \sum_0^\infty a(3m+2)x^m = 3\prod_1^\infty \frac{(1-x^{3n})^3(1-x^{6n})^3}{(1-x^n)^4(1-x^{2n})^4}.$$

Incidentally (3.2) implies, by a familiar argument,

$$\tag{3.3} a(9m+8) \equiv 0 \pmod 9.$$

Since (see for example [4, Chapter 21] for the theta formulas cited here and below)

$$(3.4)\qquad \vartheta_2 = \vartheta_2(q) = 2q^{\frac{1}{4}}\prod_1^\infty(1-q^{2n})(1+q^{2n})^2 = 2q^{\frac{1}{4}}\prod_1^\infty\frac{(1-q^{4n})^2}{1-q^{2n}}$$

the right members of (2.4) and (2.5) become $\frac{1}{16}\vartheta_2^2(x^{1/2})\vartheta_2^2(x^{3/2})$ and $\frac{1}{16}\vartheta_2^6(x^{3/2})/\vartheta_2^2(x^{1/2})$, respectively. In particular (2.6) becomes

$$(3.5)\qquad 16x\prod_1^\infty\frac{(1-x^{3n})^3(1-x^{6n})^3}{(1-x^n)(1-x^{2n})} = \vartheta_2^2(x^{\frac{1}{2}})\vartheta_2^2(x^{\frac{3}{2}}) - \vartheta_2^6(x^{\frac{3}{2}})\vartheta_2^{-2}(x^{\frac{1}{2}}).$$

4. Comparison of the Fourier series and the infinite products for the Jacobi functions also leads to interesting identities. In particular the results are rather simple for the function $\operatorname{cs} u = \operatorname{cn} u/\operatorname{sn} u$. We have

$$(4.1)\qquad \frac{2K}{\pi}\operatorname{cs} u = \cot x - 4\sum_1^\infty\frac{q^{2n}\sin 2nx}{1+q^{2n}},$$

where $u = 2Kx/\pi$; also

$$(4.2)\qquad \operatorname{cs} u = k'^{\frac{1}{2}}\cot x\prod_1^\infty\frac{1+2q^{2n}\cos 2x+q^{4n}}{1-2q^{2n}\cos 2x+q^{4n}}.$$

Since

$$\frac{2K}{\pi} = \vartheta_3^2 = \prod_1^\infty(1-q^{2n})^2(1+q^{2n-1})^4$$

and

$$k'^{\frac{1}{2}} = \frac{\vartheta_4}{\vartheta_3} = \prod_1^\infty\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^2,$$

we get from (4.1) and (4.2)

$$(4.3)\qquad \begin{aligned} &\frac{1+a}{1-a} - 2\sum_1^\infty\frac{q^{2n}}{1+q^{2n}}(a^n-a^{-n}) \\ &= \frac{1+a}{1-a}\prod_1^\infty\frac{(1-q^{2n})^4}{(1-q^{4n})^2}\,\frac{(1+q^{2n}a)(1+q^{2n}a^{-1})}{(1-q^{2n}a)(1-q^{2n}a^{-1})},\end{aligned}$$

where $a = e^{2ix}$.

In (4.3) take $q^2 = x^3, a = x$; we get after a little manipulation

$$(4.4)\qquad \frac{1+x}{1-x} - 2\sum_1^\infty\frac{x^{3n}}{1+x^{3n}}(x^n-x^{-n}) = \prod_1^\infty\frac{(1-x^{3n})^6(1-x^{2n})}{(1-x^{6n})^3(1-x^n)^2}.$$

Again if we take $q^2 = x, a = \omega = e^{2\pi i/3}$, we find without much trouble that

$$(4.5)\qquad 1-6\sum_1^\infty\left(\frac{n}{3}\right)\frac{x^3}{1+x^n} = \prod_1^\infty\frac{(1-x^n)^6(1-x^{6n})}{(1-x^{2n})^3(1-x^{3n})^2},$$

where $(n/3)$ is the Legendre symbol.

