# ON A SYSTEM OF MODULAR FUNCTIONS CONNECTED WITH THE RAMANUJAN IDENTITIES

N. J. FINE[1)]

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**1. Introduction.** In the study of the congruence properties of the partition function $p(n)$, it is natural to use the device of dissecting a power series according to the residue class of the exponent. For example, to study the properties of $p(5n+r)$, $r=0,1,2,3,4$, one considers the generating function

$$\prod_{n\geq 1}(1-x^n)^{-1}=\sum_{n\geq 0}p(n)x^n$$

and attempts to write it in the form

$$A_0(x^5)+xA_1(x^5)+x^2A_2(x^5)+x^3A_3(x^5)+x^4A_4(x^5),$$

where the $A_k(y)$ are power series in $y$. It is more convenient to work with the reciprocal of the generating function, for which we have the identity of Euler,

$$\prod_{n\geq 1}(1-x^n)=\sum_{-\infty}^{+\infty}(-1)^n x^{n(3n+1)/2}.$$

The elements in the dissection are not defficult to compute, and this has in fact been done for special cases by Ramanujan [7], Darling [3], and Watson [9], and in general by Atkin and Swinnerton-Dyer [1].

It is the purpose of this note to study the behavior of the elements $W_k(\tau)$ in the dissection of the related function $\eta(\tau/q)/\eta(q\tau)$, where

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{n\geq 1}(1-e^{2\pi in\tau})$$

and $q=6\lambda\pm 1$. From the results of [1],

$$\psi(\tau)\equiv(-1)^{\lambda}\frac{\eta(\tau/q)}{\eta(q\tau)}=1+\sum_{k=1}^{\nu}W_k(\tau),$$

where $\nu=(q-1)/2$,

$$W_k(\tau)=x^{\frac{6k^2}{q}-k}\frac{C_{4k}(x)}{C_{2k}(x)},$$

$$C_k(x)=\prod_{n\geq 1}(1-x^{qn-k})(1-x^{qn-q+k}),$$

and $x=\exp(2\pi i\tau), I(\tau)>0$. In § 2 I discuss the action of certain modular

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subgroups on $W_k(\tau)$. These subgroups are defined by

$$\Gamma_0(q)\ : c\equiv 0 \pmod q,$$
$$\Gamma_0^0(q)\ : b\equiv c\equiv 0 \pmod q,$$
$$\Gamma(q)\ : a\equiv d\equiv 1,\ b\equiv c\equiv 0 \pmod q,$$

where $a, b, c, d$, are the integral elements of the modular transformation

$$\tau'=\frac{a\tau+b}{c\tau+d}=M\tau$$

and $ad-bc=1$. The complete result is that

$$W_k(M\tau)=\exp\Big(\frac{12\pi i k^2 ab}{q}\Big)W_{ak}(\tau) \qquad (M\in\Gamma_0(q)).$$

It follows that each $W_k$ is invariant under $\Gamma(q)$, that the set $\{W_k\}$ $k=1,\ldots,\nu$, is permuted by $\Gamma_0^0(q)$, and that the vector-space spanned by this set is invariant under $\Gamma_0(q)$. An immediate corollary is the result that $\psi(\tau)$ is invariant under $\Gamma_0^0(q)$, or equivalently, that $\psi(q\tau)$ is invariant under $\Gamma_0(q^2)$. This last result, for $q$ a prime power, has been proved by Lehner [4], using the transformation equation for $\eta(\tau)$ and the theory of Dedekind sums.

In § 3 I apply the previous theory to the functions $\psi_r(\tau)=\psi(\tau+r)$, $r=0,1,\ldots,q-1$, and show that

$$\psi_r(M\tau)=\psi_{ra^2+ba}(\tau) \qquad (M\in\Gamma_0(q)).$$

This permits the deduction of the modular equation of Ramanujan-Watson type [9], an algebraic relation between $\mu(\tau)=\eta(\tau/q)/\eta(\tau)$ and $\mu(q\tau)$, which Watson used to obtain the congruence properties of $p(n)$ for the moduli $5^\alpha$ and $7^\beta$. The theory developed here makes transparent the cyclic nature of certain identities which appear in Watson's work. The modular equation for $q=11$ is given here (in parametric form), in the hope that it may be useful in settling the question for powers of 11. In addition, an identity of Ramanujan type is obtained which yields a proof of the congruence $p(11n+6)\equiv 0 \pmod{11}$. This identity is probably equivalent to one of Lehner's [4], who used a different basis, however.

In §4 I make further applications to prove two hitherto unpublished and unproved identities of Ramanujan. Identities of Slater and Newman are also derived. Finally in § 5, I prove an identity of Rademacher which he deduced by subjecting Ramanujan's

$$\sum_{n\ge 0}p(5n+4)x^n=5\prod_{m\ge 1}\frac{(1-x^{5m})^5}{(1-x^m)^6}$$

to the modular transformation $\tau'=-1/\tau$.

**2. Properties of $W_k(\tau)$.** Let $q$ be a positive integer of the form $6\lambda\pm 1$ $(\lambda>0)$. For $k\not\equiv 0 \pmod q$, we define

$$C_k(x;q)=C_k(x)=\prod_{n\ge 1}(1-x^{nq-k})(1-x^{nq-q+k}), \tag{2.1}$$

where $x=\exp(2\pi i\tau)$, $I(\tau)>0$. The following properties of the $C_k$ are easily established:

(2. 2) $$C_{k+q} = C_{-k} = -x^{-k}C_k; \qquad C_{q-k} = C_k.$$

Now we define

(2. 3) $$W_k(\tau; q) = W_k(\tau) = x^{6k^2/q-k}\frac{C_{4k}(x)}{C_{2k}(x)}.$$

From (2. 2) it follows readily that

(2. 4) $$W_{k+q}(\tau) = W_{-k}(\tau) = W_k(\tau),$$

so that there are exactly $\nu = (q-1)/2$ distinct functions $W_k(\tau)$. We may also write

