NOTE ON POSTNIKOV INVARIANTS OF A LOOP SPACE*

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1. In a recent paper, H. Suzuki [6] has proved the following:

Let X be a topological space with vanishing homotopy groups $\pi_i(X)$ for $0 \leqq i < n, n < i < q$ $(1 < n < q)$, and $\Omega(X)$ be its loop space (in the sense of J. P. Serre [4]). If we denote their Eilenberg-MacLane invariants by $k_n^{q+1}(X)$ and $k_{n-1}^q(\Omega(X))$ respectively, then there is the relation

$$k_{n-1}^q(\Omega(X)) = \Sigma\, k_n^{q+1}(X)$$

between them, where $\Sigma : H^{q+1}(\pi_n, n; \pi_q) \to H^q(\pi_n, n-1; \pi_q)$ is the cohomology suspension.

In this note, we give a proof by a different method from the original and in some general form.

2. Now, we are familiar with:

LEMMA 1. (Eilenberg-MacLane [1], see also [5]) *Let Y be a cell complex. There is a 1:1-correspondence between the homotopy classes of maps of Y into $K(\pi_q, q+1)$ and the elements of $H^{q+1}(Y, \pi_q)$, which is given by the correspondence $\{f\} \leftrightarrow f^*(u)$, where $\{f\}$ is the class represented by $f : Y \to K(\pi_q, q+1)$, $u \in H^{q+1}(\pi_q, q+1 : \pi_q)$ is fundamental class and f^* is the homomorphism induced by f.*

Let X be an arcwise connected topological space. We know that there is a contractible fiber space in the sense of J. P. Serre, in which the total space $E(X)$ is the space of all paths in X with a fixed starting point (named base point) and with compact-open topology, the base space is X and the fiber over the base point is the loop space of X. Next theorem by Nakaoka-Mizuno [2] will be used later.

LEMMA 2. *Consider the fiber space (E, p, B) satisfying the following conditions:*

(i) *The total space E is a simply connected space with vanishing homotopy groups $\pi_i(E)$ for $i > r > 1$.*

(ii) *The base space B is a space with vanishing homotopy groups $\pi_i(B)$ for $i \geqq r > 1$.*

(iii) *The projection $p : E \to B$ induces the isomorphisms*

$$\pi_i(E) \approx \pi_i(B)$$

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for $0 \leqq i < r$.

Then the image of the fundamental class u *of the fiber by transgression* τ *is equal to* $-\bar{k}_{r-1}$, *where* $\bar{k}_{r-1}$ *is the geometrical realization of Postnikov invariant* k_{r-1} *of* E. (See [3] for Postnikov invariants)

3. Now we prove:

PROPOSITION 1. *Let* (B, p, Y) *be the induced fibre space from the contractible fiber space* $(E(X), p_0, X; \Omega(X))$ *over 2-connected space* X *by a continuous map* $f: Y \to X$, *and let* B' *be the total space of the fiber space which is induced from* $(E(\Omega(X)), q, \Omega(X); \Omega^2(X))$ *by* $f_\Omega: \Omega(Y) \to \Omega(X)$, *where* f_Ω *is the continuous map induced by* f. *Then* B' *is homeomorphic to* $\Omega(B)$.

PROOF. By the definition of the induced fiber space,

$$B = \{(y, u) \mid f(y) = p_0(u),\ y \in Y, u \in E(X)\}.$$

Let $\bar{f}$ be the fiber map of B into $E(X)$ and let p the projection of B into Y. For $\varphi \in \Omega(B)$, we define a map

$$\varphi \to (p\varphi, \bar{f}\varphi)$$

by
$$\varphi(t) \to (p\varphi(t), \bar{f}\varphi(t)), \qquad t \in I = [0, 1].$$

Since $p\varphi(0) = p\varphi(1)$ and $\bar{f}\varphi(0) = \bar{f}\varphi(1)$, we have $p\varphi \in \Omega(Y)$ and $\bar{f}\varphi \in \Omega(E(X))$. Moreover, $fp\varphi(t) = p_0\bar{f}\varphi(t)$. Therefore, if we denote by p_Ω the map $\Omega(E(X)) \to \Omega(X)$ induced by $p_0: E(X) \to X$, then $f_\Omega p\varphi = p_\Omega \bar{f}\varphi$. Conversely, it is clear that the elements $(y', u') \in \Omega(Y) \times \Omega(E(X))$, satisfying the condition $f_\Omega y' = p_\Omega u'$, belong to $\Omega(B)$. So we can identify $\Omega(B)$ with a subset of $\Omega(Y) \times \Omega(E(X))$. On the other hand, by the definition

$$B' = \{(y', v') \mid f_\Omega y' = qv', y' \in \Omega(Y), v' \in E(\Omega(X))\}.$$

