POSITIVE DEFINITE FUNCTION AND DIRECT PRODUCT HILBERT SPACE

HISAHARU UMEGAKI

(Received October 5, 1955)

1. Introduction. A function $V(\omega, \omega')$ defined on a product set $\Omega \times \Omega$ with range in the ring $B(H)$ of all bounded operators on a Hilbert space H is called positive definite provided that $V(\omega, \omega')$ is a bounded operator on H for any ω, ω' in Ω and satisfies the conditions that $V(\omega, \omega') = V(\omega', \omega)^*$ and

$$\sum_{j,k=1}^{n} (\xi_j, V(\omega_j, \omega_k)\xi_k) \geqq 0, \tag{1}$$

for any finite sets of $\omega_j \in \Omega$, $\xi_j \in H$ $(j = 1, 2, \ldots, n)$.

Now let $F(\Omega)$ be the vector space of all finite-valued numerical functions on Ω and $F(\Omega)\otimes H$ the algebraic direct product of $F(\Omega)$ and H. Putting $\delta_{\omega_0}(\omega) = 1$ for $\omega = \omega_0$, $= 0$ for $\omega \neq \omega_0$, and denoting $\delta_{\omega_0}(\omega)\otimes\xi$ as $\omega_0 \times \xi$ conveniently, all elements $\omega \times \xi$, $\omega \in \Omega$, $\xi \in H$, generate a vector subspace of $F(\Omega)\otimes H$. We shall denote this subspace as $\Omega \odot H$ throughout this paper.

For a given positive definite operator-valued function $V(\omega, \omega')$ we introduce an inner product in $\Omega\odot H$ such that, for any element $\sum \omega_j \times \xi_j$ and $\sum \omega_k' \times \xi_k'$,

$$<\sum \omega_j \times \xi_j,\ \sum \omega_k' \times \xi_k'> \ = \ \sum_{j,k} (\xi_j, V(\omega_j, \omega_k')\xi_k'). \tag{2}$$

It is clear that it satisfies the properties of an inner product except that

$$<\sum \omega_j \times \xi_j,\ \sum \omega_j \times \xi_j> \ = 0 \text{ implies } \sum \omega_j \times \xi_j = 0.$$

Let N_V be the set of all expressions $\sum \omega_j \times \xi_j$ with this condition, then the quotient vector space $(\Omega\odot H)/N_V$ is a prehilbert space. The completion of this space we shall call the direct product Hilbert space of Ω and H with respect to V, and denote it by $\Omega\otimes_V H$. It is obvious that we have

$$<\omega \times \xi, \omega' \times \xi'> \ = (\xi, V(\omega, \omega')\xi') \tag{3}$$

for any ω, $\omega' \in \Omega$ and ξ, $\xi' \in H$.[1]

Sometimes when we want to treat a positive definite operator-valued functions, the above consideration may be useful and convenient, which we shall show in the following for the case that Ω is a group.

1) When $V(\omega, \omega')=1$ for $\omega=\omega'$ and $=0$ for $\omega \neq \omega'$, then $\Omega\otimes_V H$ is isometrically isomorphic to $l^2(\Omega)\otimes H$ in the sense of Murray-von Neumann [3].

2. **Neumark-Sz. Nagy's Theorem.** Let V_s be a positive definite operator-valued function on a group G into $B(H)$, that is, when we put $V(s, t) = V_{s^{-1}t}$ for each $s, t \in G$, $V(s,t)$ satisfies the condition (1) for $G \times G$ and H. Such a function V_s satisfies always the equality $V_{s^{-1}} = V_s^*$, i.e., $V(s,t) = V(t,s)^*$. For, since $\varphi_\xi(s) = (\xi, V_{s^{-1}}\xi)$ is a positive definite function on G, $\overline{\varphi_\xi(s^{-1})} = \varphi_\xi(s)$ and $(V_s\xi, \xi) = (\xi, V_{s^{-1}}\xi)$, and this implies $(V_s\xi, \eta) = (\xi, V_{s^{-1}}\eta)$ for any $\xi, \eta \in H$. Hence $V(s,t)$ is a positive definite operator-valued function on $G \times G$ into $B(H)$, and we can construct a Hilbert space $G \otimes_V H$.

The following Theorem 1 is a generalization, due to Sz. Nagy, of a theorem of M. A. Neumark [7] for a positive definite operator-valued function of a locally compact abelian group.[2)]

THEOREM 1. *Let V_s be a positive definite operator-valued function on a group G. Then there exists a unitary representation $(U_s, G \otimes_V H)$ of G such that*

$$T^* U_s T = V_s \tag{4}$$

where T is a bounded linear transformation from H to $G \otimes_V H$.