(2. 5) $$W_k(\tau) = \exp\left(\frac{12\pi i k^2\tau}{q}\right)\cdot\frac{\vartheta_1(4k\pi\tau \mid q\tau)}{\vartheta_1(2k\pi\tau \mid q\tau)},$$

or by use of the transformation

(2. 6) $$\vartheta_1(z \mid -1/\tau) = -i\sqrt{-i\tau}\,\exp\left(\frac{iz^2\tau}{\pi}\right)\cdot\vartheta_1(z\tau \mid \tau),$$

(2. 7) $$W_k(\tau) = \frac{\vartheta_1\left(\frac{4k\pi}{q} \,\middle|\, -\frac{1}{q\tau}\right)}{\vartheta_1\left(\frac{2k\pi}{q} \,\middle|\, -\frac{1}{q\tau}\right)}.$$

We now proceed to study the behavior of

(2. 8) $$g(z, w \mid \tau) = \frac{\vartheta_1(z \mid \tau)}{\vartheta_1(w \mid \tau)}$$

under an arbitrary transformation of the full modular group, given by

(2. 9) $$\tau' = M\tau = \frac{a\tau + b}{c\tau + d},$$

where $a, b, c, d$ are integers and $ad - bc = 1$. We shall show that

(2.10) $$g(z, w \mid M\tau) = \exp\left\{\frac{i(z^2 - w^2)c(c\tau + d)}{\pi}\right\}\cdot g((c\tau + d)z, (c\tau + d)w \mid \tau).$$

Since $g$ is a function of $x = \exp(2\pi i\tau)$, (2.10) is true for

$$M = S \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and (2. 6) shows that it is also true for

$$M = T \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose now that (2.10) holds for $M$, and let $M' = SM$. Then

$$g(z, w \mid M'\tau) = g(z, w \mid M\tau),$$

which is given by (2.10). But

$$M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix},$$

so that $c' = c$, $d' = d$, and (2.10) holds for $M'$. Again, suppose that it holds for $M$, and let

$$M' = TM = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

Then, with $\tau' = M\tau$,

$$g(z, w \mid M'\tau) = g(z, w \mid T\tau') = \exp\left[\frac{i\tau'(z^2 - w^2)}{\pi}\right] \cdot g(z\tau', w\tau' \mid M\tau),$$

by (2. 6). Applying the induction hypothesis with $z$ replaced by $z\tau'$, $w$ by $w\tau'$, we find

$$\begin{aligned} g(z, w \mid M'\tau) &= \exp\left[\frac{i(z^2 - w^2)H}{\pi}\right] \cdot g((c\tau + d)\tau' z, (c\tau + d)\tau' w \mid \tau) \\ &= \exp\left[\frac{i(z^2 - w^2)H}{\pi}\right] \cdot g((c'\tau + d')z, (c'\tau + d')w \mid \tau), \end{aligned}$$

where

$$\begin{aligned} H &= \tau' + \tau' c \cdot \tau'(c\tau + d) = \tau' + c\tau'(c'\tau + d') \\ &= \frac{a\tau + b}{c\tau + d}[1 + c(c'\tau + d')] \\ &= -\frac{(c'\tau + d')}{(a'\tau + b')}\left\{1 - a'(c'\tau + d')\right\} \\ &= c'(c'\tau + d'), \end{aligned}$$

since $a'd' - b'c' = 1$. This shows that (2.10) is true for $M'$. Since the transformations $S$ and $T$ generate the modular group, (2.10) is completely proved.

We observe also that

$$g(z + m\pi, w + n\pi \mid \tau) = (-1)^{m+n} g(z, w \mid \tau). \tag{2.11}$$

Now, by (2. 7) and (2. 8),

$$W_k(\tau) = g\left(\frac{4k\pi}{q}, \frac{2k\pi}{q} \,\Big|\, -\frac{1}{q\tau}\right). \tag{2.12}$$

In order to apply (2.10), let

$$N = \begin{pmatrix} \alpha & \beta \\ \gamma q & \delta \end{pmatrix} \in \Gamma_0(q).$$

Then, writing $\tau' = q\tau$, we have

$$-\frac{1}{qN\tau} = M\tau', \tag{2.13}$$

where

$$M = \begin{pmatrix} -\gamma & -\delta \\ \alpha & \beta q \end{pmatrix}. \tag{2.14}$$

Hence, applying (2.10), we obtain

$$\begin{aligned} W_k(N\tau) &= g\left(\frac{4k\pi}{q}, \frac{2k\pi}{q} \,\Big|\, M\tau'\right) \\ &= \exp\left[\frac{12k^2\pi i}{q}\alpha(\alpha\tau + \beta)\right] \cdot g(4k\pi(\alpha\tau + \beta), 2k\pi(\alpha\tau + \beta) \mid \tau') \\ &= \exp\left[\frac{12k^2\pi i}{q}\alpha(\alpha\tau + \beta)\right] \cdot g(4k\alpha\pi\tau, 2k\alpha\pi\tau \mid \tau'), \end{aligned}$$

the last by virtue of (2.11). But $\tau' = T\tau_1$, where $\tau_1 = -1/q\tau$, so (again by (2.10)),

$$W_k(N\tau) = \exp\left(\frac{12\pi i k^2\alpha\beta}{q}\right)\cdot g\left(-\frac{4k\alpha\pi}{q},\ -\frac{2k\alpha\pi}{q}\ \Big|\ -\frac{1}{q\tau}\right)$$

$$=\exp\left(\frac{12\pi i k^2\alpha\beta}{q}\right)\cdot W_{-\alpha k}(\tau).$$

By (2. 4), we have the final result

$$W_k(N\tau) = \exp\left(\frac{12\pi i k^2\alpha\beta}{q}\right)\cdot W_{\alpha k}(\tau) \qquad (N \in \Gamma_0(q)). \tag{2.15}$$

*In particular,* $W_k$ *is invariant under* $N \in \Gamma(q)$, and the set $\{W_k\}$ is permuted by $\Gamma_0^0(q)$:

$$W_k(N\tau) = W_{\alpha k}(\tau) \qquad (N \in \Gamma_0^0(q)). \tag{2.16}$$

We shall say that $F(\tau) = \Phi(W_1(\tau), \cdots, W_\nu(\tau))$ is *cyclic* if $\Phi$ is invariant under $k \to \alpha k$ for every $\alpha$ prime to $q$. If $\Phi$ is a polynomial, we shall call it $(q, m)$*-isobaric* whenever each term has weight $\equiv m \pmod q$, provided that $W_k$ is assigned weight $k^2$. Then (2.15) shows that *every cyclic function is invariant under* $\Gamma_0^0(q)$, *and every cyclic,* $(q, 0)$*-isobaric polynomial is invariant under* $\Gamma_0(q)$.

