ON ABSOLUTE CESÀRO SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES

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1. Let $f(t)$ be a summable function, periodic with period 2π. Let its Fourier series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t).$$

We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

$$\Phi_\alpha(t) = \left(\frac{1}{\Gamma(\alpha)}\right)\int_0^t (t-u)^{\alpha-1}\varphi(u)du \qquad (\alpha > 0),$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t) \qquad (\alpha > 0),$$

$$A_n^\alpha = \binom{n+\alpha}{n} \cong n^\alpha/\Gamma(\alpha+1).$$

THEOREM 1. *If*

$$\int_0^\pi t^{-\gamma+\beta}\,|\,d\varphi_\beta(t)\,| < \infty,$$

then the series $\sum n^{\gamma-\beta}A_n(t)$ *is summable* $|\,C, \alpha\,|$ *at* $t = x$, *where* $1 > \alpha > \gamma \geqq \beta \geqq 0$.

When $\gamma = \beta$, this theorem reduces to the following theorem for the case $1 > \beta \geqq 0$.

BOSANQUET'S THEOREM [2]. *If* $\varphi_\beta(t)$ *is of bounded variation in* $(0,\pi)$, *then the Fourier series of* $f(t)$ *is summable* $|\,C, \alpha\,|$ *at the point* $t = x$, *where* $\alpha > \beta \geqq 0$.

Further Theorem 1 generalizes the following theorem.

MOHANTY'S THEOREM [3]. *If* $0 < \alpha < 1$, *and*

$$\int_0^\pi t^{-\alpha}\,|\,d\varphi(t)\,| < \infty,$$

then $\sum n^\alpha A_n(t)$ *is summable* $|\,C, \beta\,|$ *for* $\beta > \alpha$, *at* $t = x$.

THEOREM 2. *If*

$$\int_0^{\pi} | d(t^{-\gamma}\Phi_\beta(t)) | < \infty,$$

then the series

$$\sum_{n=0}^{\infty} \frac{n^{\gamma-\beta}}{\{\log (n+2)\}^{1+\varepsilon}} A_n(t)$$

is summable $|C, \gamma|$ *at* $t=x$, *where* $1 \geqq \gamma \geqq \beta \geqq 0$, *and* $\varepsilon > 0$.

This theorem is a generalization of the following theorem.

CHENG'S THEOREM [4]. *If* $\varphi_\alpha(t)$, $0 \leqq \alpha \leqq 1$ *is of bounded variation in* $(0, \pi)$, *then* $\sum A_n(t)/(\log n)^{1+\varepsilon}$ *is summbale* $|C, \alpha|$ *at the point* $t=x$.

2. Proof of Theorem 1. We require the following lemmas.

LEMMA 1. *Let*

$$S_k(n, t) = \sum_{\nu=0}^{k} A_{n-\nu}^{\alpha-1} \sin \nu t \qquad (1 > \alpha > 0),\ (k \leqq n),$$

then we have

$$S_k(n, t) = O\{k(n-k)^{\alpha-1}\} = O\{t^{-1}(n-k)^{\alpha-1}\} \qquad (k < n),$$

$$S_n(n, t) = O(n^\alpha) = O(t^{-\alpha}).$$

PROOF. The first result is obtained by Abel's transformation, the second has been given by Obrechkoff [1].

LEMMA 2. *Let*

$$S_k^\lambda(n, t) = \left(\frac{d}{dt}\right)^\lambda S_k(n, t),$$

then we have

$$S_k^\lambda(n, t) = O\{k^{\lambda+1}(n-k)^{\alpha-1}\} = O\{t^{-1}k^\lambda(n-k)^{\alpha-1}\} \qquad (k < n)$$

$$S_n^\lambda(n, t) = O(n^{\alpha+\lambda}) = O(n^\lambda t^{-\alpha}).$$

The proof is the same as Lemma 1.

LEMMA 3. *Let*

$$H^\alpha(n, t) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu^\delta \sin \nu t \qquad (\delta = \gamma - \beta),$$

then we have

$$H^\alpha(n, t) = O(n^\delta) = O(t^{-1}n^{\delta-1} + t^{-\alpha}n^{\delta-\alpha}).$$

Proof. By Abel's transformation,

$$H^{\alpha}(n, t) = \frac{1}{A_n^{\alpha}}\left\{\sum_{\nu=0}^{n-1} S_{\nu}(n, t)\Delta\nu^{\delta} + S_n(n, t)n^{\delta}\right\}.$$

From Lemma 1,

$$\begin{aligned} S_{\nu}(n, t)\Delta\nu^{\delta} &= O\{\nu(n-\nu)^{\alpha-1}\nu^{\delta-1}\} \\ &= O\{t^{-1}(n-\nu)^{\alpha-1}\nu^{\delta-1}\} \end{aligned}$$

$$\begin{aligned} \sum_{\nu=0}^{n-1} S_{\nu}(n, t)\Delta\nu^{\delta} &= O\left\{\int_0^n (n-\nu)^{\alpha-1}\nu^{\delta}d\nu\right\} \\ &= O(n^{\delta+\alpha}), \\ &= O\left\{t^{-1}\int_0^n (n-\nu)^{\alpha-1}\nu^{\delta-1}d\nu\right\} \\ &= O(t^{-1}n^{\delta+\alpha-1}). \end{aligned}$$

Substituting these values into $H^{\alpha}(n, t)$, we get Lemma 3.