PROOF. Put $U_s'(\Sigma s_j \times \xi_j) = \Sigma s s_j \times \xi_j$, then U_s' is an additive operator on $G \odot H$, and satisfies

$$
\begin{aligned}
(5)\quad <U_s'(\Sigma s_j \times \xi_j), \Sigma t_k \times \eta_k> &= <\Sigma s s_j \times \xi_j, \Sigma t_k \times \eta_k> = \Sigma_{j,k}(\xi_j, V(ss_j, t_k)\eta_k) \\
&= \Sigma_{j,k}(\xi_j, V_{s_j^{-1}s^{-1}t_k}\eta_k) = \Sigma_{j,k}(\xi_j, V(s_j, (s^{-1}t_k))\eta_k) \\
&= \Sigma_{j,k}<s_j \times \xi_j, s^{-1}t_k \times \eta_k> = <\Sigma s_j \times \xi_j, \Sigma s^{-1}t_k \times \eta_k> \\
&= <\Sigma s_j \times \xi_j, U_{s^{-1}}'(\Sigma t_k \times \eta_k)>
\end{aligned}
$$

and

$$<U_s'(\Sigma s_j \times \xi_j), U_s'(\Sigma t_k \times \eta_k)> = <\Sigma s_j \times \xi_j, \Sigma t_k \times \eta_k>.$$

Thus the subspace

$$N_V = \{\Sigma s_j \times \xi_j \mid <\Sigma s_j \times \xi_j, \Sigma s_j \times \xi_j> = 0\}$$

is invariant under the operation U_s'.

Hence the operator U_s' is well-defined on $G \odot_V H = (G \odot H)/N_V$ and uniquely extended to a unitary operator U_s on $G \otimes_V H$.

Now, if we define the linear transformation T from H into $G \otimes_V H$ by

$$T : \xi \to e \times \xi$$

then we have $<T\xi, T\xi> = <e \times \xi, e \times \xi> = (\xi, V_e\xi) \leqq \|V_e\| (\xi, \xi)$, which shows the boundedness of T. Moreover, if we denote the conjugate transformation of T by T^*, we have for any $\xi, \eta \in H$,

$$
\begin{aligned}
(T^* U_s T\xi, \eta) &= <U_s T\xi, T\eta> = <U_s(e \times \xi), e \times \eta> \\
&= <s \times \xi, e \times \eta> = (\xi, V_{s^{-1}}\eta) = (V_s\xi, \eta),
\end{aligned}
$$

which implies $T^* U_s T = V_s$.

REMARK. In the case $V_e = 1$, H can be embedded into $G \otimes_V H$ by identifying $\xi \in H$ with $e \times \xi \in G \otimes_V H$, then the transformation T can be

2) Our proof is also similar to that of Godement [2], H. Nakano [6], Ky Fan [1], M. Nakamura and T. Turumaru [5] which are concerned with the numerical valued positive definite functions.

considered as the projection of $G \otimes_V H$ onto H. This is the case which Sz. Nagy discussed in [4].

COROLLARY. *If G is a topological group and (V_s, H) is weakly continuous, then the unitary representation $(U_s, G \otimes_V H)$ of G and (V_s, H) are strongly continuous.*

PROOF. Since

$$\begin{aligned} <\Sigma s_j \times \xi_j, U_s(\Sigma t_k \times \eta_k)> &= <\Sigma s_j \times \xi_j, \Sigma st_k \times \eta_k> \\ &= \Sigma_{j,k}(\xi_j, V_{s_j^{-1}st_k}\eta_k), \end{aligned}$$

the weak continuity of (V_s, H) implies the weak continuity of $(U_s, G \otimes_V H)$. Since U_s $(s \in G)$ are unitary operators $(U_s, G \otimes_V H)$ is always strongly continuous. Since $V_s = T^* U_s T$ by Theorem 1 and $(U_s, G \otimes_V H)$ is strongly continuous, (V_s, H) is also strongly continuous.

3. Positive definite function on a group and positive definite linear function on the group algebra. Let G be a locally compact group, and let $L^1(G)$ be the group algebra of all integrable functions on G with respect to the left invariant Haar measure with the convolution multiplication.

In order to discuss a positive definite operator-valued function on $L^1(G)$, we shall first define it on a *-algebra A as on the group in §2. A linear function W_x from A into $B(H)$ of a Hilbert space H is said to be *positive definite*, if $W(x, y) = W_{x^*y}$ is a positive definite operator-valued function on $A \times A$ into $B(H)$.