Now let us examine the behavior of $W_k(\tau)$ at $\tau = 0$. For this purpose, set $\tau' = -1/q\tau$. Then

$$W_k(\tau) = \frac{\sum_{n\geq 0} e^{(n^2+n)\pi i\tau'}(-1)^n \sin((2n+1)4k\pi/q)}{\sum_{n\geq 0} e^{(n^2+n)\pi i\tau'}(-1)^n \sin((2n+1)2k\pi/q)}.$$

As $\tau \to 0$, $\tau' \to i\infty$, and

$$W_k(\tau) \to 2\cos(2k\pi/q) \qquad (\tau \to 0). \tag{2.17}$$

Also, $W_k(\tau)$ has no zeroes or singularities in the finite upper half-plane $I(\tau) > 0$. In the particular case where $q = p$, a prime, the fundamental region for $\Gamma_0(p)$ has exactly two parabolic vertices $\tau = 0$, $i\infty$. Therefore, *any cyclic,* $(p, 0)$*-isobaric polynomial which is bounded at* $i\infty$ *must be a constant.* This gives us an easy method for identifying two such polynomials, by comparing the principal parts of their expansions in terms of $x = \exp(2\pi i\tau)$ (including the constant term). The expansions are easily obtained from (2. 3).

As a useful example, consider

$$F(\tau) = W_1(\tau)\cdots W_\nu(\tau).$$

Using (2. 3), (2. 2), and the fact that $2k$ and $4k$ run over a half residue-system as $k$ does, we see that $F(\tau) = \pm x^m$. But $F(\tau)$ is cyclic and $(q, 0)$-isobaric, hence invariant under $\Gamma_0(q)$. Therefore $m = 0$ and

$$F(\tau) = \pm 1 = \lim_{t\to 0} F(\tau) = \prod_{k=1}^{\nu} 2\cos(2k\pi/q).$$

Thus we have

$$W_1(\tau)\cdots W_\nu(\tau) = (-1)^{(q^2-1)/8}. \tag{2.18}$$

**3. Modular equations of Ramanujan-Watson type.** In his work on congruence properties of partitions, to the moduli $5^\alpha$ and $7^\beta$, Watson [9] dissected the Euler pentagonal series according to the residues of the exponents mod $p$ ($p = 5, 7$). This has been done in general by Atkin and Swinnerton-Dyer [1, Lemma 6]. Their result, in our notation, is

$$\psi(\tau) = 1 + \sum_{k=1}^{\nu} W_k(\tau), \tag{3. 1}$$

where

$$\psi(\tau) = (-1)^\lambda \frac{\eta(\tau/q)}{\eta(q\tau)}, \tag{3. 2}$$

$q = 6\lambda \pm 1$, and $\eta(\tau)$ is the Dedekind function. It follows immediately from the results of § 2 that $\psi(\tau)$ is invariant under $\Gamma_0^0(q)$ (see Lehner [4, Th. 3]).

Let us define

$$\psi_r(\tau) = \psi(\tau + r) = \psi(S^r\tau) \qquad (r = \text{integer}). \tag{3. 3}$$

By (3. 1) and (2.15),

$$\psi_r(\tau) = 1 + \sum_{k=1}^{\nu} \varepsilon^{rk^2} W_k(\tau), \tag{3. 4}$$

where $\varepsilon = \exp(12\pi i/q)$. Obviously $\psi_{r+q}(\tau) = \psi_r(\tau)$, so there are $q$ distinct functions $\psi_r(\tau)$, $r = 0, 1, \ldots, q-1$. Now if

$$N = \begin{pmatrix} \alpha & \beta \\ \gamma q & \delta \end{pmatrix} \in \Gamma_0(q),$$

we have, again by (2.15),

$$\psi_r(N\tau) = 1 + \sum_{k=1}^{\nu} \varepsilon^{k^2(r+\alpha\beta)} W_{\alpha k}(\tau).$$

Let $t \equiv \pm \alpha k \pmod q$ be chosen in the range $1, 2, \ldots, \nu$. Then

$$\psi_r(N\tau) = 1 + \sum_{t=1}^{\nu} \varepsilon^{t^2(r\delta^2+\beta\delta)} W_t(\tau),$$

since $\alpha\delta \equiv 1 \pmod q$. Thus

$$\psi_r(N\tau) = \psi_{r\delta^2+\beta\delta}(\tau) \qquad \left(N = \begin{pmatrix} \alpha & \beta \\ \gamma q & \delta \end{pmatrix} \in \Gamma_0(q)\right). \tag{3. 5}$$

Hence any function of the $\{\psi_r\}$ which is invariant under $r \to \delta^2 r$ (for all $\delta$ prime to $q$) will belong to $\Gamma_0^0(q)$; if in addition it is invariant under $r \to r+1$, it will belong to $\Gamma_0(q)$. In particular, *every symmetric function of the* $\{\psi_r\}$ *belongs to* $\Gamma_0(q)$.