Lemma 4.

$$\begin{aligned} \left(\frac{d}{dt}\right)^{\lambda} H^{\alpha}(n, t) &= O(n^{\lambda+\delta}), \\ &= O(t^{-1}n^{\lambda+\delta} + t^{-\alpha}n^{\lambda+\delta-\alpha}). \end{aligned}$$

Proof. From the definition

$$\left(\frac{d}{d\lambda}\right)^{\lambda} H^{\alpha}(n, t) = \frac{1}{A_n^{\alpha}}\left\{\sum_{\nu=0}^{n-1} S_{\nu}^{\lambda}(n, t)\Delta\nu^{\delta} + n^{\delta}S_n^{\lambda}(n, t)\right\}.$$

Using Lemma 2,

$$\sum_{\nu=0}^{n-1} S_{\nu}^{\lambda}(n, t)\Delta\nu^{\delta} = \begin{cases} O\left\{\int_0^n \nu^{\lambda+1}(n-\nu)^{\alpha-1}\nu^{\delta-1}d\nu\right\} = O(n^{\lambda+\delta+\alpha}), \\ \\ O\left\{t^{-1}\int_0^n \nu^{\lambda}(n-\nu)^{\alpha-1}\nu^{\delta-1}d\nu\right\} = O(t^{-1}n^{\lambda+\delta+\alpha-1}). \end{cases}$$

Then, by the above equality we get Lemma 4.

Lemma 5. *Let*

$$J(n, u) = \int_u^{\pi} (t-u)^{-\beta}\frac{d}{dt}H^{\alpha}(n, t)\,dt$$

then, we have

$$\begin{aligned} J(n, u) &= O(n^{\alpha+\beta}), \\ &= O(u^{-1}n^{\delta+\beta-1} + u^{-\alpha}n^{\delta-\alpha+\beta}). \end{aligned}$$

PROOF. We divide the interval of integration into two parts and put

$$J(n, u) = \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi} = J_1 + J_2.$$

By the aid of Lemma 4, and by the second mean value theorem, we have,

$$\begin{aligned} J_1(n, u) &= \int_u^{u+n^{-1}} (t-u)^{-\beta} \cdot O(n^{1+\delta})\, dt \\ &= O(n^{\delta+\beta}). \\ J_2(n, u) &= \int_{u+n^{-1}}^{\pi} (t-u)^{-\beta} \frac{d}{dt} H^{\alpha}(n, t)\, dt \\ &= n^{\beta} \int_{u+n^{-1}}^{\pi} \frac{d}{dt} H^{\alpha}(n, t)\, dt \\ &= O\{n^{\beta} H^{\alpha}(n, u)\} \\ &= O(n^{\delta+\beta}), \qquad (u + n^{-1} \leqq \xi \leqq \pi). \end{aligned}$$

By the same way, we have

$$\begin{aligned} J_1(n, u) &= \int_u^{u+n^{-1}} (t-u)^{-\beta} \cdot O(t^{-1}n^{\delta} + t^{-\alpha}n^{1+\delta-\alpha})\, dt \\ &= O\left\{ n^{\delta} \int_u^{u+n^{-1}} (t-u)^{-\beta} t^{-1}\, dt + n^{1+\delta-\alpha} \int_u^{u+n^{-1}} (t-u)^{-\beta} t^{-\alpha}\, dt \right\} \\ &= O(u^{-1}u^{\delta+\beta-1} + u^{-\alpha}n^{\delta+\beta-\alpha}). \\ J_2(n, u) &= O\{n^{\beta} H^{\alpha}(n, u)\} \\ &= O(u^{-1}n^{\delta+\beta-1} + u^{-\alpha}n^{\delta+\beta-\alpha}). \end{aligned}$$

LEMMA 6. *Let*

$$I(n, u) = \int_0^u v^{\beta} \frac{d}{dv} J(n, v)\, dv,$$

then we have

$$I(n, u) = O(u^{\beta} n^{\alpha+\beta}).$$

PROOF. We use the second mean value theorem and Lemma 5.

$$I(n, u) = u^\beta \int_\eta^u \frac{d}{dv} J(n, v)\, dv$$

$$= u^\beta [J(n, v)]_\eta^u$$

$$= O(u^\beta n^{\delta+\beta}).$$

LEMMA 7. *Let*

$$K(n, u) = \int_u^\pi v^\beta \frac{d}{dv} J(n, v)\, dv,$$

then we have

$$K(n, u) = O(n^{\delta+\beta-\alpha} + u^{-1} n^{\delta-1} + u^{-\alpha} n^{\delta-\alpha} + u^{-1+\beta} n^{\delta-1+\beta} + u^{-\alpha+\beta} n^{\delta-\alpha+\beta}).$$

PROOF. By integration by parts we have

$$K(n, u) = [v^\beta J(n, v)]_u^\pi - \beta \int_u^\pi v^{\beta-1} J(n, v) dv = K_1 + K_2,$$

say. By Lemma 5 we have

$$K_1 = \pi^\beta J(n, \pi) - u^\beta J(n, u)$$
$$= O(n^{\delta+\beta-\alpha} + u^{\beta-1} n^{\delta+\beta-1} + u^{\beta-\alpha} n^{\delta+\beta-\alpha}).$$

For the part K_2, we use the definition of $J(n, u)$ and Lemma 3, then

$$K_2 = \int_u^\pi v^{\beta-1} J(n, v)\, dv$$

$$= \int_u^\pi v^{\beta-1} \int_v^\pi (t-v)^{-\beta} \frac{d}{dt} H^\alpha(n, t)\, dt\, dv$$

$$= \int_u^\pi \frac{d}{dt} H^\alpha(n, t) \int_u^t v^{\beta-1}(t-v)^{-\beta} dv\, dt$$

$$= \int_u^\pi \frac{d}{dt} H^\alpha(n, t) \int_{u/t}^1 z^{\beta-1}(1-z)^{-\beta}\, dz\, dt$$

$$= \left\{\int_{u/\pi}^1 z^{\beta-1}(1-z)^{-\beta} dz\right\} \int_\zeta^\pi \frac{d}{dt} H^\alpha(n, t)\, dt$$

$$= O\left\{[H^\alpha(n, t)]_\zeta^\pi\right\}$$

$$= O\left\{H^\alpha(n, \pi) - H^\alpha(n, \zeta)\right\}$$

$$= O(n^{\delta-\alpha} + u^{-1} n^{\delta-1} + u^{-\alpha} n^{\delta-\alpha}),$$

since $u \leqq \zeta \leqq \pi$.

