ON PSEUDORECURRENCE IN TOPOLOGICAL DYNAMICAL SYSTEMS

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Periodicity, recurrence and other properties in topological dynamical systems have been considered by Gottschalk-Hedlund [1][1] and the theorem which is topological analogy of the Poincaré recurrence theorem has been obtained by the same authors. Williams [3] who has utilized the same method has obtained some extensions. In this paper we shall define pseudorecurrence by transformation semigroups instead of transformation groups in [1] and [3], and consider some analogous questions.

In §1, we shall define the terms which we shall use afterwards, and in §2, we shall consider some analogies of the Poincaré recurrence theorem. In §3, we shall characterize pseudorecurrence by incompressibility properties, and in §4, we shall consider an analogy of stability.

1. Let X be a topological space and T a multiplicative commutative topological semigroup acting as a transformation semigroup on X,[2] i. e. to $x \in X$ and $t \in T$ is assigned a point of X denoted by xt such that: (1) $(xt)s = x(ts)$ $(x \in X;\ t,\ s \in T)$, (2) the function xt defines a continuous transformation of $X \times T$ into X. Suppose that T has arbitrary numbers of subsemigroups in T.

If a family of semigroups in T has a countable base, it is said to be *admissible,* and every semigroup which belongs to this family is said to be admissible semigroup and we shall denote it by A-semigroup. Hereafter we assume that the family of semigroups in T is admissible.

Let x be a point of X and x is said to be *pseudorecurrent under* T (or T is said to be *pseudorecurrent at* x) provided that $U_xS \ni x$ for every neighborhood U_x of x, and every A-semigroup S. Let S be a A-semigroup in T, a point x is said to be *pseudorecurrent relative* to S provided that $U_xS \ni x$ for every neighborhood U_x of x and x is said to be *semipseudorecurrent under* T (or T is said to be *semipseudorecurrent at* x) provided that x is pseuporecurrent relative to one A-semigroup at least. Clearly the point x which is semipseudorecurrent relative to all A-semigroups in T is pseudorecurrent.

If T is pseudorecurrent (resp. semipseudorecurrent) at every point of X, then T is said to be pointwise pseudorecurrent (resp. pointwise semipseudorecurrent). Let x be a point of X, x is said to be *regionally recurrent under* T (or T is said *regionally recurrent* at x) provided that $U_x \cap U_xS \neq \phi$ for every A-semigroups $S \subset T$ and for every neighborhood U_x of x. Similarly

1) Numbers in brackets refer to the bibliography at the end of the paper.
2) See Gottschalk-Hedlund [1].

we can define the notion of *semiregionally recurrence*. The point which is not regionally recurrent under T is said to be *wandering*, and the point which is not semiregionally recurrent is said to be *essentially wandering*.

2. The topological analogy of recurrence theorem is obtained by Gottschalk-Hedlund [1] and the other type by Williams [3]. Here let us prove similar theorems with the new method.

THEOREM 1. *Let X be a topological space sastisfying the second axiom of countability, let $\mathfrak{R}$ be a set of pseudorecurrent points and let W be a set of wandering points, then $\mathfrak{R} \cup W$ is a residual set in X, i.e., $X-(\mathfrak{R} \cup W)$ is a set of the first category.* [3]

PROOF. Let U_m be a neighborhood belonging to the countable base and let S_n be an A-semigroup of the countable base of A-semigroups in T. Let B_{mn} be a set of points such that $x \in U_m$ and $x \notin U_m S_n$. Let D_{mn} be a set of points that belong to B_{mn} and are not wandering. It is clear that $(X-\mathfrak{R}) \cap (X-W) \subset \bigcup_{m,n=1}^{\infty} D_{mn}$. D_{mn} is a closed set. For, if $y \in \overline{D}_{mn}$ and $y \notin D_{mn}$, there is a neighborhood U_y of y and an A-semigroup S such that $U_y \cap U_y S = \phi$. Now $U_y \cap D_{mn} \neq \phi$, then we can select a neighborhood $V_x \subset U_y$, where $x \in U_y \cap D_{mn}$. But this is impossible, because $V_x \cap V_x S \neq \phi$. Hence D_{mn} is a closed set.