We give a correspondence between $v' \in E(\Omega(X))$ and $\xi v' \in \Omega(E(X))$ as follows:

$$(\xi v'(t))(s) = (v'(s))(t) \qquad s, t \in \mathrm{I} = [0, 1].$$

Choosing the base points suitably, we have a continuous map

$$\xi: v' \to \xi v'.$$

For, by [4] p. 474, $(v'(s))(t)$ is continuous relative to both s and t. Therefore, $\xi v'(t)$ is an element of $E(X)$ and $\xi v'$ is an element of $\Omega(E(X))$, and it is clear that ξ is also continuous. By similar arguments (with slight modifications), we can get a continuous map of $\Omega(E(X))$ into $E(\Omega(X))$

$$\eta: u' \to \eta u'$$

defined by
$$(u'(t))(s) = (\eta u'(s))(t).$$

Now we define the correspondences

$$B' \underset{\bar{\eta}}{\overset{\bar{\xi}}{\rightleftarrows}} \Omega(B)$$

by
$$\bar{\xi}(y', v') = (y', \xi v'),$$
$$\bar{\eta}(y', u') = (y', \eta u').$$

(Since $(p_\Omega(\xi v'))(t) = p(\xi v'(t))(\xi v'(t))(1) = (v'(1))(t) = (qv')(t) = f_\Omega y'(t)$, $(y', \xi v') \in \Omega(B)$, etc.)

It is clear that $\bar{\xi}$ and $\bar{\eta}$ are continuous, and that $\bar{\xi}\,\bar{\eta}$, $\bar{\eta}\,\bar{\xi}$ are both identities. Therefore, B' is homeomorphic with $\Omega(B)$.

Let $\mathcal{E}$ be a 1-connected cell complex with i-th homotopy groups π_i for $i = 2, 3, .., r$, and $\pi_i(\mathcal{E}) = 0$ for $i > r$, and with Postnikov invariants $k_2, .. ., k_{r-1}$.

By Cartan-Serre's construction, we have a fiber space $(\mathcal{E}_0, p, Y)$ satisfying the conditions of Lemma 2. $\mathcal{E}_0$ has the same homotopy type with $\mathcal{E}$ and Y has the same invariants $k_2, .., k_{r-2}$ as $\mathcal{E}_0$ (or $\mathcal{E}$)

Let X be a space $K(\pi_r, r+1), r > 1$. By Lemma 1, for $\bar{k}_{r-1} \in H^{r+1}(Y, \pi_r)$ there is a continuous map

$$f: Y \to X$$

such that
$$f^* u_{r+1} = -\bar{k}_{r-1},$$

where u_{r+1} is the fundamental class of $H^{r+1}(X, \pi_r)$.

Since the transgression in $(E(X), p_0, X; \Omega(X))$ is translated by f^* to that of (B, p, Y), if we denote by u_r the fundamental class of $H^r(\Omega(X), \pi_r)$ and by τ the transgression in (B, p, Y), then, by Lemma 2, $\tau u_r = -\bar{k}_{r-1}$. Therefore, B and $\mathcal{E}_0$ (and $\mathcal{E}$) have the same homotopy groups and invariants. This implies that their singular polytopes have the same homotopy types, and so do their loop spaces. Accordingly, the invariants of $\Omega(\mathcal{E})$ are equal to the corresponding invariants of $\Omega(B)$ which are equal to that of B' in Proposition 1.

But we have:

PROPOSITION 2. *Invariants* $\bar{k}_i'$ *of* B' *are of the forms* $\Sigma \bar{k}_{i+1}$, *where* Σ: $H^{i+3}(Y, \pi_{i+2}) \to H^{i+2}(\Omega(Y), \pi_{i+2})$ *is the cohomology suspension in the fiber space* $(E(Y), p, Y; \Omega(Y))$ *over* Y.

PROOF. Invariants of B' are of the forms $-f_\Omega^* u_i = -f_\Omega^* \Sigma u_{i+1}$. Since the homomorphism of cubical singular chain groups $C(\Omega(Y)) \to C(Y)$ defined by

$$\sigma y(t_1, \ldots, t_n) = (y(t_2, \ldots, t_n))(t_1) \qquad t_j \in I$$

induces the suspension and this σ commutes with the chain homomorphisms induced by f and f_Ω, $-f_\Omega^* \Sigma u_{i+1} = -\Sigma f^* u_{i+1} = -\Sigma(-\bar{k}_{i+1}) = \Sigma \bar{k}_{i+1}$.

By above discussions, we have

THEOREM. *The Postnikov invariants of the loop space of a* 1*-connected cell complex* $\mathcal{E}$ *are the images of that invariants of* $\mathcal{E}$ *by the cohomology suspension defined in the fiber space* $(E(\mathcal{E}), p, \mathcal{E})$ *over* $\mathcal{E}$.

REMARK. In the case where $\mathcal{E}$ is a "space", we can replace $\mathcal{E}$ with its singular polytope without changing proofs. So we have the theorem in this case too.

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