PROOF OF THEOREM 1. It is sufficient to prove that

$$\sum_{n=1}^{\infty} |\zeta_n^\alpha|/n < \infty,$$

where

$$\zeta_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\nu\cdot\nu^{\gamma-\beta}A_\nu(x).$$

Using the notations in above lemmas we obtain

$$\zeta_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\nu\,\nu^{\gamma-\beta}\frac{2}{\pi}\int_0^\pi \varphi(t)\cos\nu t\,dt$$

$$= \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\nu^{\gamma-\beta}\nu\cos\nu t\,dt$$

$$= \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{d}{dt}\Big(\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\nu^{\gamma-\beta}\sin\nu t\Big)dt$$

$$= \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{d}{dt}H^\alpha(n,t)\,dt$$

$$= \frac{2}{\pi}\int_0^\pi \frac{d}{dt}H^\alpha(n,t)\left\{\frac{1}{\Gamma(1-\beta)}\int_0^t (t-u)^{-\beta}d\Phi_\beta(u)\right\}dt$$

$$= \frac{2}{\pi\Gamma(1-\beta)}\int_0^\pi d\Phi_\beta(u)\int_u^\pi (t-u)^{-\beta}\frac{d}{dt}H^\alpha(n,t)\,dt$$

$$= \frac{2}{\pi\Gamma(1-\beta)}\int_0^\pi J(n,u)d\Phi_\beta(u)$$

$$= \frac{2}{\pi\Gamma(1-\beta)}[\Phi_\beta(u)J(n,u)]_0^\pi - \frac{2}{\pi\Gamma(1-\beta)}\int_0^\pi \Phi_\beta(u)\frac{d}{du}J(n,u)du$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\int_0^\pi u^{-\beta}\Phi_\beta(u)u^\beta\frac{d}{du}J(n,u)\,du$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\Big([u^{-\beta}\Phi_\beta(u)I(n,u)]_0^\pi - \frac{1}{\Gamma(\beta+1)}\int_0^\pi I(n,u)d\Big\{\Phi_\beta(u)\cdot u^{-\beta}\Gamma(\beta+1)\Big\}\Big)$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\pi^{-\beta}\Phi_\beta(\pi)I(n,\pi) + \frac{2}{\pi\Gamma(1-\beta)\Gamma(1+\beta)}\int_0^\pi I(n,u)d\varphi_\beta(u).$$

If we suppose that $\varphi(t) \equiv 1$, then $\zeta_n^\alpha = 0$, $\varphi_\beta(u) = 1$ and we obtain $I(n, \pi) = 0$. Thus we have

$$\zeta_n^\alpha = \frac{2}{\pi\Gamma(1-\beta)\Gamma(1+\beta)} \int_0^\pi I(n, u)\, d\varphi_\beta(u).$$

Hence

$$\sum_{n=1}^\infty \frac{|\zeta_n^\alpha|}{n} = O\Big\{\int_0^\pi \sum_{n=1}^\infty \frac{I(n, u)}{n} d\varphi_\beta(u)\Big\}$$

$$= O\Big\{\int_0^\pi \Big|\sum_{n=1}^\infty \frac{I(n, u)}{n}\Big| \Big|d\varphi_\beta(u)\Big|\Big\}.$$

We divide $\sum$ into two parts such that

$$\sum_{n=1}^\infty \frac{I(n, u)}{n} = \sum_{n<u^{-1}} \frac{I(n, u)}{n} + \sum_{n\geq u^{-1}} \frac{I(n, u)}{n}$$

$$= M_1 + M_2,$$

say. From Lemma 6 we have

$$M_1 = O\Big(\sum_{n<u^{-1}} \frac{u^\beta n^{\delta+\beta}}{n}\Big) = O(u^\beta \int_0^{u^{-1}} y^{\delta+\beta-1}\, dy) = O(u^{-\delta}).$$

For the part M_2 we use Lemma 7. Since

$$I(n, u) = I(n, \pi) - K(n, u) = - K(n, u),$$

we have

$$M_2 = O\Big\{\sum_{n\geq u^{-1}} \frac{-K(n, u)}{n}\Big\}$$

$$= O\Big\{\sum_{n\geq u^{-1}} n^{\delta+\beta-\alpha-1} + u^{-1}n^{\delta-2} + u^{-\alpha}n^{\delta-\alpha-1} + u^{-1+\beta}n^{\delta-2+\beta}$$

$$+ u^{-\alpha+\beta}n^{\delta-\alpha+\beta-1}\Big\}$$

$$= O(u^{-\delta}).$$

Thus

$$\sum_{n=1}^\infty I(n, u)/n = O(u^{-\delta}),$$

and then

$$\sum_{n=1}^\infty |\zeta_n^\alpha|/n = O\Big(\int_0^\pi u^{-\delta} |d\varphi_\beta(u)|\Big) = O\Big(\int_0^\pi u^{-\gamma+\beta} |d\varphi_\beta(u)|\Big) < \infty$$

since $\delta = \gamma - \beta$.

This completes the proof of Theorem 1.

Similarly we can prove the following theorem.

THEOREM 3. *If*

$$\int_0^{\pi} t^{-\gamma} |d\Phi_\beta(t)| < \infty, \qquad \Phi_\beta(+0) = 0,$$

then the series

$$\sum_{n=0}^{\infty} n^{\gamma-\beta} A_n(t)$$

is summable $|C, \alpha|$ *at* $t = x$, *where* $1 > \alpha > \gamma \geqq \beta \geqq 0$.

3. Proof of Theorem 2. We shall prove here the theorem for the case $\gamma > \beta$ only.

Firstly we suppose $\beta > 0$. We require the following lemmas.

LEMMA 8. *Let*

$$S_k(n, t) = \sum_{\nu=0}^{k} A_{n-\nu}^{\gamma-1} \sin \nu t \qquad (k \leqq n,\ 1 \geqq \gamma > 0),$$

then we have

$$S_k(n, t) = O\{k(n-k)^{\gamma-1}\}, \quad = O\{t^{-1}(n-k)^{\gamma-1}\} \qquad (k < n),$$

and

$$S_n(n, t) = O(n^\gamma), \quad = O(t^{-\gamma}).$$

LEMMA 9. *Let*

$$S_k^\lambda(n, t) = \left(\frac{d}{dt}\right)^\lambda S_k(n, t) \qquad (\lambda \geqq 1,\ k \leqq n),$$

then we have

$$S_k^\lambda(n, t) = O\{k^{\lambda+1}(n-k)^{\gamma-1}\},$$

$$= O\{t^{-1}k^\lambda(n-k)^{\gamma-1}\}, \qquad (k < n),$$

and

$$S_n^\lambda(n, t) = O(n^{\gamma+\lambda}),$$

$$= O(n^\lambda t^{-\gamma}).$$

These are similarly proved as Lemmas 1, 2.

LEMMA 10. *Let*

$$H^\gamma(n, t) = \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \frac{\nu^\delta}{\{\log(\nu+2)\}^{1+\varepsilon}} \sin \nu t \qquad (\delta = \gamma - \beta),$$

then we have

$$H^\gamma(n, t) = O\left\{\frac{n^\delta}{(\log n)^{1+\varepsilon}}\right\},$$

$$= O\left\{t^{-1}\frac{n^{\delta-1}}{(\log n)^{1+\varepsilon}} + t^{-\gamma}\frac{n^{\delta-\gamma}}{(\log n)^{1+\varepsilon}}\right\}.$$

PROOF. By Abel's transformation we have

$$H^{\gamma}(n,t)=\frac{1}{A_n^{\gamma}}\left\{\sum_{\nu=0}^{n-1}S_\nu(n,t)\varDelta\frac{\nu^\delta}{\{\log(\nu+2)\}^{1+\varepsilon}}+S_n(n,t)\frac{n^\delta}{\{\log(n+2)\}^{1+\varepsilon}}\right\}.$$

From Lemma 8

$$\begin{aligned}S_\nu(n,t)\varDelta\frac{\nu^\delta}{\{\log(\nu+2)\}^{1+\varepsilon}}&=O\left\{\nu(n-\nu)^{\gamma-1}\frac{\nu^{\delta-1}}{(\log\nu)^{1+\varepsilon}}\right\},\\&=O\left\{t^{-1}(n-\nu)^{\gamma-1}\frac{\nu^{\delta-1}}{(\log\nu)^{1+\varepsilon}}\right\},\end{aligned}$$

hence

$$\sum_{\nu=1}^{n-1}S_\nu(n,t)\varDelta\frac{\nu^\delta}{\{\log(\nu+2)\}^{1+\varepsilon}}$$

$$=\begin{cases}O\left\{\displaystyle\int_0^n(n-\nu)^{\gamma-1}\frac{\nu^\delta}{(\log\nu)^{1+\varepsilon}}d\nu\right\}=O\left\{\dfrac{n^{\delta+\gamma}}{(\log n)^{1+\varepsilon}}\right\},\\[2ex] O\left\{\displaystyle\int_0^n(n-\nu)^{\gamma-1}\frac{\nu^{\delta-1}}{(\log\nu)^{1+\varepsilon}}d\nu\cdot t^{-1}\right\}=O\left\{t^{-1}\dfrac{n^{\delta+\gamma-1}}{(\log n)^{1+\varepsilon}}\right\}.\end{cases}$$

LEMMA 11.

$$\begin{aligned}\left(\frac{d}{dt}\right)^{\lambda}H^{\gamma}(n,t)&=O\left\{\frac{n^{\lambda+\delta}}{(\log n)^{1+\varepsilon}}\right\},\\&=O\left\{t^{-1}\frac{n^{\lambda-1+\delta}}{(\log n)^{1+\varepsilon}}+t^{-\gamma}\frac{n^{\lambda+\delta-\gamma}}{(\log n)^{1+\varepsilon}}\right\}.\end{aligned}$$

PROOF. By Abel's transformation

$$\left(\frac{d}{dt}\right)^{\lambda}H^{\gamma}(n,t)=\frac{1}{A_n^{\gamma}}\left\{\sum_{\nu=0}^{n-1}S_\nu^{\lambda}(n,t)\varDelta\frac{\nu^\delta}{\{\log(\nu+2)\}^{1+\varepsilon}}+S_n^{\lambda}(n,t)\frac{n^\delta}{\{\log(n+2)\}^{1+\varepsilon}}\right\}.$$

By Lemma 9 we obtain

$$\sum_{\nu=0}^{n-1}S_\nu^{\lambda}(n,t)\varDelta\frac{\nu^\delta}{(\log\nu)^{1+\varepsilon}}$$

$$=\begin{cases}O\left\{\displaystyle\int_0^n\nu^{\lambda+1}(n-\nu)^{\gamma-1}\frac{\nu^{\delta-1}}{(\log\nu)^{1+\varepsilon}}d\nu\right\}=O\left\{\dfrac{n^{\delta+\lambda-\gamma}}{(\log n)^{1+\varepsilon}}\right\},\\[2ex] O\left\{\displaystyle\int_0^n\nu^{\lambda}(n-\nu)^{\gamma-1}\frac{\nu^{\delta-1}}{(\log\nu)^{1+\varepsilon}}d\nu\cdot t^{-1}\right\}=O\left\{t^{-1}\dfrac{n^{\lambda+\delta+\gamma-1}}{(\log n)^{1+\varepsilon}}\right\}.\end{cases}$$

LEMMA 12. *Let*

$$J(n, u) = \int_u^\pi (t-u)^{-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt,$$

then we have

$$J(n, u) = O\left\{\frac{n^{\delta+\beta}}{(\log n)^{1+\varepsilon}}\right\}$$

$$= O\left\{u^{-1}\frac{n^{\delta+\beta-1}}{(\log n)^{1+\varepsilon}} + u^{-\gamma}\frac{n^{\delta+\beta-\gamma}}{(\log n)^{1+\varepsilon}}\right\}.$$

PROOF. We split up the interval into two parts, i.e.

$$J(n, u) = \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi} = J_1 + J_2,$$

say. Then as $\lambda = 1$, Lemma 4 gives

$$J_1 = \begin{cases} \displaystyle\int_u^{u+n^{-1}} (t-u)^{-\beta}\, O\left\{\frac{n^{1+\delta}}{(\log n)^{1+\varepsilon}}\right\} dt, \\ \displaystyle\int_u^{u+n^{-1}} (t-u)^{-\beta}\, O\left\{t^{-1}\frac{n^{\delta}}{(\log n)^{1+\varepsilon}} + t^{-\gamma}\frac{n^{-\gamma+1+\delta}}{(\log n)^{1+\varepsilon}}\right\} dt; \end{cases}$$

$$= \begin{cases} \displaystyle O\left\{\frac{n^{1+\delta}}{(\log n)^{1+\varepsilon}} \int_u^{u+n^{-1}} (t-u)^{-\beta} dt\right\}, \\ \displaystyle O\left\{\frac{n^{\delta}}{(\log n)^{1+\varepsilon}} \int_u^{u+n^{-1}} (t-u)^{-\beta} t^{-1}\, dt + \frac{n^{1+\delta-\gamma}}{(\log n)^{1+\varepsilon}} \int_u^{u+n^{-1}} (t-u)^{-\beta} t^{-\gamma}\, dt\right.; \end{cases}$$

$$= O\left\{\frac{n^{\delta+\beta}}{(\log n)^{1+\varepsilon}}\right\}, \quad = O\left\{u^{-1}\frac{n^{\delta+\beta-1}}{(\log n)^{1+\varepsilon}} + u^{-\gamma}\frac{n^{\delta+\beta-\gamma}}{(\log n)^{1+\varepsilon}}\right\}.$$

For the second part J_2 we use the second mean value theorem and Lemma 10, then

$$J_2 = \int_{u+n^{-1}}^{\pi} (t-u)^{-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt = n^\beta \int_{u+n^{-1}}^{\xi} \frac{d}{dt} H^\gamma(n, t) dt$$

$$= n^\beta [H^\gamma(n, t)]_{u+n^{-1}}^{\xi}$$

$$= O\{n^{\delta+\beta}(\log n)^{-1-\varepsilon}\},$$

$$= O\{u^{-1}n^{\delta+\beta-1}(\log n)^{-1-\varepsilon} + u^{-\gamma}n^{\delta+\beta-\gamma}(\log n)^{-1-\varepsilon}\}$$

since by $u + n^{-1} \leqq \xi \leqq \pi$ we may regard $[H^\gamma(n, t)]_{u+n^{-1}}^{\xi} = O\{H^\gamma(n, u)\}$.

LEMMA 13. *Let*

$$I(n, u) = \int_0^u v^\gamma \frac{d}{dv} J(n, v)\, dv,$$

then we have

$$I(n, u) = O\{u^\gamma n^{\delta+\beta}/(\log n)^{1+\varepsilon}\}.$$

PROOF. We use the first value of $J(n, u)$ in Lemma 12.

$$I(n, u) = u^\gamma \int_0^\eta \frac{d}{dv} J(n, v)\, dv = u^\gamma \Big[J(n, v) \Big]_0^\eta = O\Big\{ u^\gamma n^{\delta+\beta}/(\log n)^{1+\varepsilon} \Big\}.$$

LEMMA 14. *Let*

$$K(n, u) = \int_u^\pi v^\gamma \frac{d}{dv} J(n, v) dv,$$

then we have

$$K(n, u) = O\Big\{ \frac{n^{\delta+\beta-\gamma}}{(\log n)^{1+\varepsilon}} + u^{\gamma-1} \frac{n^{\delta+\beta-1}}{(\log n)^{1+\varepsilon}} + u^{-1+\gamma-\beta} \frac{n^{\delta-1}}{(\log n)^{1+\varepsilon}} + u^{-\beta} \frac{n^{\delta-\gamma}}{(\log n)^{1+\varepsilon}} \Big\}.$$

PROOF. We integrate by parts,

$$K(n, u) = \Big[v^\gamma J(n, v) \Big]_u^\pi - \gamma \int_u^\pi v^{\gamma-1} J(n, v) dv = K_1 + K_2,$$

say. Then by the second estimation of $J(n, u)$ in Lemma 12,

$$K_1(n, u) = \pi^\gamma J(n, \pi) - u^\gamma J(n, u) = O\Big\{ \frac{n^{\delta+\beta-\gamma}}{(\log n)^{1+\varepsilon}} + u^{-1+\gamma} \frac{n^{\delta+\beta-1}}{(\log n)^{1+\varepsilon}} \Big\}.$$

For the part K_2, from the definition of $J(n, u)$ and the second estimation of $H^\gamma(n, t)$ in Lemma 10,

$$\begin{aligned} K_2 &= \int_u^\pi v^{\gamma-1} \int_v^\pi (t-v)^{-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt\, dv \\ &= \int_u^\pi \frac{d}{dt} H^\gamma(n, t) \int_u^t v^{\gamma-1}(t-u)^{-\beta} dv\, dt \\ &= \int_u^\pi \frac{d}{dt} H^\gamma(n, t) t^{\gamma-\beta} \int_{u/t}^1 z^{\gamma-1}(1-z)^{-\beta}\, dz\, dt \\ &= \int_{u/\pi}^1 z^{\gamma-1}(1-z)^{-\beta} dz \int_\zeta^\pi t^{\gamma-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt \\ &= O\Big\{ \int_\zeta^\pi t^{\gamma-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt \Big\} \end{aligned}$$

$$= O\left\{\left[H^{\gamma}(n,t)t^{\gamma-\beta}\right]_{\zeta}^{\pi} - \int_{\zeta}^{\pi} H^{\gamma}(n,t)t^{\gamma-\beta-1}\,dt\right\}.$$

We substitute the second estimation of Lemma 3 for $H^{\gamma}(n,t)$, then we get easily

$$K_2 = O\left\{u^{\gamma-\beta-1}\frac{n^{\delta-1}}{(\log n)^{1+\varepsilon}} + u^{-\beta}\frac{n^{\delta-\gamma}}{(\log n)^{1+\varepsilon}}\right\}.$$

LEMMA 15. *When $nu \geqq 1$, we have $K(n,u) = O\{1/(\log n)^{1+\varepsilon}\}$.*

PROOF. If we remember that $\delta = \gamma - \beta$, we obtain from Lemma 14

$$K(n,u) = O\left\{\frac{1}{(\log n)^{1+\varepsilon}} + \frac{(un)^{\gamma-1}}{(\log n)^{1+\varepsilon}} + \frac{(un)^{-1+\gamma-\beta}}{(\log n)^{1+\varepsilon}} + \frac{(un)^{-\beta}}{(\log n)^{1+\varepsilon}}\right\}.$$

From the assumption, $nu \geqq 1$, and $\gamma - 1$, $-1+\gamma-\beta$, $-\beta$ are all non-positive. Hence we get easily the lemma.

Now we shall prove Theorem 2 for the case $\beta > 0$. Proof runs quite similary as that of Theorem 1.

It is sufficient to prove that

$$\sum_{n=1}^{\infty} |\zeta_n^{\gamma}|/n < \infty,$$

where ζ_n^{γ} is the n-th Cesàro mean of order γ of the sequence $\{n \cdot n^{\gamma-\beta}A_n(x)/(\log n)^{1+\varepsilon}\}$. Using the notations in the above lemmas we have

$$\zeta_n^{\gamma} = \frac{1}{A_n^{\gamma}}\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\nu\cdot\nu^{\gamma-\beta}\frac{1}{\{\log(\nu+2)\}^{1+\varepsilon}}A_n(x)$$

$$= \frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{d}{dt}\left\{\frac{1}{A_n^{\gamma}}\sum_{\nu=0}^{n}A_{n-\nu}^{\gamma-1}\frac{\nu^{\gamma-\beta}}{\{\log(\nu+2)\}^{1+\varepsilon}}\sin\nu t\,dt\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{d}{dt}H^{\gamma}(n,t)\,dt$$

$$= \frac{2}{\pi\Gamma(1-\beta)}\int_0^{\pi}d\Phi_{\beta}(u)\int_u^{\pi}(t-u)^{-\beta}\frac{d}{dt}H^{\gamma}(n,t)\,dt$$

$$= \frac{2}{\pi\Gamma(1-\beta)}\int_0^{\pi}J(n,u)\,d\Phi_{\beta}(u)$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\int_0^{\pi}\Phi_{\beta}(u)\frac{d}{du}J(n,u)\,du$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\left(\left[u^{-\gamma}\Phi_{\beta}(u)I(n,u)\right]_0^{\pi} - \int_0^{\pi}I(n,u)\,d\left\{\Phi_{\beta}(u)u^{-\gamma}\right\}\right)$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\left[\pi^{-\gamma}\Phi_\beta(\pi)I(n,\pi) - \int_0^\pi I(n,u)\, d\left\{u^{-\gamma}\Phi_\beta(u)\right\}\right].$$

Thus it suffices for us to prove that

$$\text{(A)} \qquad \sum_{n=1}^{\infty} I(n,\pi)/n < \infty,$$

and

$$\text{(B)} \qquad \sum_{n=1}^{\infty}\left|\int_0^\pi I(n,v)\, d\left\{u^{-\gamma}\Phi_\beta(u)\right\}\right|/n < \infty.$$

Proof of (A). We have

$$\zeta_n^\gamma = \frac{1}{A_n^\gamma}\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\nu\cdot\nu^{\gamma-\beta}\frac{1}{\{\log(n+2)\}^{1+\varepsilon}}\cdot\frac{2}{\pi}\int_0^\pi \varphi(t)\cos\nu t\, dt$$

$$= -\frac{2}{\pi\Gamma(1-\beta)}\left[\pi^{-\gamma}\Phi_\beta(\pi)I(n,\pi) - \int_0^\pi I(n,u)\, d\left\{u^{-\gamma}\Phi_\beta(u)\right\}\right]$$

for any integrable even function $\varphi(t)$. If we put $\varphi(t) = t^{\gamma-\beta}$, then we easily get

$$\Phi_\beta(t) = \frac{\mathrm{B}(\beta, \gamma-\beta+1)}{\Gamma(\beta)}\, t^\gamma,$$

therefore

$$d\{u^{-\gamma}\Phi_\beta(u)\} = 0.$$

Further, we have [5]

$$\int_0^\pi \varphi(t)\cos\nu t dt = \int_0^\pi t^{\gamma-\beta}\cos\nu t dt$$

$$\cong \nu^{-1-\gamma+\beta}\Gamma(\gamma-\beta+1)\cos\frac{1}{2}\pi(\gamma-\beta+1) \qquad (|\gamma-\beta| \leqq 1).$$

Hence, when $\varphi(t) = t^{\gamma-\beta}$, we have

$$I(n,\pi) = O\left\{\frac{1}{A_n^\gamma}\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\frac{1}{\{\log(\nu+2)\}^{1+\varepsilon}}\right\}.$$

Now, since

$$\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\frac{1}{\{\log(\nu+2)\}^{1+\varepsilon}} = O\left\{\int_0^n (n-\nu)^{\gamma-1}\frac{d\nu}{(\log\nu)^{1+\varepsilon}}\right\}$$

$$= O\left\{n^\gamma\frac{1}{(\log n)^{1+\varepsilon}}\right\},$$

we obtain

$$I(n,\pi) = O\{1/(\log n)^{1+\varepsilon}\}.$$

Hence the proof of (A) follows immediately, that is,

$$\sum_{n=1}^{\infty} I(n,\pi)/n = O\left\{\sum_{n=2}^{\infty} 1/n(\log n)^{1+\varepsilon}\right\} < \infty.$$

PROOF OF (B). By the assumption

$$\int_0^{\pi} \left| d\{u^{-\gamma}\Phi_\beta(u)\} \right| < \infty,$$

and obviously

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \int_0^{\pi} I(n,u)\, d\{u^{-\gamma}\Phi_\beta(u)\} \right| \leqq \int_0^{\pi} \left| \sum_{n=1}^{\infty} \frac{1}{n} I(n,u) \right| \left| d\{u^{-\gamma}\Phi_\beta(u)\} \right|,$$

hence it suffices for us to prove

$$\sum_{n=1}^{\infty} \frac{1}{n} \, |I(n,u)| = O(1)$$

uniformly with respect to u. We now divide the sum into two parts such that

$$\sum_{n=1}^{\infty} = \sum_{n<u^{-1}} + \sum_{n\geqq u^{-1}} = M_1 + M_2,$$

say. In the estimation of M_1, we use Lemma 13, then

$$\begin{aligned} M_1 &= O\left\{ \sum_{n<u^{-1}} \frac{1}{n} \left(u^\gamma \frac{n^\gamma}{(\log n)^{1+\varepsilon}} \right) \right\} \\ &= O\left\{ u^\gamma \sum_{n<u^{-1}} \frac{n^{\gamma-1}}{(\log n)^{1+\varepsilon}} \right\} \\ &= O\left\{ u^\gamma \int_0^{u^{-1}} \frac{x^{\gamma-1}}{(\log x)^{1+\varepsilon}} dx \right\} \\ &= O\left\{ 1/(\log n)^{1+\varepsilon} \right\} \\ &= O(1). \end{aligned}$$

Remembering that $I(n,u) = I(n,\pi) - K(n,u)$, we have

$$M_2 \leqq \sum_{n\geqq u^{-1}} \frac{1}{n} \, |I(n,\pi)| + \sum_{n\geqq n^{-1}} \frac{1}{n} \, |K(n,u)|,$$

where the first sum on the right side is finite as was proved and, by Lemma 15,

$$\begin{aligned} \sum_{n\geqq u^{-1}} \frac{1}{n} \, |K(n,u)| &= O\left\{ \sum \frac{1}{n(\log n)^{1+\varepsilon}} \right\} \\ &= O(1). \end{aligned}$$

Thus Theorem 2 is proved for the case $\beta > 0$.

For the case $\beta = 0$, we shall only sketch the proof since this case is rather simple. It is sufficient to prove that, if

$$\int_0^\pi |d\{t^{-\gamma}\varphi(t)\}| < \infty,$$

where $0 < \gamma \leqq 1$, then the series

$$\sum_{n=0}^\infty n^\gamma A_n(t)/\{\log (n+2)\}^{1+\varepsilon}$$

is summable $|C, \gamma|$ at $t = x$.

Now

$$\zeta_n^\gamma = \frac{1}{A_n^\gamma} \sum_{\nu=0}^n A_{n-\nu}^{\gamma-1} \nu \cdot \nu^\gamma \frac{A_\nu(x)}{\{\log(\nu+2)\}^{1+\varepsilon}}$$

$$= O\left\{ \int_0^\pi v^\gamma \frac{d}{dv} H^\gamma(n, v) dv + \int_0^\pi d\{t^{-\gamma}\varphi(t)\} \int_0^t v^\gamma \frac{d}{dt} H^\gamma(n, v)\, dv \right\}.$$

We put

$$\int_0^t v^\gamma \frac{d}{dv} H^\gamma(n, v)\, dv = I(n, t),$$

then it is sufficient to prove that

$$\sum_{n=1}^\infty |I(n, t)|/n = O(1),$$

for $0 \leqq t \leqq \pi$.

We begin to prove this for $t = \pi$. Putting $\varphi(t) = t^\gamma$, we get

$$I(n, \pi) = O\{1/(\log n)^{1+\varepsilon}\},$$

and then

$$\sum_{n=1}^\infty |I(n, \pi)|/n < \infty.$$

Next we write

$$\sum_{n=1}^\infty |I(n, t)|/n = \sum_{n<t^{-1}} |I(n, t)|/n + \sum_{n \geq t^{-1}} |I(n, \pi) - K(n, t)|/n.$$

By the second mean value theorem and the first part of Lemma 10 we have

$$I(n, t) = \int_0^t v^\gamma \frac{d}{dv} H^\gamma(n, v)\, dv = O\left\{t^\gamma n^\gamma/(\log n)^{1+\varepsilon}\right\},$$

therefore

$$\sum_{n<t^{-1}} |I(n, t|/n = O(1).$$

By integration by parts we have

$$K(n,t) = \int_t^\pi v^\gamma \frac{d}{dv} H^\gamma(n,v)\,dv$$

$$= O\left\{1/(\log n)^{1+\varepsilon} + t^{\gamma-1}n^{\gamma-1}/(\log n)^{1+\varepsilon}\right\} - \gamma \int_t^\pi v^{\gamma-1} H^\gamma(n,v)\,dv$$

where we used for $H^\gamma(n,v)$ the second value of Lemma 10. The second term of the right is

$$\int_t^\pi v^{\gamma-1} H^\gamma(n,v)dv = \frac{1}{A_n^\gamma} \sum_{\nu=0}^n A_{n-\nu}^{\gamma-1} \frac{\nu^\gamma}{\{\log(\nu+2)\}^{1+\varepsilon}} \int_t^\pi v^{\gamma-1} \sin vdv$$

$$= O\left\{t^{\gamma-1}n^{\gamma-1}/(\log n)^{1+\varepsilon}\right\}.$$

Hence when $n \geqq t^{-1}$, remembering $\gamma \leqq 1$, we have $K(n,t) = O\{1/(\log n)^{1+\varepsilon}\}$. The proof of

$$\sum_{n \geq t^{-1}} |K(n,t)|/n = O(1)$$

is now in hand.

Thus Theorem 2 is proved completely.

4. We shall end this paper by showing theorems, which will clarify the relation between Theorem 1 and Theorem 2.

THEOREM 4. *Under the condition*

$$\int_0^\pi |d(u^{-\gamma}\Phi_\beta(u))| < \infty,$$

we cannot conclude the $|C, \alpha|$ *summability of* $\sum n^{\gamma-\beta}A_n(x)$, *however large* α *may be, where* $\alpha > \gamma > \beta$.

PROOF. For the proof we shall give a negative example. Let $\varphi(t) = t^{\gamma-\beta}$ $(\gamma > \beta)$, then

$$\int_0^\pi |d\{u^{-\gamma}\Phi_\beta(u)\}| = 0,$$

but as we have already shown

$$\zeta_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} \nu \cdot \nu^{\gamma-\beta} \cdot \frac{2}{\pi} \int_0^\pi \varphi(t) \cos \nu t\,dt$$

$$\cong \frac{C}{A_n^\alpha} \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} = C$$

where C is a constant. Hence

$$\sum_{n=1}^{\infty} |\zeta_n^{\alpha}|/n = \infty.$$

THEOREM 5. *Let* $\eta > 0$. *In order that*

(i) $t^{-\gamma}\Phi_\beta(t)$ *is of bounded variation in* $(0, \eta)$

and

(ii) $t^{-\gamma-1}|\Phi_\beta(t)|$ *is integrable in* $(0, \eta)$,

it is necessary and sufficient that

(iii) $$\int_0^\eta t^{-\gamma+\beta}|d\varphi_\beta(t)| < \infty,$$

and

(iv) $\varphi_\beta(+0) = 0.$

PROOF. The condition is sufficient. For, if (iii) and (iv) hold, then we have

$$\int_0^\eta \frac{|\Phi_\beta(t)|}{t^{1+\gamma}}\,dt = \int_0^\eta \frac{|\varphi_\beta(t)|}{\Gamma(\beta+1)t^{1+\gamma-\beta}}\,dt = \int_0^\eta \frac{dt}{\Gamma(\beta+1)t^{1+\gamma-\beta}}\left|\int_0^t d\varphi_\beta(u)\right|$$

$$\leqq \int_0^\eta \frac{dt}{\Gamma(\beta+1)t^{1+\gamma-\beta}}\int_0^t |d\varphi_\beta(t)|$$

$$= \int_0^\eta \frac{|d\varphi_\beta(u)|}{\Gamma(\beta+1)}\int_u^\eta \frac{dt}{t^{1+\gamma-\beta}}$$

$$= \int_0^\eta \frac{|d\varphi_\beta(u)|}{\Gamma(\beta+1)}\left(\frac{u^{-\gamma+\beta}-\eta^{-\gamma+\beta}}{\gamma-\beta}\right) < \infty$$

from which (ii) follows; and then (i) follows from

$$\int_0^\eta |d\{t^{-\gamma}\Phi_\beta(t)\}| = \int_0^\eta \left|d\left\{\frac{t^{-\gamma+\beta}}{\Gamma(1+\beta)}\varphi_\beta(t)\right\}\right|$$

$$\leqq \int_0^\eta \frac{t^{-\gamma+\beta}}{\Gamma(\beta+1)}|d\varphi_\beta(t)| + \int_0^\eta \frac{\gamma-\beta}{\Gamma(1+\beta)}|\varphi_\beta(t)|t^{-\gamma+\beta-1}dt$$

$$= \int_0^\eta \frac{t^{-\gamma+\beta}}{\Gamma(1+\beta)}|d\varphi_\beta(t)| + \int_0^\eta (\gamma-\beta)\frac{|\Phi_\beta(t)|}{t^{\gamma+1}}\,dt < \infty.$$

The condition is necessary. For, suppose that (i) and (ii) hold, we now obtain (iii) from

$$\int_0^\eta t^{-\gamma+\beta}\,|d\varphi_\beta(t)|\leqq\int_0^\eta |d\{t^{-\gamma+\beta}\varphi_\beta(t)\}|+(\gamma-\beta)\int_0^\eta |\varphi_\beta(t)|\,t^{-\gamma+\beta-1}dt$$

$$=\int_0^\eta |d\{\Gamma(\beta+1)t^{-\gamma}\Phi_\beta(t)\}|+(\gamma-\beta)\int_0^\eta \Gamma(\alpha+1)t^{-\gamma-1}|\Phi_\beta(t)|\,dt<\infty.$$

(iv) is obvious from (i).

Thus Theorem 5 is proved.

From the proof of Theorem 5, we can also conclude that the condition

$$\int_0^\eta t^{-\gamma+\beta}\,|d\varphi_\beta(t)|<\infty,\qquad \varphi_\beta(+0)=0$$

is equivalent to the condition of Theorem 3, that is,

$$\int_0^\eta t^{-\gamma}\,|d\Phi_\beta(t)|<\infty,\qquad \Phi_\beta(+0)=0.$$

References

[1] N. Obrechkoff, Sur la sommation des séries trigonométriques de Fourier par les moyennes arithmétiques, Bull. de la Soc. Math. de France, 62(1934), 84-109 et 167-184.

[2] L.S. Bosanquet, The absolute Cesàro summability of Fourier series, Proc. London Math. Soc., 41(1936), 517-528.

[3] R. Mohanty, The absolute Cesàro summability of some series associated with a Fourier series and its allied series, Journ. London Math. Soc., 25(1950), 63-67.

[4] M.T. Cheng, Summability factors of Fourier series at a given point, Duke Math. Journ., 15(1948), 29-36.

[5] E.C. Titchmarsh, The theory of functions, Oxford, 1932.

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