ON THE REPRESENTATION THEOREM BY THE LAPLACE TRANSFORMATION OF VECTOR-VALUED FUNCTIONS

ISAO MIYADERA

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1. Introduction. The theory of the Laplace integral of vector-valued functions has been developed by E. Hille [2]. But, there is no theorem giving conditions that a vector-valued function is represented as the Laplace transformation of a function in $B_p([0, \infty); X)$.

P. G. Rooney [4] has recently developed this representation theory in terms of an inversion operator given by the formula

$$L_{k,t}[f] = (ke^{2k}/\pi t)\int_0^\infty s^{-\frac{1}{2}} \cos(2ks^{\frac{1}{2}}) f(k(s+1)/t)\, ds, \tag{1. 1}$$

where the integral is Bochner integral.

Since Rooney's method is quite general, his argument can be used for Widder's operator given by the formula

$$L_{k,t}[f] = \frac{(-1)^k}{k!} f^{(k)}(k/t)\,(k/t)^{k+1}, \tag{1. 2}$$

where $f^{(k)}(t)$ denotes the k-th strong derivative of $f(t)$.

In his argument, the basic space X is a *reflexive* Banach space for $1 < p < \infty$ and is a *uniformly convex* Banach space for $p = \infty$.

The main purpose of the present paper is to give a necessary and sufficient condition in order that a function $f(s)$ is the Laplace transformation of a function in $B_\infty([0, \infty); X)$ where X is a *reflexive* Banach space. The representation theorems are stated in terms of the Widder operator (1. 2) and we can obtain the theorems similarly as in the numerically-valued case.

2. Preliminary theorems. Let X be a Banach space.

DEFINITION. A vector-valued function $f(s)$ on $(0, \infty)$ into X is said to belong to $B_p([0, \infty); X)$ $(1 \leqq p < \infty)$ if $f(s)$ is Bochner measurable and

$$\int_0^\infty \|f(s)\|^p\, ds < \infty.$$

Similarly $f(s)$ is said to belong to $B_\infty([0, \infty); X)$ if it is Bochner measurable in $(0, \infty)$ and $\|f(s)\|$ is bounded except in a null set.

It is obvious that the class $B_p([0, \infty); X)$ becomes a Banach space under the norm

$$\|f(\cdot)\|_p = \left\{\int_0^\infty \|f(s)\|^p\, ds\right\}^{1/p} (1 \leq p < \infty), \; \|f(\cdot)\|_\infty = \text{ess sup} \|f(s)\|.$$

The following inversion formula has been proved by E. Hille [2, Theorem 10.3.4]:

THEOREM 1. *If $\varphi(t)$ is in $B_1([0, \omega]; X)$ for each $\omega > 0$, and if the integral*

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\, dt$$

converges for some s, then

$$\lim_{k\to\infty} L_{k,t}[f] = \varphi(t)$$

in the Lebesgue set of $\varphi(t)$.

The following theorem which is fundamental in the representation theory, is proved similarly as in the numerically-valued case, so that we omit the details.

THEOREM 2. *If for each positive integer k*

$$\left\| \int_0^s L_{k,t}[f]\, dt \right\| = O(s) \qquad (s \to \infty),$$

then $f(\infty)$ exists and

$$\lim_{k\to\infty} \int_0^\infty e^{-st} L_{k,t}[f] dt = f(s) - f(\infty) \qquad (0 < s < \infty).$$

3. Representation of vector-valued functions by Laplace transformations of functions in $B_p([0, \infty); X)$, $1 < p < \infty$.

THEOREM 3. *If X is a reflexive Banach space, then a necessary and sufficient condition that $f(s)$ can be expressed in the form*

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\, dt \qquad (s > 0),$$

where $\varphi(t) \in B_p([0, \infty); X)$, p fixed, $1 < p < \infty$, is that

(i) *$f(s)$ has strong derivatives of all orders in $0 < s < \infty$ and $f(\infty) = 0$,*
(ii) *there exists a constant M such that*

$$\left(\int_0^\infty \|L_{k,t}[f]\|^p\, dt\right)^{1/p} \leq M \qquad (k = 1, 2, \dots).$$