Now consider

$$\Phi(z\,|\,\tau) = (z - \psi_0(\tau))(z - \psi_1(\tau)) \cdots (z - \psi_{q-1}(\tau)) \tag{3.6}$$
$$= z^q - \sigma_1(\tau) z^{q-1} + \sigma_2(\tau) z^{q-2} - \cdots - \sigma_q(\tau).$$

We have just proved that all the $\sigma_r(\tau)$ belong to $\Gamma_0(q)$. An easy computation with the infinite products shows that

$$\sigma_q(\tau) = (-1)^\lambda \left(\frac{\eta(\tau)}{\eta(q\tau)}\right)^{q+1}. \tag{3.7}$$

Since $\Phi(\psi(\tau)\,|\,\tau) = 0$, we have

$$(-1)^\lambda \left(\frac{\eta(\tau)}{\eta(q\tau)}\right)^{q+1} = \psi^q(\tau) - \sigma_1(\tau)\psi^{q-1}(\tau) + \cdots + \sigma_{q-1}(\tau)\psi(\tau). \tag{3.8}$$

If we eliminate the $\sigma_r(\tau)$ $(r = 1, \ldots, q-1)$ by means of the algebraic equations connecting them with $\sigma_q(\tau)$ (as modular functions belonging to the same subgroup $\Gamma_0(q)$), we obtain an equation between $\psi(\tau)$ and $\sigma_q(\tau)$. This is essentially the modular equation connecting $\mu(\tau) = \eta(\tau/q)/\eta(\tau)$ and $\mu(q\tau)$, since

$$\psi(\tau) = (-1)^\lambda \mu(\tau)\mu(q\tau), \tag{3.9}$$

$$\sigma_q(\tau) = (-1)^\lambda \mu^{q+1}(q\tau). \tag{3.10}$$

Such equations were discussed by Watson [9], and were obtained explicitly by him for $q = 5,7$.

In the construction of the modular equation, it is somewhat easier to work with the equation $\Phi(z+1\,|\,\tau) = 0$, with roots $\psi_r(\tau) - 1$, $r = 0, 1, \ldots, q-1$. Thus

$$\Phi(z+1\,|\,\tau) = z^q - \sigma_1^*(\tau) z^{q-1} + \cdots - \sigma_q^*(\tau).$$

Instead of computing the elementary symmetric functions $\sigma_r^*(\tau)$ directly, we go over to the power-sums

$$S_m^*(\tau) = \sum_{r=0}^{q-1} (\psi_r - 1)^m$$
$$= \sum_{r=0}^{q-1} \left(\sum_{k=1}^{\nu} \varepsilon^{rk^2} W_k\right)^m$$
$$= \sum_{r=0}^{q-1} \sum_{s_1+\cdots+s_\nu=m} \binom{m}{s_1, \ldots, s_\nu} \varepsilon^{r(s_1+s_2\cdot 2^2+\cdots+s_\nu\cdot\nu^2)} W_1^{s_1} \ldots W_\nu^{s_\nu}$$
$$= \sum_{s_1+\cdots+s_\nu=m} \binom{m}{s_1, \ldots, s_\nu} W_1^{s_1} \ldots, W_\nu^{s_\nu} \sum_{r=0}^{q-1} \varepsilon^{r(s_1+\cdots+s_\nu\cdot\nu^2)},$$

$$S_m^*(\tau) = q \sum_{\substack{s_1+\cdots+s_\nu=m \\ s_1+\cdots+s_\nu\nu^2\equiv 0 \pmod q}} \binom{m}{s_1, \ldots, s_\nu} W_1^{s_1} \ldots W_\nu^{s_\nu}. \tag{3.11}$$

The transition to the $\sigma_r^*$ is then made by the formulas

$$\sigma_1^* = S_1^*,$$

$$(3.12)\qquad 2\sigma_2^* = \sigma_1^* S_1^* - S_2^*,$$

$$\cdots\cdots$$

$$(q-1)\sigma_{q-1}^* = \sigma_{q-2}^* S_1^* - \sigma_{q-3}^* S_2^* + \cdots - S_{q-1}^* .$$

It is not necessary to compute $\sigma_q^*$, since we need only $\sigma_1, \ldots, \sigma_{q-1}$ in (3.8), and these are given by

$$(3.13)\qquad \sigma_r = \sum_{m=0}^{q-r} \binom{q-m}{r} \sigma_m^* \qquad (\sigma_0 = \sigma_0^* = 1).$$

This program is easy to carry out for $q = 5, 7$. For $q = 5$ we get $S_1^* = 0$, $S_2^* = 10W_1W_2 = -10$ (by (2.18)), $S_3^* = 0$, $S_4^* = 30W_1^2W_2^2 = 30$. Then $\sigma_1^* = 0$, $\sigma_2^* = 5$, $\sigma_3^* = 0$, $\sigma_4^* = 5$, and from (3.13), $\sigma_1 = \sigma_2 = 25$, $\sigma_3 = 15$, $\sigma_4 = 5$. Hence

$$(3.14)\qquad \sigma_5 = \psi^5 - 5\psi^4 + 15\psi^3 - 25\psi^2 + 25\psi,$$

or, putting

$$v = -\sigma_5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6,$$

$$u = -\psi = \left(\frac{\eta(\tau/5)}{\eta(5\tau)}\right),$$

we have

$$(3.141)\qquad v = u^5 + 5u^4 + 15u^3 + 25u^2 + 25u.$$

This is essentially the modular equation as given by Watson.