Now, let V_s be a positive definite operator-valued function defined on G whose range is in $B(H)$ then we can construct the unitary representation $(U_s, G \otimes_V H)$ of G as in the preceding section.

Since V_s and U_s are strongly continuous, for each $x \in L^1(G)$ the operator-valued functions $x(s)V_s$ and $x(s)U_s$ are Bochner integrable with respect to the Haar measure ds. Denote their Bochner integrals $\int x(s)V_s ds$ and $\int x(s)U_s ds$ by $\overline{V}_x$ and $\overline{U}_x$ which are acting on the Hilbert spaces H and $G \otimes_V H$ respectively. Then $\|\overline{V}_x\|, \|\overline{U}_x\| \leqq M\|x\|_1$ where $\| \|_1$ denotes the L^1-norm and M is a constant. Taking the bounded transformation T from H into $G \otimes_V H$ as in Theorem 1 such that $T\xi = e \times \xi$, then *$(\overline{U}_x, G \otimes_V H)$ is a bounded *-respresentation of $L^1(G)$ and satisfies $T^*\overline{U}_x T = \overline{V}_x$ for all $x \in L^1(G)$.* In fact, since $(U_s, G \otimes_V H)$ is a strongly continuous unitary representation of G, the first part is clear. The second part follows from the fact that

$$<\overline{U}_x(e \times \xi), e \times \eta> = \int x(s) <s \times \xi, e \times \eta> ds = \int x(s)(V_s\xi, \eta) ds = (\overline{V}_x \xi, \eta).$$

Now, it is clear that $\overline{V}_x$ is a positive definite operator-valued function on $L^1(G)$. With respect to the $\overline{V}_x$, we can construct a Hilbert space $L^1(G) \otimes_{\overline{V}} H$ in our sense (§1) which is the same as the Hilbert space constructed by

Stinespring as algebraic tensor product between a C^*-algebra and a Hilbert space with respect to a positive definite operator-valued function [10]. [3] Put $x*\xi = \overline{U}_x(e\times\xi) = \int x(s)(s\times\xi)ds$ (in the sense of Bochner integral),which belongs to $G\otimes_V H$. then the set $L^1(G)*H = \{x*\xi;\ x\in L^1(G),\ \xi\in H\}$ is a subspace of $G\otimes_V H$.

For any finite subsets $\{x_j\}$, $\{y_k\}$ of $L^1(G)$ and $\{\xi_j\}$, $\{\eta_k\}$ of H, we have

$$<\sum x_j*\xi_j, \sum_k y^*\eta_k> = \sum_{j,k}<x_j*\xi_j, y_k*\eta_k> = \sum_{j,k}<\overline{U}_{y_k^* x_j}(e\times\xi_j), e\times\eta_k>$$
$$= \sum_{j,k}\int y_k^* x_j(s)<s\times\xi_j, e\times\eta_k>ds = \sum_{j,k}\int y_k^* x_j(s)(V_s\xi_j, \eta_k)ds$$
$$= \sum_{j,k}(Vy_k^* x_j\xi_j, \eta_k) = \sum_{j,k}<x_j\times\xi_j, y_k\times\eta_k> = <\sum x_j\times\xi_j, \sum y_k\times\eta_k>.$$

Hence the mapping $\phi: \sum x_j*\xi_j \to \sum x_j\times\xi_j$ is a unitary transformation from $L^1(G)*H$ to $L^1(G)\otimes_V H$. Using the approximate identity $\{e_\alpha\}$ in $L^1(G)$ corresponding to a complete system of neighborhoods of the unit e of G and by the definition of the inner product $<\cdot,\cdot>$ in $G\otimes_V H$, $\{e_\alpha^s\times\xi\}$ converges to $s\times\xi$, and this implies that $L^1(G)*H$ is dense in $G\otimes_V H$. [4] Therefore, we obtain that *the Hilbert spaces $G\otimes_V H$ and $L^1(G)\otimes_{\overline{V}} H$ are isomorphic by an isomorphism ϕ which maps the element $x*\xi$ of $G\otimes_V H$ to the element $x\times\xi$ in $L^1(G)\otimes_{\overline{V}} H$.*

In the above, we have seen that there can be defined a positive definiteo perator-valued linear function $\overline{V}_x$ on $L^1(G)$ for a given positive definite operator-valued function V_s on G. We can also show the converse case.

THEOREM 2. *If a positive definite operator-valued bounded linear function W_x on $L^1(G)$ into $B(H)$ is given, then there exists a unique strongly continuous positive definite operator-valued function V_s on G into $B(H)$ such that*

$$W_x = \int_G x(s)V_s ds \qquad \textit{for all } x\in L^1(G) \tag{6}$$

where the integral is in the sense of Bochner.