PROOF OF NECESSITY. Suppose

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\, dt \qquad (s > 0),$$

and $\varphi(t) \in B_p([0, \infty); X)$. Then using Hölder's inequality we have

$$\| f(s) \| \leqq \int_0^\infty e^{-st} \| \varphi(t) \| dt \leqq \Big(\int_0^\infty \| \varphi(t) \|^p dt\Big)^{1/p} (sq)^{-1/q},$$

where $1/p + 1/q = 1$, so that (i) is necessary. Another application of Hölder's inequality gives

$$\| L_{k,t}[f] \|^p = \Big\| \int_0^\infty e^{-ku/t} \frac{u^k}{k!} \Big(\frac{k}{t}\Big)^{k+1} \varphi(u)\, du \Big\|^p$$

$$\leqq \int_0^\infty e^{-ku/t} \frac{u^k}{k!} \Big(\frac{k}{t}\Big)^{k+1} \| \varphi(u) \|^p du \Big(\int_0^\infty e^{-ku/t} \frac{u^k}{k!} \Big(\frac{k}{t}\Big)^{k+1} du\Big)^{p/q}$$

$$\leqq \int_0^\infty e^{-ku/t} \frac{u^k}{k!} \Big(\frac{k}{t}\Big)^{k+1} \| \varphi(u) \|^p du,$$

so that

$$\int_0^\infty \| L_{k,t}[f] \|^p dt \leqq \int_0^\infty \| \varphi(u) \|^p u^k\, du \Big(\int_0^\infty e^{-ku/t} \frac{1}{k!} \Big(\frac{k}{t}\Big)^{k+1} dt\Big)$$

$$= \int_0^\infty \| \varphi(u) \|^p du.$$

Hence (ii) is necessary.

Proof of Sufficiency. By (ii) and Hölder's inequality

$$\int_0^s \| L_{k,t}[f] \| dt \leqq M s^{(p-1)/p} \leqq M s$$

for every $s > 1$ and for each positive integer k. Thus we obtain by Theorem 2 and (i)

$$\text{(3. 1)} \qquad \lim_{k\to\infty} \int_0^\infty e^{-st} L_{k,t}[f] dt = f(s) \qquad (0 < s < \infty).$$

Since X is a reflexive Banach space, $B_p([0, \infty); X)$, $1 < p < \infty$, is reflexive (see S. Bochner and A. E. Taylor [1] and B. J. Pettis [3]). Therefore $B_p([0, \infty); X)$ is locally weakly compact. Since

$$\Big(\int_0^\infty \| L_{k,t}[f] \|^p dt\Big)^{1/p} \leqq M,$$

there exists an element $\varphi(t)$ of $B_p([0, \infty); X)$ and an increasing sequence $\{k_i\}$ of positive integers such that for every y^* in $B_p^*([0, \infty); X)$

$$\lim_{k_i\to\infty} y^*(L_{k_i, \cdot}[f]) = y^*(\varphi(\cdot)).$$

Let x^* be an arbitrary element of X^*. Then if $g(t)$ is an arbitrary element of $B_p([0, \infty); X)$,

$$x^*\Big(\int_0^\infty e^{-st}g(t)\ dt\Big) = y_s^*(g(\cdot))$$

defines an element in $B_p^*([0, \infty); X)$ for each $s > 0$. Thus we have

$$y_s^*(\varphi(\cdot)) = x^*\Big(\int_0^\infty e^{-st}\varphi(t)dt\Big) = \lim_{k_i\to\infty} y_s^*(L_{k_i}, \cdot [f])$$

$$= \lim_{k_i\to\infty} x^*\Big(\int_0^\infty e^{-st} L_{k_i, t}[f]\,dt\Big),$$

so that by (3. 1)

$$x^*(f(s)) = x^*\Big(\int_0^\infty e^{-st}\varphi(t)\ dt\Big).$$

Hence

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\ dt \qquad (0 < s < \infty),$$

and the theorem is proved.

4. Representation of vector-valued functions by Laplace transformations of functions in B_∞ ([0; ∞); X).

The following Lemma is due to P. G. Rooney [4]:

LEMMA. *If $\{T_\sigma;\ 0 < \sigma < \infty\}$ is a set of bounded linear operators on a separable Banach space X into a reflexive Banach space Y, and if $\|T_\sigma\| \leqq M$ independently of σ for all $\sigma > 0$, then there exists an increasing unbounded sequence $\{\sigma_i\}$ and a linear operator T on X into Y with $\|T\| \leqq M$, such that*

$$\lim_{\sigma_i\to\infty} y^*(T_{\sigma_i}(x)) = y^*(T(x))$$

for every x in X and every y^* in Y^*.

THEOREM 4. *If X is a reflexive Banach space, then a necessary and sufficient condition that $f(s)$ can be expressed in the form*

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\ dt \qquad (s > 0),$$

where $\varphi(t) \in B_\infty([0, \infty); X)$, is that

(i′) *$f(s)$ has strong derivatives of all orders in $0 < s < \infty$,*

(ii′) *there exists a constant M such that for $0 < s < \infty$*

$$\frac{s^{k+1}}{k!}\|f^{(k)}(s)\| \leqq M \qquad (k = 0, 1, 2, \ldots).$$

PROOF OF NECESSITY. Suppose

$$f(s)=\int_0^\infty e^{-st}\varphi(t)dt \qquad (s>0),$$

and $\varphi(t)$ is in $B_\infty([0, \infty); X)$. Then (i′) is obvious. Since

$$\frac{s^{k+1}}{k!}\| f^{(k)}(s)\| \leqq \frac{s^{k+1}}{k!}\int_0^\infty t^k \|\varphi(t)\| e^{-st}dt \leqq \text{ess sup } \|\varphi(t)\|,$$

(ii′) is necessary.

Proof of Sufficiency. By (i′) and (ii′)

$$t\|f(t)\|\leqq M,$$

and for $0<t<\infty$

$$\|L_{k,t}[f]\|\leqq M \qquad (k=1,2,\ldots.),$$

so that $f(\infty)=0$ and

$$\|\int_0^s L_{k,t}[f]\, dt\| = O(s) \qquad (s\to\infty).$$

Thus we have by Theorem 2

$$\lim_{k\to\infty}\int_0^\infty e^{-st}L_{k,t}[f]\ dt = f(s) \qquad (0<s<\infty). \tag{4. 1}$$

Let $\psi(t)$ be in $L_1(0,\infty)$. Define

$$T_k(\psi)=\int_0^\infty \psi(t)L_{k,t}[f]dt.$$

It is obvious that $\{T_k\}$ is a set of bounded linear operators on a separable Banach space $L_1(0,\infty)$ into a reflexive Banach space X and $\|T_k\|\leqq M$. Thus, by the preceding lemma, there exists an increasing sequence $\{k_i\}$ of positive integers and a bounded linear operator T on $L_1(0,\infty)$ into X with $\|T\|\leqq M$, such that for every x^* in X^* and every ψ in $L_1(0,\infty)$,

$$\lim_{k_i\to\infty} x^*(T_{k_i}(\psi)) = x(T(\psi)). \tag{4. 2}$$

Let ω be an arbitrary positive integer. Since $B_2([0, \omega]; X)$ is reflexive and

$$\Big(\int_0^\omega \|L_{k_i,t}[f]\|^2\, dt\Big)^{1/2}\leqq M\omega^{1/2},$$

there exists an element $\varphi^{(\omega)}(t)$ of $B_2([0,\omega]; X)$ and a sequence $\{k_i'\}(\subset\{k_i\})$ such that for every $y^*_{(\omega)}$ in $B^*([0,\omega]; X)$

$$\lim_{k_i'\to\infty} y^*_{(\omega)}(L_{k_i',\cdot}[f]) = y^*_{(\omega)}(\varphi^{(\omega)}(\cdot)).$$

Let x^* be an arbitrary element of X^*. Then if $\psi(t)$ is an arbitrary

Lebesgue measurable function such that $\int_0^\omega |\psi(t)|^2 dt < \infty$ and $\psi(t) = 0$ for $t > \omega$, and if $g(t)$ is an arbitrary element of $B_2([0, \omega]; X)$, then

$$x^*\Big(\int_0^\omega \psi(t) g(t) dt\Big)$$

defines an element in $B_2^*([0, \omega]; X)$. Therefore we have

$$\lim_{k_i' \to \infty} x^*\Big(\int_0^\infty \psi(t) L_{k_i', t}[f] dt\Big) = \lim_{k_i' \to \infty} x^*\Big(\int_0^\omega \psi(t) L_{k_i', t}[f] dt\Big)$$

$$= x^*\Big(\int_0^\omega \psi(t) \varphi^{(\omega)}(t) dt\Big).$$

On the other hand, since such $\psi(t)$ belongs to $L_1(0, \infty)$,

$$\lim_{k_i \to \infty} x^*(T_{k_i}(\psi)) = \lim_{k_i \to \infty} x^*\Big(\int_0^\infty \psi(t) L_{k_i, t}[f] dt\Big) = x^*(T(\psi)).$$

Thus

$$x^*\Big(\int_0^\omega \psi(t) \varphi^{(\omega)}(t)\, dt\Big) = x^*(T(\psi)),$$

so that

$$T(\psi) = \int_0^\omega \psi(t) \varphi^{(\omega)}(t)\, dt \tag{4. 3}$$

for every $\psi(t)$ such that $\psi(t) \in L_2(0, \omega)$ and $\psi(t) \equiv 0$ for $t > \omega$. It is easy that if $\omega' > \omega$, then $\varphi^{(\omega)}(t) = \varphi^{(\omega')}(t)$ for almost all t in $(0, \omega)$.

We now define a function $\varphi(t)$ on $(0, \infty)$ into X by

$$\varphi(t) = \varphi^{(\omega)}(t) \qquad \text{for } \omega - 1 \leqq t < \varphi \qquad (\omega = 1, 2, 3, \ldots.).$$

From the definition of $\varphi(t)$, it is obvious that $\varphi(t) = \varphi^{(\omega)}(t)$ for almost all t in $(0, \omega)$, $\varphi(t)$ is Bochner measurable and $\| \varphi(t) \|^2$ is integrable in any finite interval.

Hence (4. 3) may be written as follows:

$$T(\psi) = \int_0^\infty \psi(t) \varphi(t)\, dt \tag{4. 4}$$

for every $\psi(t)$ such that $\psi(t) \in L_2(0, \omega)$ and $\psi(t) = 0$ for $t > \omega$.

Let us put

$$\psi_{\xi, h}(t) = \begin{cases} 1/h & \text{for } \xi \leqq t < \xi + h, \\ 0 & \text{otherwise,} \end{cases}$$

where ξ and h are any positive number. By (4. 4), we have

$$T(\psi_{\xi,h}) = \frac{1}{h}\int_{\xi}^{\xi+h} \varphi(t)\, dt.$$

Since $\| T(\psi_{\xi,h}) \| \leqq M \int_0^{\infty} | \psi_{\xi,h}(t) |\, dt = M$, we get

$$\| \frac{1}{h}\int_{\xi}^{\xi+h} \varphi(t)\, dt \| \leqq M.$$

Thus

$$\| \varphi(\xi) \| = \lim_{h\to 0} \| \frac{1}{h}\int_{\xi}^{\xi+h} \varphi(t) dt \| \leqq M$$

for almost all $\xi > 0$, so that $\varphi(t)$ is an element in $B_\infty([0, \infty) ; X)$.

If we now define the new operator T' on $L_1(0, \infty)$ into X by

$$T'(\psi) = \int_0^{\infty} \psi(t)\varphi(t)dt,$$

where $\psi(t) \in L_1(0, \infty)$, then T' is a bounded linear operator $L_1(0, \infty)$ into X. The set $D \equiv \bigcup_{\omega=1}^{\infty} \{\psi(t) ; \psi(t) \in L_2(0, \omega) \text{ and } \psi(t) = 0 \text{ for } t > \omega\}$ is dense in $L_1(0, \infty)$ and $T(\psi) = \int_0^{\infty} \psi(t)\, \varphi(t)\, dt = T'(\psi)$ for any $\psi \in D$, so that $T = T'$.

Thus we get

$$T(\psi) = \int_0^{\infty} \psi(t)\varphi(t)dt \tag{4. 5}$$

for each $\psi(t) \in L_1(0, \infty)$.

Let $\psi(t) = e^{-st}$. Then, by (4. 1) and (4. 5), for each $x^* \in X^*$ and for each $s > 0$

$$\begin{aligned} x^*(f(s)) &= \lim_{k_i \to \infty} x^*\Big(\int_0^{\infty} e^{-st} L_{k_i, t}[f] dt\Big) \\ &= \lim_{k_i \to \infty} x^*(T_{k_i}(e^{-st})) = x^*(T(e^{-st})) \\ &= x^*\Big(\int_0^{\infty} e^{-st}\varphi(t) dt),\Big) \end{aligned}$$

so that

$$f(s) = \int_0^{\infty} e^{-st}\varphi(t)\, dt \qquad (s > 0).$$

Thus the theorem is proved.

Since the above method is quite general, Rooney's result [4; Theorem 9. 2] is also true for a reflexive Banach space.

Theorem 4 shows that Hille's condition [2; Theorem 10.3.5, (10.3.15)] is also sufficient to represent the function $f(\lambda)$ as a Laplace transform when X is reflexive (= locally weakly compact).

We shall show by an example that the reflexivility is essential for the sufficiency of Theorems 3 and 4.

Exsample. Let $C[0, \infty]$ be the family of all real-valued continuous functions $x(u)$ of u on the closed interval $[0, \infty]$. The norm of $x(u)$ is defined as the maximum of its absolute value in $[0, \infty]$. $C[0, \infty]$ is obviously a Banach space. Let X be the family of all bounded linear transformations on $C[0, \infty]$ into itself. It is well known that X is a Banach algebra.

We define

$$T(t)[x(u)] = e^{-t}x(u+t), \qquad (t \geqq 0). \tag{4. 6}$$

Then $\{T(t);\, t \geqq 0\}$ is a semi-group of operators satisfying the following conditions:

$$(C_1) \qquad T(t) \in X \text{ and } T(t+s) = T(t)T(s) \qquad (t,\, s \geqq 0),$$

$$(C_2) \qquad \| T(t) \| \leqq e^{-t} \qquad (t \geqq 0),$$

$$(C_3) \qquad \lim_{h \to 0} \| T(t+h)x - T(t)x \| = 0 \qquad (t \geqq 0,\ x \in C[0, \infty]).$$

Furthermore we have

$$(C_4) \qquad \| T(t+h) - T(t) \| \geqq e^{-t} \qquad (t+h > 0,\ t > 0 \text{ and } h \neq 0).$$

In fact, we can always find the element of $C[0, \infty]$ such that $x(t) = 1$, $x(t+h) = -1$ and $\max |x(t)| = 1$ for any given t and h, where $t > 0, t+h > 0$ and $h \neq 0$. For such an element x we have $\| T(t+h)x - T(t)x \| \geqq e^{-t}$, so that (C_4) holds.

We denote the infinitesimal generator of $T(t)$ by A and the resolvent of A by $R(s; A)$. From the theory of semi-group of operators,

$$R(s; A)x = \int_0^\infty e^{-st}T(t)x\, dt \tag{4. 7}$$

for all $s > 0$ and for all $x \in C[0, \infty]$. $R(s; A)$ has derivatives (strong derivatives in the sense of X-norm) of all orders by the resolvent equation $R(s; A) - R(t; A) = -(s-t)R(s; A)R(t; A)$. By (4. 7)

$$R^{(k)}(s; A)x = \int_0^\infty (-t)^k e^{-st}T(t)x dt$$

and

$$L_{k,t}[R(\cdot\,; A)x] = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty u^k e^{-ku/t}T(u)x du,$$

so that

$$\frac{s^{k+1}}{k!} \| R^{(k)}(s; A) \| \leqq \frac{s^{k+1}}{k!} \int_0^\infty t^k e^{-st} dt = 1 \qquad (k = 0, 1, 2, \dots), \tag{4. 8}$$

and

$$\int_0^\infty \| L_{k,t}[R(\cdot\,;A)] \|^p \,dt \leqq \int_0^\infty \| T(u) \|^p \,du = 1/p \qquad (1 < p < \infty;\ k = 1, 2, \dots). \tag{4. 9}$$

Thus, the vector-valued function $R(s;A)$ on $(0, \infty)$ into X satisfies the sufficient conditions of Theorems 3 and 4.

If there exists an element $\varphi(t)$ of $B_p([0, \infty)\,; X)$, $1 < p \leqq \infty$, such that

$$R(s;A) = \int_0^\infty e^{-st}\varphi(t)\,dt \qquad (s > 0),$$

then, by Theorem 1,

$$\lim_{k\to\infty} \| L_{k,t}[R(\cdot\,;A)] - \varphi(t) \| = 0$$

for almost all $t > 0$.

On the other hand, we have from (4. 7), (C_3) and Theorem 1

$$\lim_{k\to\infty} \| L_{k,t}[R(\cdot\,;A)x] - T(t)x \| = 0$$

for all $t > 0$ and for all $x \in C[0, \infty]$.

Thus

$$T(t) = \varphi(t)$$

for almost all $t > 0$, so that $T(t)$ is a Bochner measurable function on the interval $(0, \infty)$ into X such that for $0 < t,\ s < \infty$

$$T(t+s) = T(t)T(s).$$

Then, by Hille's theorem [2, Theorem 8. 3. 1],

$$\lim_{h\to 0} \| T(t+h) - T(t) \| = 0$$

for all $t > 0$. This is contrary to the condition (C_4). Thus X is not reflexive and $R(s;A)$ can not be represented as Laplace transformations of function in $B_p([0, \infty)\,;X)$ for each p, $1 < p \leqq \infty$.

5. Representation of vector-valued functions by Laplace transformations of functions in $B_1([0, \infty)\,; X)$.

THEOREM 5. *Let X be a Banach space. A necessary and sufficient condition that $f(s)$ can be expressed in the form*

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\,dt \qquad (s > 0),$$

where $\varphi(t) \in B_1([0, \infty)\,; X)$, is that

(i″) *$f(s)$ has strong derivatives of all orders in $0 < s < \infty$ and $f(\infty) = 0$,*

$$\int_0^\infty \| L_{k,t}[f] \| \,dt < \infty \qquad (k = 1, 2, \dots), \tag{ii″}$$

(iii″) $$\lim_{j,k\to\infty}\int_0^\infty \| L_{k,t}[f] - L_{j,t}[f] \| \, dt = 0.$$

PROOF OF SUFFICIENCY. By (iii″), there exists an element $\varphi(t)$ of $B_1([0, \infty); X)$ such that

$$\lim_{k\to\infty}\int_0^\infty \| L_{k,t}[f] - \varphi(t) \| \, dt = 0.$$

Then there exists a positive integer k_0 such that

$$\int_0^\infty \| L_{k,t}[f] \| \, dt \leqq 1 + \int_0^\infty \| \varphi(t) \| \, dt$$

for $k \geqq k_0$.

Let x^* be an arbitrary element in X^*. By Widder's theorem [5; Chap. VII, Theorem 12a], there exists a function of bounded variation $\alpha_{x^*}(t)$ such that

$$x^*[f(s)] = \int_0^\infty e^{-st} \, d\alpha_{x^*}(t).$$

On the other hand, we obtain from the inversion formula

$$\alpha_{x^*}(t) - \alpha_{x^*}(0+) = \lim_{k\to\infty}\int_0^t x^*(L_{k,u}[f]) \, du$$

$$= \int_0^t x^*(\varphi(u)) \, du.$$

$f(\infty) = 0$ implies $\alpha_{x^*}(0+) = 0$, so that

$$\alpha_{x^*}(t) = \int_0^t x^*(\varphi(u)) \, du.$$

Thus

$$x^*(f(s)) = \int_0^\infty e^{-st} x^*(\varphi(t)) \, dt = x^*\Big(\int_0^\infty e^{-st}\varphi(t) \, dt\Big),$$

so that

$$f(s) = \int_0^\infty e^{-st}\varphi(t) \, dt \qquad (s > 0).$$

The necessity of the theorem may be proved similarly as in numerically-valued case.

REFERENCES

[1] S. BOCHNER AND A. E. TAYLOR, Linear functionals on certain spaces of abstractly-valued functions, Ann. of Math., 39(1938), 913-944.
[2] E. HILLE, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publications, New York(1948).
[3] B. J. PETTIS, Differentiation in Banach space, Duke Math. Journ., 5(1939), 254-269.
[4] P. G. ROONEY, An inversion and representation theory for the Laplace integral of abstractly-valued functions, Canadian Journ. of Math., 5(1954), 190-209.
[5] D. V. WIDDER, The Laplace transform, Princeton(1941).

MATHEMATICAL INSTITUTE, TOKYO METROPOLITAN UNIVERSITY.

ADDED IN PROOF. The formula (4.5) can also be obtained from the following theorem.

If T is a bounded linear operator on $L_1(0, \infty)$ into a reflexive Banach space X, then there exists an element $\varphi(t) \in B_\infty([0, \infty); X)$ such that

$$T(\psi)=\int_0^\infty \psi(t)\varphi(t)dt, \qquad \psi(t) \in L_1(0, \infty).$$