On the other hand if in (4.3) we take $q^2 = x^3$, $a = -x$, or $q^2 = x, a = -\omega$ we get

$$(4.6) \qquad 1+2\sum_{1}^{\infty}(-1)^n\left(\frac{n}{3}\right)\frac{x^n}{1+x^n} = \prod_{1}^{\infty}\frac{(1-x^n)^2(1-x^{3n})^2}{(1-x^{2n})(1-x^{6n})}.$$

In view of the formula

$$(4.7) \qquad \vartheta_4 = \vartheta_4(q) = \prod_{1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2 = \prod_{1}^{\infty}\frac{(1-q^n)^2}{1-q^{2n}},$$

(4.6) becomes

$$(4.8) \qquad 1+2\sum_{1}^{\infty}(-1)^n\left(\frac{n}{3}\right)\frac{x^n}{1+x^n} = \vartheta_4(x)\vartheta_4(x^3).$$

We may also write (4.4) in the form

$$(4.9) \qquad 1+2\sum_{1}^{\infty}(-1)^{n-1}\left(\frac{n}{3}\right)\frac{x^n}{1-x^n} = \frac{\vartheta_4^3(x^3)}{\vartheta_4(x)}.$$

In terms of

$$\vartheta_3 = \vartheta_3(x) = \sum_{-\infty}^{\infty} x^{n^2},$$

(4.8) and (4.9) become

$$(4.8)' \qquad 1+2\sum_{1}^{\infty}\left(\frac{n}{3}\right)\frac{x^n}{1+(-x)^n} = \vartheta_3(x)\vartheta_3(x^3),$$

$$(4.9)' \qquad 1-2\sum_{1}^{\infty}\left(\frac{n}{3}\right)\frac{x^n}{1-(-x)^n} = \frac{\vartheta_3^3(x^3)}{\vartheta_4(x)};$$

(4.8)′ is well known.

5. We now put

$$(5.1) \qquad \frac{1}{\vartheta_4(x)} = \prod_{1}^{\infty}\frac{1-x^{2n}}{(1-x^n)^2} = \sum_{0}^{\infty} b(m)x^m.$$

Then consideration of the terms in (4.4) with exponents divisible by 3 gives

$$(5.2) \qquad \sum_{0}^{\infty} b(3m)x^m = \prod_{1}^{\infty}\frac{(1-x^{3n})^6(1-x^{2n})^4}{(1-x^{6n})^3(1-x^n)^8} = \frac{\vartheta_4^3(x^3)}{\vartheta_4^4(x)}.$$

This identity evidently implies

$$(5.3) \qquad b(3m) \equiv b(m) \pmod 3.$$

We next put

$$(5.4) \qquad \vartheta_4^3(x) = \prod_{1}^{\infty}\frac{(1-x^n)^6}{(1-x^{2n})^3} = \sum_{0}^{\infty} c(m)x^m.$$

Then (4.5) implies the identity

$$(5.5) \qquad \sum_{0}^{\infty} c(3m)x^m = \prod_{1}^{\infty}\frac{(1-x^n)^8(1-x^{6n})}{(1-x^{2n})^4(1-x^{3n})^2} = \frac{\vartheta_4^4(x)}{\vartheta_4(x^3)}.$$

Since

$$\vartheta_4(x) = \sum_{-\infty}^{\infty} (-1)^n x^{n^2},$$

it is clear from (5.4) that

$$c(m) = \begin{cases} (-1)^n \pmod 3 & (m = 3n^2) \\ 0 \pmod 3 & (\text{otherwise}). \end{cases} \tag{5.6}$$

We also observe that comparison of (5.2) with (5.5) yields

$$\sum_0^{\infty} b(3m)x^m \sum_0^{\infty} c(3m)x^m = \vartheta_4^2(x^3). \tag{5.7}$$

Note that by (5.4), $c(m) = (-1)^m r_3(m)$, where $r_3(m)$ is the number of representations of m as a sum of three squares.