LEMMA 2.1. *There exists a function $f(\xi, s)$ defined for $\xi\in H$ and $s\in G$ and with range in $L^1(G)\otimes_W H$ such that, for each fixed $s\in G$, $f(\xi, s)$ is a bounded linear transformation from H into $L^1(G)\otimes_W H$, and for each fixed $\xi\in H$, $f(\xi, s)$ is strongly continuous on G, and moreover $<f(\xi, s), f(\eta, t)> = <f(\xi, e), f(\eta, s^{-1}t)>$ for any $\xi, \eta\in H$ and $s, t\in G$.*

PROOF. For each $\xi\in H$, put $\sigma_\xi(x) = (W_x\xi, \xi)$, then $\sigma_\xi(x)$ is a bounded linear functional on $L^1(G)$ such that $\sigma_\xi(x^*x)\geqq 0$ for all $x\in L^1(G)$. Hence there exists a continuous positive definite function $\varphi_\xi(s)$ on G such that

3) The material of the present paper is obtained independently to W. F. Stinespring [10]. The author is awared Stinespring's paper when he visits the Tôhoku University in the summer. Stinespring calls *completely positive* instead of positive definite.

4) We denote the function $x(s^{-1}t)$ of t by $x^s(t)$.

$$\sigma_\xi(x) = \int_G x(s)\varphi_\xi(s)ds.$$

For the approximate identity $\{e_\alpha\}$,

$$\sigma_\xi(e^s_\alpha e^s_\beta) = \int e_\alpha e_\beta(t^{-1})\varphi_\xi(t)dt \to \varphi_\xi(e).$$

Therefore

$$\begin{aligned}\|e^s_\alpha \times \xi - e^s_\beta \times \xi\|^2 &= \langle e^s_\alpha \times \xi, e^s_\alpha \times \xi\rangle + \langle e^s_\beta \times \xi, e^s_\beta \times \xi\rangle \\ &\quad - 2R\langle e^s_\alpha \times \xi, e^s_\beta \times \xi\rangle \\ &= \sigma_\xi(e^{s*}_\alpha e^s_\alpha) + \sigma_\xi(e^{s*}_\beta e^s_\beta) - 2R\sigma_\xi(e^{s*}_\alpha e^s_\beta) \\ &= \sigma_\xi(e_\alpha e_\alpha) + \sigma_\xi(e_\beta e_\beta) - 2R\sigma_\xi(e_\alpha e_\beta) \to 0\end{aligned}$$

and the strong limit of $e^s_\alpha \times \xi$ exists in $L^1(G)\otimes_W H$. Denote it by $f(\xi, s)$. For any finite set $\xi_j \in H$ and any $s \in G$, we have

$$\begin{aligned}f(\textstyle\sum \xi_j, s) &= \lim_\alpha e^s_\alpha \times \textstyle\sum \xi_j = \lim_\alpha \textstyle\sum (e^s_\alpha \times \xi_j) \\ &= \textstyle\sum \lim_\alpha (e^s_\alpha \times \xi_j) = \textstyle\sum f(\xi_j, s)\end{aligned} \tag{7}$$

while,

$$\begin{aligned}\varphi_\xi(s) &= \lim_\alpha \int e^{s*}_\alpha\, e_\alpha(t)\varphi_\xi(t)dt = \lim_\alpha \sigma_\xi(e^{s*}_\alpha\, e_\alpha) = \lim\ \sigma_\xi(e_\alpha e^{s^{-1}}_\alpha) \\ &= \lim_\alpha \langle e_\alpha \times \xi, e^{s^{-1}}_\alpha \times \xi\rangle = \langle f(\xi, e), f(\xi, s^{-1})\rangle,\end{aligned}$$

$$\begin{aligned}\langle f(\xi, s), f(\eta, t)\rangle &= \lim\langle e^s_\alpha \times \xi, e^t_\alpha \times \eta\rangle \\ &= \lim\langle e_\alpha \times \xi, e^{s^{-1}t}_\alpha \times \eta\rangle = \langle f(\xi, e), f(\eta, s^{-1}t)\rangle.\end{aligned} \tag{8}$$

The strong continuity of $f(\xi, s)$ (for fixed $\xi \in H$) follows easily from the above fact and the construction of it. Since for any $\xi \in H$ and $s \in G$,

$$\|f(\xi, s)\|^2 = \|f(\xi, e)\|^2 = \lim_\alpha \|e_\alpha \times \xi\|^2 = \lim_\alpha \langle \xi, W_{e_\alpha^* e_\alpha}\xi\rangle \leqq M\|\xi\|^2, \tag{9}$$

for fixed $s \in G$, $f(\xi, s)$ is a bounded linear transformation from H into $L^1(G)\otimes_W H$, where M is a constant such that $\|W_x\| \leqq M\|x\|_1$ for all $x \in L^1(G)$.