Suppose that there is an inner point x of D_{mn}, we have a neighborhood $V_x \subset D_{mn}$. Since x belongs to D_{mn}, $V_x \cap V_x S \neq \phi$ holds for every A-semigroup S and for every neighborhood V of x. On the other hand $V_x \subset B_{mn}$ and $B_{mn} S_n \cap B_{mn} = \phi$, hence we have $V_x S_n \cap V_x = \phi$, this is also impossible.

Hence D_{mn} is nowhere dence. Therefore $\mathfrak{R} \cup W$ is a residual set in X.

COROLLARY 1. *If T is pointwise regionally recurrent, that is $W=\phi$, then $\mathfrak{R}$ is a residual set in X.* [4]

THEOREM 2. *Let X be a topological space satisfying the second axiom of countability, let $\mathfrak{R}'$ be a set of semipseudorecurrent point, and let W' be a set of essentially wandering points. Then $\mathfrak{R}' \cup W'$ is a residual set in X.*

We can prove this theorem similarly and the proof is omitted.

COROLLARY 2. *If T is pointwise semiregionally recurrent, that is $W'=\phi$, then $\mathfrak{R}'$ is a residual set in X.*

3. In this section we characterize pseudorecurrence by incompressibility properties.

PROPOSITION 1. *In order that T is pointwise pseudorecurrent, it is necessary*

3) Cf. Theorem 4.10 in [3].
4) Cf. Theorem 3 in [1].

and sufficient that if O is an open subset of X and S an A-semigroup in T such that $OS \subset O$, then $O - OS = \phi$.

PROOF. We first establish the necessity of the condition. Let O be an open subset of X and S an A-semigroup in T such that $OS \subset O$. A point belonging to O is pseudorecurrent, hence $U_x S \ni x$ for every neighborhood U_x. Now O is an open set which contains x, therefore O is considered as a neighborhood of x. That is $OS \ni x$, hence $OS \supset O$ and $O = OS$.

To show that the condition is sufficient, we define $O = V_x \cup V_x S$, where V_x is a neighborhood of x. Hence O is an open set and $OS = V_x S \subset O$. By virtue of the condition of the proposition we have $O - OS = \phi$, that is $O = OS$, and $V_x S = V_x \cup V_x S$. Hence $V_x S \supset V_x \ni x$. But we can choose V_x and S arbitrarily, hence x is pseudorecurrent under T. Again x is an arbitrary point of X, therefore T is pointwise pseudorecurrent. Q. E. D.

In generally the transformation semigroup is not pointwise pseudorecurrent. However, it is still possible to characterize in terms of an incompressibility property.

LEMMA 1. *Let $\mathfrak{R}$ be a set of all pseudorecurrent points. If $\mathfrak{R}$ is a residual set in X, then $O - OS$ is a set of the first category, where O is an open subset of X and S an A-semigroup in T such that $OS \subset O$.*

PROOF. Let O be an open subset of X and S an A-semigroup in T such that $OS \subset O$. Now, let x be a point of O which is pseudorecurrent, then $U_x S \ni x$ holds for every neighborhood U_x of x. Hence $x \in OS$ and $O - OS \subset X - \mathfrak{R}$. By the assumption that $\mathfrak{R}$ is residual in X, $O - OS$ is a set of the first category.

LEMMA 2. *Let O be any open subset of X and S any A-semigroup in T such that $OS \subset O$. If $O - OS$ is always nowhere dense, then $\mathfrak{R}$ is a residual set in X.*

PROOF. Let B_{mn} be a set of all point x such that x belongs to U_m and holds $U_m S_n \not\ni x$. Then B_{mn} is a border set. For if B_{mn} contains an inner point x, we have a neighborhood V_x of x such that $V_x \subset B_{mn}$. Since $U_m S_n \cap B_{mn} = \phi$, $B_{mn} S_n \cap B_{mn} = \phi$, hence $V_x S_n \cap V_n = \phi$. Now, we put $O = V_x \cup V_x S_n$, then O is an open set and $OS \subset O$. Hence by virtue of the assumption, $O - OS$ is nowhere dense. Since $O - OS \supset V_x$, this is impossible. Hence B_{mn} is a border set.

B_{mn} is $U_m \cap \overline{B}_{mn}$. For, first of all, $B_{mn} \subset U_m \cap \overline{B}_{mn}$ is clear. Conversely, y is any point of $U_m \cap \overline{B}_{mn}$ and if $y \notin B_{mn}$, then $U_m S_n \ni y$ and $y \in U_m$, or, $y \notin U$ and $U_m S_n \not\ni y$. But the latter is out of the case, hence we consider the former alone. On the other hand we have a point x such that $x \in B_{mn}$ and $x \in U_y$ for every neighborhood U_y of y. Since $U_m S_n \supset y$, we can get $s_0 \in S_n$ such that $y = us_0$, but $U_y s_0 \not\subset U_m S_n$ for every neighborhood U_y of y. This contradicts to the continuity of s at y, hence $B_{mn} \supset U_m \cap \overline{B}_{mn}$. Therefore $B_{mn} = U_m \cap \overline{B}_{mn}$.

Since $\overline{B}_{mn} = (U_m \cap \overline{B}_{mn}) \cup (\overline{B}_{mn} - U_m)$, $\overline{B}_{mn} \cap U_m$ is a closed set, then $\overline{B}_{mn} - U_m$ is nowhere dense. On the other hand $B_{mn} = U_m \cap \overline{B}_{mn}$ and B_{mn} is a border set, hence $\overline{B}_{mn}$ is a closed border set. Therefore B_{mn} is a nowhere dense set.

Finally we shall prove $X - \mathfrak{R} \subset \bigcup_{m,n=1}^{\infty} B_{mn}$. Let x be a point of $X - \mathfrak{R}$, and $x \notin \mathfrak{R}$, then there is an A-semigroup S and a neighborhood U_x such that $U_xS \not\ni x$. Since we can choose U_i which belongs to the base such that $U_i \subset U_x$ and S_j which belong to the base such that $S_j \subset S$, hence we have $x \notin U_iS_j$ and $x \in B_{ij}$, hence x is a point of $\bigcup_{m,n=1}^{\infty} B_{mn}$. Therefore $\mathfrak{R}$ is a residual set in X.

From the last two lemmas we can obtain the following theorem by employing the property that a set of the first category is a border set when the space is complete metric (Baire-Hausdorff's theorem).

THEOREM 3. *Let X be a complete metric separable space and $\mathfrak{R}$ a set of all pseudorecurrent points. In order that $\mathfrak{R}$ is a residual set in X, it is necessary and sufficient that for any open set $O \subset X$ and any A-semigroup $S \subset T$ such that $OS \subset O$, $O - OS$ is always a set of the first category.*

4. Let X be a locally compact regular space. For any x there is a closure-compact neighborhood V_x of x, i.e. V_x is a neighborhood of x and the closure $\overline{V}_x$ is compact. Now x is said to be *n-stable under* S provided that the closure of VS is compact for any closure-compact neighborhood V_x of x. Let $\mathfrak{S}(S)$ be a set of all points of X which are n-stable under S. Let $\mathfrak{R}(S)$ be a set of all points of X which are pseudorecurrent under S, and $\mathfrak{U}(S)$ be $X - \mathfrak{S}(S)$, where x is said to be *pseudorecurrent under* S provided that $U_xS^* \ni x$ for every A-semigroup S^* such that $S^* \subset S$ and for every neighborhood U_x of x.

THEOREM 4. *In order that $X = \mathfrak{U}(S) \cup \mathfrak{R}(S)$, it is necessary and sufficient that for any closure-compact openset $O \subset X$ and any A-semigroup $S^* \subset S$ such that $OS^* \subset O$, $O - OS^* = \phi$ holds good.*

PROOF. First of all, we can easily verify that $X = \mathfrak{U}(S) \cup \mathfrak{R}(S)$ is equivalent to $\mathfrak{R}(S) \supset \mathfrak{S}(S)$, then for the purpose of proving the theorem, it is sufficient if we only show that in order to be $\mathfrak{R}(S) \supset \mathfrak{S}(S)$ it is necessary and sufficient that the condition of the theorem holds.

The condition is necessary. For, let O be any closure-compact open set in X and S^* be any A-semigroup in S such that $OS^* \subset O$. Suppose that $O - OS^* \neq \phi$ and contains some point x. Then we have $x \notin \mathfrak{R}(S)$. On the other hand, for x we can choose a closure compact neighborhood V_x such that $V_x \subset O$. Since $V_xS \subset OS \subset O$, $\overline{V_xS} \subset \overline{O}$. Hence $\overline{V_xS}$ is compact, i.e. $x \in \mathfrak{S}(S)$. This is impossible since $x \notin \mathfrak{R}(S)$ and $\mathfrak{R}(S) \supset \mathfrak{S}(S)$. Therefore we have $O - OS^* = \phi$.

The condition is sufficient. For let x be a point of X such that $x \notin \mathfrak{R}(S)$ and that $x \in \mathfrak{S}(S)$. Then we have $x \notin \overline{U_x S^*}$ for some S^* and for some U_x. On the other hand, there exists a closure-compact neighborhood V_x of x such that $\overline{V}_x \subset U$ and that $V_x S^*$ compact. For this V_x, $x \notin \overline{V_x S}$ holds. Now we put $O = V_x \cup V_x S^*$, then we have $OS^* = V_x S^* \subset O$. By the assumption, $O - OS^* = \phi$. But this is impossible since $x \notin V_x S$ and $x \in O$. Therefore we have $\mathfrak{R}(S) \supset \mathfrak{S}(S)$. Q. E. D.

A point x is said to be *n-stable* provided that $\overline{V_x T}$ is compact, for any closure-compact neighborhood V_x of x. Let $\mathfrak{S}$ be a set of all points of X which are n-stable, and $\mathfrak{U}$ be $X - \mathfrak{S}$, and $\mathfrak{D}$ be the set of all points of X which belong to $\mathfrak{U}(S)$ for every A-semigroup S in T.

COROLLARY 1. *If $X = \mathfrak{R} \cup \mathfrak{D}$, then, for any closure-compact open set $O \subset X$ and any A-semigroyp $S \subset T$ such that $OS \subset O$, $O - OS = \phi$ holds good.*

COROLLARY 2. *For every closure compact open set $O \subset X$ and for every A-semigroup $S \subset T$ for which $OS \subset O$, $O - OS = \phi$ holds, then $X = \mathfrak{R} \cup \mathfrak{U}$.*

COROLLARY 3. *If O is a closure-compact open set in X and S and A-semigroup in T such that $OS \subset O$, and if $\mathfrak{R} \cup \mathfrak{D}$ is a residual set in X, then $O - OS$ is a set of the first category.*

Lastly, we shall refer to the relation between the results considered in this paper and that of [1] or [3]. We can verify that "pseudorecurrence" which we defined becomes "recurrence" in [1] or [3] by modifying "transformation semigroups" with "transformation groups" and "A-semigroups" with "replete semigroups"[5]. For let x be a pseudorecurrent point under T (transformation group), i. e. $x \in \overline{U_x S}$ for every neiborhood U_x of x and for every replete semigroup S. On the other hand, if S runs over all replete semigroups in T, it is clear that S^{-1} runs over all replete semigroups in T. Now fixing U_x and S, we can choose $u \in U_x$ and $s \in S$ such that $x = us$, then $u = xs^{-1}$. Hence $U_x \cap xS \neq \phi$. Now it holds for arbitrary U_x's and S's, i. e. x is a recurrent point under T[6].

And we can point out the difference between Theorem 1 and the similar theorems in [1] or [3] in point of their assumptions. The property that the family of replete semigroups in T has a countable base is proved in [1] by adding some condition. Then we can get following remarks as corollaries of Theorem 1 and 2 under the same additional condition:

REMARK 1. *Let X be a topological space satisfying the second axiom of*

5) A semigroup $S \subset T$ is said to be a replete semigroup provided that S contains some translate of each compact subset of T. See [1] or [3].

6) A point $x \in X$ is said to be recurrent under T provided that to each neighborhood U_x of x there corresponds an extensive set $A \subset T$ such that $xA \subset U_x$, where a subset $A \subset T$ is said to be an extensive set provided that A intersects every replete semigroup in T. On the other hand, it is known that in order that a point $x \in X$ be recurrent under T it is necessary and sufficient that $x \in \overline{xS}$ for every replete semigroup $S \subset T$. Then above x becomes a recurrent point in this sense.

countability, let R be a set of recurrent points and let W be a set of wandering points, then $R \cup W$ is residual set in X.

REMARK 2. *Let X be a topological space satisfying the second axiom of countability, Let R' be a set of semirecurrent points*[7] *and let W' be a set of essentially wandering points, then $R' \cup W'$ is a residual set in X.*

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7) A point $x \in X$ is said to be a semirecurrent point provided that to each neighborhood U_x of x there corresponds a replete semigroup $S \subset T$ such that $xS \cap U_x \neq \phi$.