6. The formula [3, p. 106]

$$\frac{1}{2}\frac{\vartheta_1'(0, q)}{\vartheta_3(z, q)} = \sum_1^{\infty} (-1)^{n-1} q^{\left(n-\frac{1}{2}\right)^2} \frac{1 - q^{4n-2}}{1 - 2q^{2n-1}\cos 2z + q^{2(2n-1)}} \tag{6.1}$$

implies

$$\begin{aligned} &\prod_1^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1}a)(1 - q^{2n-1}a^{-1})} \\ &\qquad = \sum_1^{\infty} (-1)^{n-1} q^{n(n-1)} \frac{1 - q^{4n-2}}{(1 - q^{2n-1}a)(1 - q^{2n-1}a^{-1})}. \end{aligned} \tag{6.2}$$

Consequently if we take $q^2 = x^3$, $a^2 = x$ in (6.2), we get

$$\prod_1^{\infty} \frac{(1 - x^{3n})^3}{1 - x^n} = \sum_1^{\infty} (-1)^{n-1} x^{\frac{3}{2}n(n-1)} \frac{1 - x^{3(2n-1)}}{(1 - x^{3n-1})(1 - x^{3n-2})}. \tag{6.3}$$

Similarly using

$$\frac{1}{4}\frac{\vartheta_1'(0, q)}{\vartheta_1(z, q)} = \frac{1}{4\sin z} + \sin z \sum_1^{\infty} (-1)^n \frac{q^{n^2+n} + q^{n^2+3n}}{1 - q^{2n}\cos 2z + q^{4n}} \tag{6.4}$$

we get

$$\begin{aligned} &\prod_1^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n}a)(1 - q^{2n}a^{-1})} \\ &\qquad = 1 + 4\sin^2 z \sum_1^{\infty} (-1)^n \frac{q^{n^2+n} + q^{n^2+3n}}{(1 - q^{2n}a)(1 - q^{2n}a^{-1})} \end{aligned} \tag{6.5}$$

where $a = e^{2iz}$. In (6.5) take $q = x$, $a = e^{2\pi i/3}$ and we find that

$$\prod_1^{\infty} \frac{(1 - x^n)^3}{1 - x^n} = 1 + 3\sum_1^{\infty} (-1)^n x^{\frac{1}{2}n(n+1)} \frac{1 - x^{2n}}{1 - x^{3n}}. \tag{6.6}$$

We may also make use of the formulas for the square of the left members of (6.1) and (6.3) to obtain series for the squares of the left members of (6.3) and (6.4).

The identities (6.3) and (6.6) may be compared with

$$x\prod_{1}^{\infty}\frac{(1-x^{3n})^9}{(1-x^n)^3}=\frac{x(1+x)}{(1-x)^3}+\sum_{1}^{\infty}\frac{n^2x^{3n}}{1-x^{3n}}(x^n+x^{-n})$$

and

$$\prod_{1}^{\infty}\frac{(1-x^n)^9}{(1-x^{3n})^3}=1-9\sum_{1}^{\infty}\left(\frac{n}{3}\right)\frac{n^2x^n}{1-x^n},$$

which are obtained in [2] as consequences of (1. 3). We also remark that if we break up the right member of (6. 3) by means of the evident identity

$$\frac{1-x^{3(2n-1)}}{(1-x^{3n-1})(1-x^{3n-2})}=\frac{1}{1-x^{3n-1}}+\frac{x^{3n-2}}{1-x^{3n-2}}.$$

we get

$$\prod_{1}^{\infty}(1-x^n)^3\sum_{0}^{\infty}p(3m)x^m=\sum_{1}^{\infty}(-1)^{n-1}x^{\frac{1}{2}n(n-1)}\frac{1-x^{3(2n-1)}}{(1-x^{3n-1})(1-x^{3n-2})},\tag{6.7}$$

where $p(n)$ is the partition function. Using Jacobi's formula, the left member of (6. 7) may be written in the form

$$\sum_{0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)}\sum_{0}^{\infty}p(3m)x^m.$$

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