Similarly, for $q = 7$, we find $S_1^* = S_2^* = 0$, $S_3^* = 42$, $S_4^* = 28A$, $S_5^* = 70B$, $S_6^* = 42C + 630$, where

$$A = W_1^3W_2 + W_2^3W_3 + W_3^3W_1,$$

$$(3.15)\qquad B = W_1^3W_3^2 + W_2^3W_1^2 + W_3^3W_2^2,$$

$$C = W_1^5W_3 + W_2^5W_1 + W_3^5W_2 .$$

The first few terms in the expansions of these functions are

$$A = -x^{-1} - 4 - 2x + \cdots ,$$

$$(3.16)\qquad B = x^{-1} + 1 + \cdots ,$$

$$C = 3 + \cdots .$$

By the method sketched in § 2, we get $B = -A - 3$, $C = 3$. Again, putting

$$v = -\sigma_7 = \left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^8,$$

$$u = -\psi = \left(\frac{\eta(\tau/7)}{\eta(7\tau)}\right),$$

and carrying through the routine calculations of $\sigma_r$, we obtain (3.8) in the form

$$v = u^7 + 7u^6 + 21u^5 + 49u^4 + (91 - 7A)u^3 + (63 - 35A)u^2 - 49(1 + A)u . \tag{3.17}$$

Directly from the product,

$$v = x^{-2} - 8x^{-1} + 20 + \cdots ,$$

so $v = (A + 8)^2$. Thus, setting

$$w = (\eta(\tau)/\eta(7\tau))^4 = x^{-1} + \cdots ,$$

we have

$$A = -\left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^4 - 8, \tag{3.18}$$

and finally

$$w^2 - 7w(u^3 + 5u^2 + 7u) = u^7 + 7u^6 + 21u^5 + 49u^4 + 147u^3 + 343u^2 + 343u, \tag{3.19}$$

which agrees with Watson's result.

For $q = 11$ the calculations are rather lengthy but elementary, and we shall give only the final result. Let

$$v = \left(\frac{\eta(\tau)}{\eta(11\tau)}\right)^{12} = x^{-5} \prod_{n\geq 1} \left(\frac{1 - x^n}{1 - x^{11n}}\right)^{12},$$

$$u = \frac{\eta(\tau/11)}{\eta(11\tau)} = x^{-\frac{5}{11}} \prod_{n\geq 1} \left(\frac{1 - x^{n/11}}{1 - x^{11n}}\right),$$

$$\begin{aligned} \alpha &= W_1^3W_4W_5 + W_2^3W_3W_1 + W_4^3W_5W_2 + W_3^3W_1W_4 + W_5^3W_2W_3 - 17 \\ &= x^{-2} + 2x^{-1} - 12 + 5x + 8x^2 + x^3 + 4x^4 + \cdots , \\ \beta &= 2 - (W_1^6W_4 + W_2^6W_3 + W_4^6W_5 + W_3^6W_1 + W_5^6W_2) \\ &= x^{-3} + x^{-1} - 12 + 2x + 2x^2 + 16x^3 + \cdots . \end{aligned} \tag{3.20}$$

Then

$$\begin{aligned} v = u^{11} &- 11u^{10} + 5\cdot 11u^9 - 11^2u^8 - 11^2u^7 + 11(11^2 - 2\alpha)u^6 \\ &- 11^2(11 - 2\alpha)u^5 - 11(11^3 + 126\alpha + 2\beta)u^4 \\ &+ 11^2(5\cdot 11^2 + 38\alpha + 2\beta)u^3 - 11(11^4 + 72\cdot 11\alpha - \alpha^2 + 9\cdot 11\beta)u^2 \\ &+ 11^2(11^3 + 8\cdot 11\alpha + \alpha^2 + 11\beta)u. \end{aligned} \tag{3.21}$$

There are the further relations

$$v = (11^2 + \alpha)\beta - 3\cdot 11^2\alpha - 14\alpha^2 \tag{3.22}$$

$$v^2 + (11^4 + 13\cdot 11^2\alpha + 34\alpha^2)v = \alpha^5 + 9(11^2 + \alpha)^2. \tag{3.23}$$

The auxiliary functions are connected by

$$\alpha^3 + 38\alpha^2 + 3\cdot 11^2\alpha + 9 = \beta^2 + 6\alpha\beta + 11^2\beta. \tag{3.24}$$

Elimination of $\alpha$ and $\beta$ from (3.21), (3.22), and (3.23) will yield the modular equation, which will be of degree 55 in $u$. This fact makes it difficult to apply the methods of Watson, since the conjugates of the root $u = \psi(\tau)$

consist not only of the translates $\psi(\tau+r)$, $r=0$, $r=0, 1, 2, \cdots, 10$, but also of others, the factor 5 being accounted for by the degree of (3.23) in $\alpha$.

Equation (3.21) enables us, however, to prove the Ramanujan congruence $p(11n+6)\equiv 0 \pmod{11}$. In fact, we get an identity for the generating function. To see this, we consider

$$\frac{1}{11}\sum_{r=0}^{10}\psi_r^{-1}(\tau)=\prod_{n\geq 1}(1-x^{11n})\sum_{m\geq 0}p(11m+6)x^{m+1}.$$

On the other hand, from (3.21),

$$\frac{1}{11}\sum_{r=0}^{10}\psi_r^{-1}(\tau)=\frac{1}{11}\frac{\sigma_{10}}{\sigma_{11}}=11x^5\prod_{n\geq 1}\left(\frac{1-x^{11n}}{1-x^n}\right)^{12}\cdot(11^3+8\cdot 11\alpha+\alpha^2+11\beta).$$

Hence

$$\sum_{m\geq 0}p(11m+6)x^{m+1}=11x^5\prod_{n\geq 1}\frac{(1-x^{11n})^{11}}{(1-x^n)^{12}}(11^3+8\cdot 11\alpha+\alpha^2+11\beta). \tag{3.25}$$

Since $\alpha$ and $\beta$ have integral coefficients, the result follows. For a similar identity, see Lehner [4].

**4. Two identities of Remanujan for $q=7$.** Bailey [2] has given a proof of the identity

$$x\prod_{m\geq 1}\frac{(1-x^{5m})^5}{(1-x^m)}=\sum_{n\geq 1}\left(\frac{n}{5}\right)\frac{x^n}{(1-x^n)^2}, \tag{4. 1}$$

which appears, unproved, in Ramanujan's notebooks. It is easy to see that (4. 1) implies the Ramanujan identity

$$\sum_{n\geq 0}p(5n+4)x^n=5\prod_{m\geq 1}\frac{(1-x^{5m})^5}{(1-x^m)^6}. \tag{4. 2}$$

Indeed, if we denote by $f(x)=\sum A_nx^n$ the left side of (4. 1), and set

$$f^*(x)=\sum A_{5n}x^n, \tag{4. 3}$$

we have

$$f^*(x)=\prod_{m\geq 1}(1-x^m)^5\cdot\sum_{n\geq 0}p(5n+4)x^{n+1}. \tag{4. 4}$$

But the right side of (4. 1) is easily seen to be

$$\sum_{N\geq 1}x^N\cdot N\sum_{n\mid N}\frac{1}{n}\left(\frac{n}{5}\right). \tag{4. 5}$$

Thus $A_{5N}=5A_N$, and

$$f^*(x)=5f(x)=5x\prod_{m\geq 1}\frac{(1-x^{5m})^5}{(1-x^m)}, \tag{4. 6}$$

from which (4. 2) follows directly.

The elegance of this proof of the Ramanujan congruence and identity for $p=5$ led me to seek a similar one for $p=7$. Let

$$F(x) = x^2 \prod_{m\geq 1} \frac{(1-x^{7m})^7}{(1-x^m)} = \sum B_n x^n, \tag{4.7}$$

$$F^*(x) = \sum B_{7n} x^n = \prod_{m\geq 1} (1-x^m)^7 \sum_{n\geq 0} p(7n+5)x^{n+1}. \tag{4.8}$$

The classical identity of Ramanujan,

$$\sum_{n\geq 0} p(7n+5)x^n = 7\prod_{m\geq 1} \frac{(1-x^{7m})^3}{(1-x^m)^4} + 49x\prod_{m\geq 1} \frac{(1-x^{7m})^7}{(1-x^m)^8}, \tag{4.9}$$

is then equivalent to

$$F^*(x) = 7(Q(x) + 7F(x)), \tag{4.10}$$

where

$$Q(x) = x\prod_{m\geq 1} (1-x^{7m})^3(1-x^m)^3. \tag{4.11}$$

I had hoped to prove (4.10) using some elementary methods in elliptic functions which I had developed. Instead, the following identity came forth:

$$F^*(x) = 7(S(x) - F(x)), \tag{4.12}$$

where

$$S(x) = \sum_{N\geq 1} x^N \cdot N^2 \sum_{n\mid N} \frac{1}{n^2}\left(\frac{n}{7}\right). \tag{4.13}$$

Of course, (4.12) proves the congruence $p(7n+5) \equiv 0 \pmod 7$, but it was not clear how it related to (4.10). The two together yield

$$S(x) = Q(x) + 8F(x). \tag{4.14}$$

Conversely, (4.14) easily implies both (4.10) and (4.12). For, equating coefficients of $x^{7n}$, we obtain

$$S^*(x) = Q^*(x) + 8F^*(x). \tag{4.15}$$

Now it is easy to see that

$$S^*(x) = 49S(x) \tag{4.16}$$

and

$$Q^*(x) = -7Q(x). \tag{4.17}$$

Hence

$$49S(x) = -7Q(x) + 8F^*(x). \tag{4.18}$$

Elimination of $Q$ from (4.18) and (4.14) yields (4.12), and elimination of $S$ yields (4.10). It was therefore highly desirable to find an independent proof of the key identity (4.14).

When I communicated these results to professor Bailey, he informed me that (4.14) appears unproved in Ramanujan's notebooks, along with the similar (also unproved) formula

$$49Q(x) + 8\prod_{m\geq 1} \frac{(1-x^m)^7}{(1-x^{7m})} = 8 - 7T(x), \tag{4.19}$$

where

$$T(x)=\sum_{n\geq 1}x^n\sum_{d\mid n}\left(\frac{d}{7}\right)d^2. \tag{4.20}$$

Using the results of § 3, I can now prove (4.14) and (4.19) without much difficulty. From (3.18),

$$\left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^4+8+A(\tau)=0. \tag{4.21}$$

Now $W_1W_2W_3=1$, by (2.18), so

$$\begin{aligned} A(\tau)&=W_1^3W_2+W_2^3W_3+W_3^3W_1 \\ &=\frac{W_1^2}{W_3}+\frac{W_2^2}{W_1}+\frac{W_3^2}{W_2} \\ &=-P\left\{x^{-1}\frac{C_2}{C_1^4}+\frac{C_4}{C_2^4}-x\,\frac{C_6}{C_3^4}\right\}, \end{aligned} \tag{4.22}$$

where

$$P=C_1C_2C_3=\prod_{n\geq 1}\frac{(1-x^n)}{(1-x^{7n})}, \tag{4.23}$$

and the $C$ are defined by (2. 1). (The identity obtained by eliminating $A(\tau)$ between (4.21) and (4.22) is stated by Slater [8,(1.3)], who also quotes (4.14) and shows that the latter implies the former.)

Now we have the formula[2]

$$K^6(x)t\prod_{n\geq 1}\frac{(1-x^nt^{-2})(1-x^{n-1}t^2)}{[(1-x^nt^{-1})(1-x^{n-1}t)]^4}=\sum_{-\infty}^{+\infty}\frac{x^mt(1+x^mt)}{(1-x^mt)^3}, \tag{4.24}$$

where

$$K(x)=\prod_{n\geq 1}(1-x^n). \tag{4.25}$$

Replacing $x$ by $x^7$ and then $t$ by $x^a$, $a=1,2,3$, in (4.24), we obtain

$$\begin{aligned} S_a\equiv x^aK^6(x^7)\frac{C_{2a}}{C_a^4}&=\sum_{-\infty}^{+\infty}\frac{x^{7m+a}(1+x^{7m+a})}{(1-x^{7m+a})^3} \\ &=\sum_{m\geq 0}\frac{x^{7m+a}(1+x^{7m+a})}{(1-x^{7m+a})^3}-\sum_{m\geq 0}\frac{x^{7m+a'}(1-x^{7m+a'})}{(1-x^{7m+a})^3}, \end{aligned} \tag{4.26}$$

where $a'=7-a$. Hence, from (4.22),

$$-x^2P^{-1}K^6(x^7)A(\tau)=S_1+S_2-S_3. \tag{4.27}$$

Returning to (4.21), and multiplying by

$$x^2P^{-1}K^6(x^7)=x^2\prod_{n\geq 1}\frac{(1-x^{7n})^7}{(1-x^n)}=F(x).$$

2) This is essentially the elliptic function identity

$$\wp'(u)=-\frac{\sigma(2u)}{\sigma^4(u)}.$$

we have

(4.28) $$Q(x) + 8F(x) = S_1 + S_2 - S_3\,,$$

since

$$F(x)\left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^4 = Q(x).$$

Equation (4.28) is the first of Ramanujan's identities referred to above, in the form given by him. An easy transformation shows that $S_1+S_2-S_3=S$, and (4.14) is proved.

To prove (4.19), we subject (4.21) to the transformation $\tau \to -\frac{1}{7\tau}$, to get

(4.29) $$49\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4 + 8 + A\left(-\frac{1}{7\tau}\right) = 0.$$

Now

(4.30) $$W_k\left(-\frac{1}{q\tau}\right) = \frac{\vartheta_1\left(\frac{4k\pi}{q}\,\middle|\,\tau\right)}{\vartheta_1\left(\frac{2k\pi}{q}\,\middle|\,\tau\right)} = 2b_k\frac{D_{4k}}{D_{2k}},$$

where $b_k = \cos(2k\pi/q)$ and

(4.31) $$D_k = D_k(x) = \prod_{n\geq 1}(1-\omega^k x^n)(1-\omega^{-k}n^n) \qquad (\omega = e^{\frac{2\pi i}{q}}).$$

Hence, with $q = 7$,

(4.32) $$A\left(-\frac{1}{7\tau}\right) = B_1 + B_2 + B_3\,,$$

(4.33) $$B_k = 2\frac{b_{2k}^2}{b_k}\cdot\frac{D_k^2 D_{2k}}{D_{3k}^3} = 2\frac{b_{2k}^2}{b_k}R\frac{D_k}{D_{3k}^4} = 2\frac{b_{2k}^2}{b_k}R\frac{D_{6k}}{D_{3k}^4}\,,$$

where

(4.34) $$R = D_1 D_2 D_3 = \prod_{n\geq 1}\frac{(1-x^{7n})}{(1-x^n)}\,.$$

Now put $t = \omega^{3k}$ $(k = 1, 2, 3)$ in (4.24) to get

(4.35) $$B_k = 2\frac{b_{2k}^2}{b_k}RK^{-6}\frac{(1-\omega^{3k})^4}{\omega^{3k}(1-\omega^{6k})}\sum_{-\infty}^{+\infty}\frac{\omega^{3k}x^m(1+\omega^{3k}x^m)}{(1-\omega^{3k}x^m)^3}\,.$$

Split the sum for $m = 0$, $m > 0$, and $m < 0$, then expand in powers of $x$:

(4.36) $$\sum_{-\infty}^{+\infty} = \frac{\omega^{3k}(1+\omega^{3k})}{(1-\omega^{3k})^3} + \sum_{n>0}x^n\sum_{d\mid n}(\omega^{3kd}-\omega^{-3kd})d^2.$$

Now

$$h(k) = \frac{2b_{2k}^2(1-\omega^{3k})^4}{b_k\omega^{3k}(1-\omega^{6k})} = \sum_{m=0}^{6}\left(\frac{m}{7}\right)\omega^{km},$$

and $h(-k) = -h(k)$, so

$$\sum_{k=1}^{3}h(k)(\omega^{3kd}-\omega^{-3kd}) = \frac{1}{2}\sum_{k=0}^{6}h(k)(\omega^{3kd}-\omega^{-3kd})$$
$$= \frac{7}{2}\left\{\left(\frac{-3d}{7}\right)-\left(\frac{3d}{7}\right)\right\} = 7\left(\frac{d}{7}\right).$$

Therefore

$$A\left(-\frac{1}{7\tau}\right) = B_1 + B_2 + B_3 = RK^{-6}\left(a + 7\sum_{n>0} x^n \sum_{d\mid n}\left(\frac{d}{7}\right)d^2\right). \tag{4.37}$$

Substituting in (4.29) and simplifying, we get

$$49Q + 8\Phi = -a - 7T,$$

where

$$\Phi(x) = \prod_{n\geq 1}\frac{(1-x^n)^7}{(1-x^{7n})}. \tag{4.38}$$

Equating constant terms, we find that $a = -8$, so

$$49Q + 8\Phi = 8 - 7T. \tag{4.39}$$

This completes the proof of (4.19).

If we apply the starring operation to (4.39), observing that $T^* = T$, we get

$$-343Q + 8\Phi^* = 49Q + 8\Phi,$$

or

$$\Phi^* = \Phi + 49Q. \tag{4.40}$$

Following Newman [5], we define $p_r(n)$ by

$$\prod_{n\geq 1}(1-x^n)^r = \sum_{n\geq 0} p_r(n)x^n. \tag{4.41}$$

Then

$$\Phi(x) = \prod_{m\geq 1}(1-x^{7m})^{-1}\sum_{n\geq 0} p_7(n)x^n,$$

$$\Phi^*(x) = \prod_{m\geq 1}(1-x^m)^{-1}\sum_{n\geq 0} p_7(7n)x^n.$$

Therefore (4.40) is equivalent to

$$\sum_{n\geq 0} p_7(7n)x^n = \prod_{n\geq 1}(1-x^n)^8(1-x^{7n})^{-1} + 49x\prod_{n\geq 1}(1-x^n)^4(1-x^{7n})^3. \tag{4.42}$$

This is example 3 on p. 320 of Newman's paper.

**5. Rademacher's identity.** Rademacher [6, eq. (4. 7)] has derived the following interesting identity by subjecting (4. 2) to the transformation $\tau \to -\tau^{-1}$, using the transformation theory of $\eta(\tau)$ and some results on Dedekind sums:

$$\sum_{n\geq 0} p(n)x^{25n} - 5\sum_{n\geq 1}\left(\frac{n}{5}\right)p(n-1)x^n = \prod_{m\geq 1}\frac{(1-x^m)^5}{(1-x^{5m})^6}. \tag{5. 1}$$

There are several other methods of deriving (5. 1), of which I select the following for its elementary nature, and because it illustrates the utility of the systems of functions introduced here.

We have

$$\left(\frac{n}{5}\right)=\frac{1}{\sqrt{5}}\sum_{r=1}^{4}\left(\frac{r}{5}\right)\omega^{rn} \qquad (\omega=\exp(2\pi i/5)).$$

Hence

$$\sum_{n\geq 1}p(n-1)\left(\frac{n}{5}\right)x^n=\frac{1}{\sqrt{5}}\sum_{r=1}^{4}\left(\frac{r}{5}\right)\sum_{n\geq 0}p(n)(\omega^r x)^{n+1}$$
$$=-\prod_{m\geq 1}(1-x^{25m})^{-1}\sum_{r=1}^{4}\frac{1}{\sqrt{5}}\left(\frac{r}{5}\right)\psi_r^{-1}(5\tau),$$

where the $\psi_r$ are defind by (3. 3) with $q=5$. Also

$$\sum_{n\geq 0}p(n)x^{25n}=\prod_{m\geq 1}(1-x^{25m})^{-1}.$$

Inserting these values in (5. 1), multiplying by $x^{-1}\prod(1-x^m)$, and expressing the products as $\eta$-functions, we find that (5. 1) is equivalent to

$$\frac{\eta(\tau)}{\eta(25\tau)}\left[1+\sqrt{5}\sum_{r=1}^{4}\left(\frac{r}{5}\right)\psi_r^{-1}(5\tau)\right]=\left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6. \tag{5. 2}$$

Recalling the definitions of $\psi(\tau)$ and $\sigma_5(\tau)$ (which we shall write as $\sigma(\tau)$), we find

$$\psi(5\tau)\left[1+\sqrt{5}\sum_{r=1}^{4}\left(\frac{r}{5}\right)\psi_r^{-1}(5\tau)\right]=\sigma(\tau). \tag{5. 2$'$}$$

Now replace $\tau$ by $\tau/5$ and observe that

$$\sigma(\tau/5)=\psi^6(\tau)/\sigma(\tau),$$

so that (5. 2′) becomes

$$\psi^5(\tau)=\sigma(\tau)\left[1+\sqrt{5}\sum_{r=1}^{4}\left(\frac{r}{5}\right)\psi_r^{-1}(\tau)\right]. \tag{5. 2$''$}$$

Now the functions $\left(\frac{r}{5}\right)\psi_r(\tau)$ $(r=1,2,3,4)$ satisfy an equation

$$z^4-A_1z^3+A_2z^2-A_3z+A_4=0, \tag{5. 3}$$

and

$$\sum_{r=1}^{4}\left(\frac{r}{5}\right)\psi_r^{-1}(\tau)=\frac{A_3}{A_4}. \tag{5. 4}$$

To determine the coefficients, we express the symmetric functions of $\psi_1$, $\psi_4$ and $\psi_2$, $\psi_3$ in terms of $W_1$, $W_2$ by (3. 4). Thus

$$\psi_1+\psi_4=2+(\omega+\omega^4)\ (W_1+W_2),$$
$$\psi_1\psi_4=1+W_1^2+W_2^2+(\omega+\omega^4)(W_1+W_2)+(\omega^2+\omega^3)W_1W_2,$$

with similar expressions for $\psi_2$, $\psi_3$. But $W_1+W_2=\psi-1$ and $W_1W_2=-1$, so

$$
(5.\ 5)\qquad
\begin{aligned}
&\psi_1 + \psi_4 = 2 + (\omega + \omega^4)(\psi - 1),\\
&\quad \psi_1\psi_4 = (\psi - 1)^2 + (\omega + \omega^4)(\psi - 1) + (3 - \omega^2 - \omega^3),\\
&\psi_2 + \psi_3 = 2 + (\omega^2 + \omega^3)(\psi - 1),\\
&\quad \psi_2\psi_3 = (\psi - 1)^2 + (\omega^2 + \omega^3)(\psi - 1) + (3 - \omega - \omega^4).
\end{aligned}
$$

An elementary calculation shows that (5. 3) is

$$
(5.\ 6)\qquad z^4 - \sqrt{5}(\psi - 1)z^3 + (3\psi^2 - 5\psi + 5)z^2 - \sqrt{5}(\psi^3 - 3\psi^2 + 5\psi - 5)z + A_4 = 0.
$$

Hence

$$
\frac{A_3}{A_4} = \sqrt{5}\,\frac{\psi^3 - 3\psi^2 + 5\psi - 5}{A_4} = \frac{\sqrt{5}}{\sigma}(\psi^4 - 3\psi^3 + 5\psi^2 - 5\psi),
$$

since $A_4 = \psi_1\psi_2\psi_3\psi_4 = \sigma/\psi$. Putting this value in (5. 2″) and simplifying, we find the modular equation (3.14). Thus (5. 1) is proved. It was of course not necessary to appeal to the previous derivation of (3.14), since $A_4 = (\psi_1\psi_4)(\psi_2\psi_3)$ can be found directly from (5. 5). It is not surprising that the symmetric functions in (5. 5) are expressible in terms of $\psi$, since they satisfy the condition immediately following (3. 5) (invariance under $\tau \rightarrow \delta^2\tau$), which guarantee that they belong to $\Gamma_0^0(5)$, as does $\psi$. In fact, $\psi$ is a Hauptmodul for this group, but this fact was not used here.

A similar, though computationally more involved, treatment should yield the corresponding identity for $q = 7$ proved by Rademacher [6, eq. (5. 7)].

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University of Pennsylvania.