PROOF OF THEOREM 2. By the above Lemma, for any $\xi, \eta \in H$ and $s \in G$,

$$|\langle f(\xi, s), f(\eta, e)\rangle| \leqq \|f(\xi, e)\|\|f(\eta, e)\| \leqq M\|\xi\|\|\eta\|$$

Hence, there exists a bounded linear operator V_s on H (depending on $s \in G$) such that $\langle f(\xi, s), f(\eta, e)\rangle = \langle V_s\xi, \eta\rangle$; by Lemma 2.1, V_s is a strongly continuous positive definite $B(H)$-valued function on G. Moreover, for $x \in L^1(G)$

$$(W_x\xi, \eta) = \int x(s)\varphi_\xi(s)ds = \int x(s)\langle f(\xi, s), f(\xi, e)\rangle ds = \int x(s)\ (V_s\xi, \xi)ds$$

and for any $\xi, \eta \in H$, $(W_x\xi, \eta) = \int x(s)(V_s\xi, \eta)ds$. Since $x(s)V_s$ is Bochner integrable, the Bochner integral $\int x(s)V_s ds$ exists and equals to W_x. The uniqueness of V_s is obvious by (6).

COROLLARY. *Let V_s and W_x be positive definite operator-valued functions in Theorem 2. Then $V_e = 1$ if and only if $\|W_x\| \leqq \|x\|_1$ and the weak closure* *of* $\{W_x;\ \|x\|_1 \leqq 1\}$ $\{W_x;\ x \in L^1(G)\}$ *contains* 1.

PROOF. "If" part: Using the notation in the proof of Theorem 2, taking a directed set $\{x_\gamma\} \subset L^1(G)$, $\|x_\gamma\|_1 \leqq 1$, such that W_{x_γ} converges weakly to 1, we have

$$\|\xi\|^2 = \lim |(W_{x_\gamma}\xi, \xi)| = \lim \Big| \int_G x_\gamma(s)\varphi_\xi(s)\,ds \Big| \leqq \varphi_\xi(e)$$

$$= \langle f(\xi, e), f(\xi, e)\rangle = \|f(\xi, e)\|^2 \ (\leqq \|\xi\|^2 \ by\ (9))$$

and hence $\|\xi\| = \|f(\xi, e)\|$ for all $\xi \in H$. Therefore for all ξ, $\eta \in H$

$$(\xi, \eta) = \langle f(\xi, e), f(\eta, e)\rangle = (\xi, V_e\eta),$$

that is, $V_e = 1$.

Conversely, if $V_e = 1$, we have, for any $\xi \in H$,

$$(W_{e_\alpha}\xi, \xi) = \int e_\alpha(s)(V_s\xi, \xi)ds \rightarrow (V_e\xi, \xi) = (\xi, \xi).$$

Since by (6) $\|W_x\| \leqq \|x_1\|$ holds we complete the proof.

REFERENCES

[1] KY FAN: On positive definite sequences, Ann. of Math., 47(1946), 593-607.

[2] R. GODEMENT: Les fonctions de type positif et la theorie des groupes, Trans. Amer. Math. Soc., 63(1945), 1-84.

[3] F. J. MURRAY AND J. VON NEUMANN: On rings of operators, Ann. of Math., 37(1936), 116-229.

[4] S. SZ. NAGY: Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe, Acta de Szeged, 15(1954), 104-114.

[5] M. NAKAMURA AND T. TURUMARU: On the representations of positive definite functions and stationary functions on a topological group, Tôhoku Math. Journ., 4(1954), 1-9.

[6] H. NAKANO: On the theory of Hilbert space (in Japanese), Sugaku, 1(1947-48), 38-42.

[7] M. NEUMARK: Positive definite functions on a commutative group, Akad. Nauk, Leningrad. Otdelnie Matematicheskaya i estestvennykh nauk. Izvestia akad. nauk, SSSR. Ser. mat., 7(1943), 237-244.

[8] M. NEUMAK: On representation of additive operator set functions, Akad. Nauk, SSSR(Doklady), 41(1943), 359-361.

[9] R. SCHATTEN: A theory of cross-spaces, Ann. of Math. Stud., 26, Princeton, 1950.

[10] W. F. STINESPRING: Positive functions on C^*-algebras, Proc. Amer. Math. Soc., 6(1955), 211-216.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY.