CORRECTION AND REMARK ON THE PAPER : ON INTEGRAL INEQUALITIES AND CERTAIN OF ITS APPLICATIONS TO FOURIER SERIES.

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1. In my previous paper, there is an essential error and here we should correct it. This error was pointed out by Dr. S. Yano, whom the author expresses hearty thanks.

The Lemma A of that paper is incorrect. In fact estimations from line 1 to 2 of p.122 are incorrect, so the conclusion may be replaced in the following form.

LEMMA 1. *Let* $\varphi(z) \in H^r$, $r > 1$, then we have

$$|\varphi'(\rho e^{i\theta+it})| \leqq C\varphi_s^*(\theta)\frac{1}{\delta}\Big(1 + \frac{|t|}{\delta}\Big)^{1/s}$$

*where* $1 < s < r$, $\delta = 1 - \rho$ *and*

$$|\varphi_s^*(\theta)| = \sup_{0<|h|<\pi}\Big(\frac{1}{h}\int_0^h |\varphi(e^{i\theta+it})|^s dt\Big)^{1/s}.$$

PROOF. We have with the same notation (2.03)-(2.05),

$$|u_\rho(\rho, \theta + t)| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + t + u)\frac{\partial}{\partial\rho}P(\rho, u)du\Big|$$

$$\leqq \frac{C}{2\pi\delta}\int_{-\pi}^{\pi} |f(\theta + t + u)|P(\rho, u)du.$$

Applying Jensen's inequality and integrating by parts successively, we have

$$|u_\rho(\rho, \theta + t)| \leqq \frac{C}{\delta}\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta + u)|^s P(\rho, u - t)du\Big\}^{1/s}$$

$$= \frac{C}{\delta} f_s^*(\theta)\Big(1 + \frac{|t|}{\delta}\Big)^{1/s}.$$

Repeating the same argument, we have

$$|u_\theta(\rho, \theta + t)| \leqq \frac{C}{\delta} f_s^*(\theta)\Big(1 + \frac{|t|}{\delta}\Big)^{1/s}.$$

Hence we have Lemma 1, and it is well known that $f_s^*(\theta)$ belongs to $L^r$

(cf. A.Zygmund [4]).

Consequently Theorem D, (1.16) is incorrect, and should be replaced by the following form.

THEOREM 1. *Let* $\varphi(z) \in H^p$, $p > 1$, *then we have for* $2 \leqq q < 2p$,

$$\int_{-\pi}^{\pi} (g_q^*(\theta))^p d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta.$$

For the proof of the theorem, we use Lemma 2 instead of Lemma 1.

LEMMA 2. *Let* $\varphi(z) \in H^r, r > 1$, *then we have*

$$|\varphi(\rho e^{i\theta+it})| \leqq C\varphi_s^*(\theta)\left(1 + \frac{|t|}{\delta}\right)^{1/s},$$

*where* $1 < s < r$, $\delta = 1 - \rho$ *and*

$$\int_{-\pi}^{\pi} (\varphi_s^*(\theta))^r d\theta \leqq A_r \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^r d\theta.$$

This lemma has been proved by A. Zygmund [4].

PROOF OF THEOREM 1. The proof will be done in three steps.

(a) the case $p = q(\geqq 2)$. By the change of order of integration, we have

$$\int_{-\pi}^{\pi} (g_q^*(\theta))^q d\theta = \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{q-1} d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi'(\rho e^{i\theta+it})|^q P(\rho, t) dt$$

$$= \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{q-1} |\varphi'(\rho e^{i\theta})|^q d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, t) dt$$

$$= \int_{-\pi}^{\pi} (g_q(\theta))^q d\theta$$

$$\leqq A_q \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^q d\theta$$

by Theorem A, (1.08) of the previous paper.

(b) the case $p > q(\geqq 2)$. In this case if we put $r = p/q > 1$, the conjugate index $s$ of $r$ is $p/(p-q)$, and for the family of functions $\{\xi(\theta)\}$, which are non-negative, belong to the class $L^s$ and $\|\xi(\theta)\| \leqq 1$, we have

$$\left\{\int_{-\pi}^{\pi} (g_q^*(\theta))^p d\theta\right\}^{q/p} = \sup_{\{\xi(\theta)\}} \int_{-\pi}^{\pi} (g_q^*(\theta))^q \xi(\theta) d\theta$$

$$= \sup_{\{\xi(\theta)\}} \int_{-\pi}^{\pi} dt \int_{0}^{1} (1-\rho)^{q-1} |\varphi'(\rho e^{it})|^q d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\theta) P(\rho, t-\theta) d\theta$$

$$= \sup_{\{\xi(\theta)\}} \int_{-\pi}^{\pi} \Xi(t)(g_q(t))^q dt,$$

where $\Xi(t) = \sup_{0<\rho<1} \xi(\rho, t)$ and $\Xi(t)$ belongs the same class of $\{\xi(\theta)\}$.

Applying Hölder's inequality and the maximal theorem of Hardy-Littlewood, we have

$$\left\{\int_{-\pi}^{\pi} (g_q^*(\theta))^p \xi(\theta) d\theta\right\}^{q/p} = \sup_{\{\xi(\theta)\}} \left\{\int_{-\pi}^{\pi} \Xi((t))^s dt\right\}^{1/s} \left\{\int_{-\pi}^{\pi} (g_q(t))^p dt\right\}^{q/p}$$

$$\leqq A_s \sup_{\{\xi(\theta)\}} \left\{\int_{-\pi}^{\pi} (\xi(t))^s dt\right\}^{1/s} \left\{\int_{-\pi}^{\pi} (g_q(t))^p dt\right\}^{q/p}$$

$$\leqq A_s \left\{\int_{-\pi}^{\pi} |\varphi(e^{it})|^p dt\right\}^{q/p}.$$

Hence we have

$$\int_{-\pi}^{\pi} (g_q^*(\theta))^p d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta.$$

(c) the case $p<q$. It is no loss of generality to suppose that $\varphi(z)$ is free from zero points, and if we put $\varphi^p(z) = \psi^q(z)$, then $\psi(z) \in H^q$ and $\varphi'(z) = q/p\ \psi^{(q-p)/p}(z)\psi'(z)$. It follows that, by Lemma 2,

$$(g_q^*(\theta, \varphi))^q = \frac{1}{2\pi} \int_0^1 \delta^{q-1} d\rho \int_{-\pi}^{\pi} \left(\frac{q}{p}\right)^q |\psi(\rho e^{i\theta+it})|^{q(q-p)/p} |\psi'(\rho e^{i\theta+it})|^q P(\rho, t) dt$$

$$\leqq A_{p,q} (\psi_k^*(\theta))^{q(q-p)/p} \int_0^1 \delta^{q-1} d\rho \int_{-\pi}^{\pi} |\psi'(\rho e^{i\theta+it})|^q \left(1 + \frac{|t|}{\delta}\right)^{q(q-p)/kp} P(\rho, t) dt$$

$$= A_{p,q} (\psi_k^*(\theta))^{q(q-p)/p} (J(\theta))^q, \qquad 1<k<q,$$

say. Hence by Hölder's inequality, we have

$$\int_{-\pi}^{\pi} (g_q^*(\theta, \varphi))^p d\theta \leqq A_{p,q} \left\{\int_{-\pi}^{\pi} (\psi_k^*(\theta))^q d\theta\right\}^{(q-p)/q} \left\{\int_{-\pi}^{\pi} (J(\theta))^q d\theta\right\}^{p/q}$$

and

$$\int_{-\pi}^{\pi} (J(\theta))^q d\theta \leqq A \int_{-\pi}^{\pi} d\theta \int_0^1 \delta^q |\psi'(\rho e^{i\theta})|^q d\rho \int_{-\pi}^{\pi} \left(1 + \frac{|t|}{\delta}\right)^{q(q-p)/kp} \frac{dt}{\delta^2 + t^2}$$

and that

$$\delta^q \int_{-\pi}^{\pi} \left(1 + \frac{|t|}{\delta}\right)^{q(q-p)/kp} \frac{dt}{\delta^2 + t^2} = 2\delta^q \left(\int_0^{\delta} + \int_{\delta}^{\pi}\right)\left(1 + \frac{|t|}{\delta}\right)^{q(q-p)/kp} \frac{dt}{\delta^2 + t^2}$$
$$= J_1 + J_2$$

say. For $J_1$:

$$J_1 \leqq 2\delta^q \int_0^{\delta} \frac{dt}{\delta^2 + t^2} \leqq 2\delta^{q-1}$$

and for the $J_2$:

$$J_2 \leqq 2\delta^q \int_{\delta}^{\pi} \left(\frac{t}{\delta}\right)^{q(q-p)/kp} t^{-2} dt$$

$$= 2\delta^{q-q(q-p)/kp} \int_{\delta}^{\pi} t^{q(q-p)/kp-2} dt$$

$$= A_{p,q}\delta^{q-1}$$

under an assumption $1 - q(q-p)/kp > 0$, and this condition is satisfied taking $k$ sufficiently near to $q$ and if $q < 2p$.

Combining these estimations we obtain

$$\int_{-\pi}^{\pi} (g_q^*(\theta, \varphi))^p d\theta \leqq A_{p,q} \left\{\int_{-\pi}^{\pi} (\psi_k^*(\theta))^q d\theta\right\}^{(q-p)/q} \left\{\int_{-\pi}^{\pi} (g_q(\theta, \psi))^q d\theta\right\}^{p/q}$$

$$= A_{p,q} \int_{-\pi}^{\pi} |\psi(e^{i\theta})|^q d\theta.$$

Thus the theorem is proved completely.

Using Theorem 1, we can not prove the theorem on strong summability in general, that is, Theorem F can not be proved by this method completely.

**2.** Remark. On the other hand, Theorem 1 can be generalized. For a function $\varphi(z) \in H^p, p > 1$, by the definition we put

$$g_{q,\alpha}^*(\theta, \varphi) = g_{q,\alpha}^*(\theta)$$

$$= \left\{\int_0^1 (1-\rho)^{q\alpha} d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi'(\rho e^{i\theta + it})|^q \frac{dt}{|1-\rho e^{it}|^{q\alpha - q + 2}}\right\}^{1/q}$$

where $q \geqq 2$ and $\alpha$ will be defined later.

Then we obtain

Theorem 2. *Let $\varphi(z) \in H^p$, $p > 1$, then we have for $q \geqq 2$,*

$$\int_{-\pi}^{\pi} (g^*_{q,\alpha}(\theta))^p d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta$$

*where*

(i) *if* $1 < p \leqq 2$ *and* $q \geqq 2$, *then* $\alpha > 1 + \dfrac{1}{p} - \dfrac{2}{q}$

(ii) *if* $p > 2$ *and* $q \geqq 2$, *then* $\begin{cases} \alpha > 1 + \dfrac{1}{p} - \dfrac{2}{q} & \text{if } p < q, \\ \alpha > 1 - \dfrac{1}{q} & \text{if } p \geqq q. \end{cases}$

In Theorem 2, if we put $q = 2$, we obtain Theorems 1 and 2 of G.Sunouchi [3].

PROOF OF THEOREM 2. Proof can be done following the same line of that of Theorem 1, and so we sketch the proof.

(a) the case $p = q(\geqq 2)$. By the change of the order of integration, we have

$$\int_{-\pi}^{\pi} (g^*_{q,\alpha}(\theta))^q d\theta = \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{q\alpha} |\varphi'(\rho e^{i\theta})|^q d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{|1-\rho e^{it}|^{q\alpha-q+2}}$$

where we have

$$\int_{-\pi}^{\pi} \frac{dt}{|1-\rho e^{it}|^{q\alpha-q+2}} = 2\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) \frac{dt}{(\delta^2+t^2)^{(q\alpha-q+2)/2}}$$
$$= I + J,$$

say. For $I$:

$$I \leqq 2\delta^{q-q\alpha-2} \int_0^{\delta} dt = 2\delta^{q-q\alpha-1}$$

and for $J$:

$$J \leqq 2 \int_{\delta}^{\pi} t^{q-q\alpha-2} dt \leqq A_{p,q} \delta^{q-q\alpha-1}$$

under an assumption $q\alpha > q-1$. This is satisfied if

$$\alpha > 1 - \frac{1}{q}. \tag{2.01}$$

Hence we have under the above condition (2.01),

$$\int_{-\pi}^{\pi} (g^*_{q,\alpha}(\theta))^q d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} (g^*_q(\theta))^q d\theta$$
$$\leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^q d\theta.$$

(b) the case $p > q(\geqq 2)$. In this case with the same notation of the case (b) of Theorem 1, we have

$$\left\{\int_{-\pi}^{\pi}(g^*_{q,\alpha}(\theta))^p d\theta\right\}^{q/p} = \sup_{\{\xi(\theta)\}}\int_{-\pi}^{\pi}(g^*_{q,\alpha}(\theta))^q\xi(\theta)d\theta$$

$$= \sup_{\{\xi(\theta)\}}\int_{-\pi}^{\pi}dt\int_0^1(1-\rho)^{q-1}|\varphi'(\rho e^{it})|^q d\rho$$

$$\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi(\theta)\frac{(1-\rho)^{q\alpha-q+1}}{|1-\rho e^{it-i\theta}|^{q\alpha-q+2}}d\theta$$

hence, if we define

$$\xi^*(t) = \sup_{0<|\theta|<\pi}\left|\frac{1}{\theta}\int_0^\theta \xi(t-u)du\right|$$

and integrating by parts, we have

$$I = \int_{-\pi}^{\pi}\xi(\theta)\frac{(1-\rho)^{q\alpha-q+1}}{|1-\rho e^{it-i\theta}|^{q\alpha-q+2}}d\theta \leqq \int_{-\pi}^{\pi}\xi(t-\theta)\frac{\delta^{q\alpha-q+1}}{(\delta^2+\theta^2)^{(q\alpha-q+2)/2}}d\theta$$

$$\leqq A_{p,q}\xi^*(t)\left(\delta^{q\alpha-q+1}+\int_{-\pi}^{\pi}\frac{\delta^{q\alpha-q+1}\theta^2 d\theta}{(\delta^2+\theta^2)^{(q\alpha-q+4)/2}}\right).$$

And if we put

$$J = \int_{-\pi}^{\pi}\frac{\delta^{q\alpha-q+1}\theta^2}{(\delta^2+\theta^2)^{(q\alpha-q+4)/2}}d\theta = 2\left(\int_0^\delta+\int_\delta^\pi\right)$$

$$= J_1 + J_2,$$

say. Concerning with $J_1$:

$$J_1 \leqq 2\int_0^\delta \frac{\delta^{q\alpha-q+1}\theta^2}{\delta^{q\alpha-q+4}}d\theta \leqq A$$

and for $J_2$:

$$J_2 \leqq 2\int_\delta^\pi \delta^{q\alpha-q+1}\theta^{q-q\alpha-2}d\theta \leqq A_{p,q}$$

under an assumption $q\alpha - q + 1 > 0$, and this is satisfied if

$$\alpha > 1 - \frac{1}{q}. \tag{2.02}$$

Then, we obtain under the above condition (2.02), applying Hölder's inequality and the maximal theorem,

$$\left\{\int_{-\pi}^{\pi}(g^*_{q,\alpha}(\theta))^p d\theta\right\}^{q/p} = \sup_{\{\xi(\theta)\}} A_{p,q}\int_{-\pi}^{\pi}\xi^*(t)(g_q(t))^q dt$$

$$= \sup_{\{\xi(\theta)\}} A_{p,q}\left\{\int_{-\pi}^{\pi}(\xi^*(t)^s dt\right\}^{1/s}\left\{\int_{-\pi}^{\pi}(g_q(t))^p dt\right\}^{q/p}$$

$$\leq A_{p,q}\left\{\int_{-\pi}^{\pi}|\varphi(e^{it})|^p dt\right\}^{q/p}.$$

Hence we obtain

$$\int_{-\pi}^{\pi}(g^*_{q,\alpha}(\theta))d\theta \leq A_{p,q}\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|^p d\theta.$$

(c) the case $p<q$. Following the same arguments such as the case (c) of Theorem 1, we have

$$(g^*_{q,\alpha}(\theta,\varphi))^q = A_{p,q}(\psi^*_k(\theta))^{q(q-p)/p}(J(\theta))^q, \qquad 1<k<q$$

say. And so by Hölder's inequality,

$$\int_{-\pi}^{\pi}(g^*_{q,\alpha}(\theta))^q d\theta \leq A_{p,q}\left\{\int_{-\pi}^{\pi}(\psi^*_k(\theta))^q d\theta\right\}^{(q-p)/q}\left\{\int_{-\pi}^{\pi}(J(\theta))^q d\theta\right\}^{p/q}$$

where

$$\int_{-\pi}^{\pi}(J(\theta))^q d\theta = \int_{-\pi}^{\pi}d\theta\int_0^1\delta^{q\alpha}|\psi'(\rho e^{i\theta})|^q d\rho\int_{-\pi}^{\pi}\left(1+\frac{|t|}{\delta}\right)^{q(q-p)/kp}\frac{dt}{|1-\rho e^{it}|^{q\alpha-q+2}}$$

and that

$$\delta^{q\alpha}\int_{-\pi}^{\pi}\left(1+\frac{|t|}{\delta}\right)^{q(q-p)/kp}\frac{dt}{|1-\rho e^{it}|^{q\alpha-q+2}} \leq 2\delta^{q\alpha}\left(\int_0^\delta+\int_\delta^\pi\right)$$

$$= J_1 + J_2,$$

say. For $J_1$:

$$J_1 \leq 2\delta^{q\alpha}\int_0^\delta\frac{dt}{(\delta^2+t^2)^{(q\alpha-q+2)/2}} \leq A_{p,q}\delta^{q-1}$$

and for $J_2$:

$$J_2 \leq 2\delta^{q\alpha}\int_\delta^\pi\left(\frac{t}{\delta}\right)^{q(q-p)/kp}t^{q-\alpha-2}dt$$

$$\leq A_{p,q}\delta^{q-1}$$

under the assumption $q\alpha - q(q-p)/kp > q-1$. And this condition is satisfied taking $k$ sufficiently near to $q$ and if

$$\alpha > 1 + \frac{1}{p} - \frac{2}{q}. \tag{2.03}$$

Combining these estimations we obtain under the above condition (2.03),

$$\int_{-\pi}^{\pi} (g^*_{q,\alpha}(\theta, \varphi))^p d\theta \leqq A_{p,q}\left\{\int_{-\pi}^{\pi} (\psi^*_k(\theta))^q d\theta\right\}^{(q-p)/q} \left\{\int_{-\pi}^{\pi} (g_{q,\alpha}(\theta))^q d\theta\right\}^{p/q}$$

$$\leqq A_{p,q} \int_{-\pi}^{\pi} |\psi(e^{i\theta})|^q d\theta$$

$$= A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta.$$

Thus we can complete the proof of Theorem 2.

As an application of Theorem 2, we can prove theorems on strong summability of Fourier series.

Let us put according to G. Sunouchi [2],

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n \rho^n e^{in\theta},$$

$$s_n(\theta, \varphi) = s_n(\theta) = \sum_{\nu=0}^{n} c_\nu e^{i\nu\theta},$$

$$t_n(\theta, \varphi) = t_n(\theta) = n c_n e^{in\theta},$$

$$\sigma^\alpha_n(\theta, \varphi) = \sigma^\alpha_n(\theta) = \frac{1}{A^\alpha_n} \sum_{\nu=0}^{n} A^{\alpha-1}_{n-\nu} s_\nu(\theta), \text{ for } \alpha > -1$$

and

$$t^\alpha_n(\theta, \varphi) = t^\alpha_n(\theta) = \frac{1}{A^\alpha_n} \sum_{\nu=0}^{n} A^{\alpha-1}_{n-\nu} t_\nu(\theta), \text{ for } \alpha > 0,$$

where $A^\alpha_n = \binom{n+\alpha}{n} \sim n^\alpha/\Gamma(\alpha+1)$.

Then we have

$$t^\alpha_n(\theta) = n\{\sigma^\alpha_n(\theta) - \sigma^\alpha_n(\theta)\} = \alpha\{\sigma^{\alpha-1}_n(\theta) - \sigma^\alpha_n(\theta)\}.$$

Now, G. Sunouchi [2,3] proved the following most general theorem.

THEOREM 3. *If $\varphi(z) \in H^p, p > 0$, then we have*

$$\int_{-\pi}^{\pi} \left\{\sum_{n=1}^{\infty} \frac{|t^\alpha_n(\theta)|^2}{n}\right\}^{p/2} d\theta \leqq A_{p,\alpha} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta$$

*where $\alpha > \frac{1}{p}$ if $0 < p \leqq 2$, and $\alpha > \frac{1}{2}$ if $p > 2$ respectively.*

We can generalize this theorem for the case $p > 1$.

THEOREM 4. *If $\varphi(z) \in H^p$, $p > 1$, then we have*

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|t_n^\alpha(\theta)|^q}{n} \right\}^{p/q} d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta$$

*where $p$, $q$ and $\alpha$ satisfy the conditions of Theorem* 2.

This theorem is immediately obtained from a following lemma.

LEMMA 3. *If $\varphi(z) \in H^p, p > 1$, then*

$$\left( \sum_{n=1}^{\infty} \frac{|t_n^\alpha(\theta)|^q}{n} \right)^{1/q} \leqq A_{p,q} g_{q,\alpha}^*(\theta), \qquad a.e.$$

This can be proved following the same argument of G. Sunouchi [3, Lemma 1] and S. Koizumi [1, p.125].

REMARK. In the case $1 < p \leqq 2$, Theorem 2 can be proved simply by reducing to Theorem 1 of G. Sunouchi [3], where we use Lemma 1 instead of Lemma 2 in this note. However in the case $p > 2$ it may not be followed but less general result, if we reduce to Theorem 2 of G. Sunouchi [3].

## LITERATURE

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