ON THE DIRECT PRODUCT OF OPERATOR ALGEBRAS IV

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M. Nakamura [2] has discussed the relation between the direct product and the generation of two sub-factors in a finite W^*-factor, and obtained a satisfactorily analogous result as the classical theory of hypercomplex numbers: that is, in a finite W^*-factor, the direct product (in W^*-sense) of elementwise commutative two sub-factors means the generation in weak operator topology and vice versa. In the present paper, we shall consider the same problem for sub-algebras in an arbitrary C^*-algebra. In the case of W^*-factor, the key point of Nakamura's argument is the multiplicativity of the faithful normal trace. In view of this fact, we introduce a notion of independence of sub-algebras in a C^*-algebra (§1, Def.): this definition seems to be artificial at a first glance but this may be considered as a generalization of the stochastic independence in the probability theory. Finally in §2, we shall state a theorem for an abelian C^*-algebra recently obtained by R. McDowell for real case [1].

1. Let A be a C^*-algebra with the unit 1, and A_i $(i = 1, 2)$ be two C^*-sub-algebras of A which contain 1. Then A_i are called *algebraically independent* if they have following properties:

(1) A_1 and A_2 commute elementwise,

(2) If $\{a_i \mid i = 1, \ldots, m\}$ and $\{b_j \mid j = 1, \ldots, n\}$ are arbitrary linearly independent set of elements of A_1 and A_2 respectively, then $\{a_i b_j \mid i = 1, \ldots, m;\ j = 1, \ldots, n\}$ is linearly independent in A.

If $A_i (i = 1, 2)$ are algebraically independent C^*-sub-algebras of a C^*-algebra A, the following facts are easily verified:

(I) The algebraic direct product $A_1 \odot A_2$ of A_1 and A_2 is isomorphic to the *-algebra A_0 generated by A_1 and A_2 algebraically and this isomorphism is given by the correspondence

$$\sum_i a_i b_i \leftrightarrow \sum_i a_i \times b_i.$$

(II) For any pair of positive linear functionals (σ, τ) where σ is defined on A_1 and τ on A_2, the functional which we call a *product functional* of σ and τ,

$$[\sigma \odot \tau]\left(\sum_i a_i b_i\right) = \sum_i \sigma(a_i)\tau(b_i), \quad \sum_i a_i b_i \in A_0$$

is well-defined on A_0 and additive, homogeneous and positive:

$$[\sigma \odot \tau]\left(\left(\sum_i a_i b_i\right)\left(\sum_i a_i b_i\right)^*\right) \geqq 0, \text{ for all } \sum_i a_i b_i \in A_0.$$

Then, *"Under what topological conditions for A_i, the product functional does become continuous?"* is an interesting problem for us. In abelian real case, R. MacDowell has discussed this problem with different aspects [1]. In §2, we shall come back to this problem for the abelian complex case. In this section we discuss the relation between this problem and the direct product $A_1 \times_\alpha A_2$(cf. [4]). For this purpose, we want to introduce the notion of independence of sub-algebras. Now, we begin with the consideration on the probability theory. If (X, F, μ) be a probability measure space, then two bounded real random variables a and b are called mutually independent if $E(f(a)g(b)) = E(f(a))E(g(b))$, where $f(\lambda)$, $g(\lambda)$ are arbitrary complex-valued Baire functions and $E(c) = \int c(x)d\mu(x)$. And moreover, the space $L^\infty(X, F, \mu)$ of all bounded random variables forms a commutative W^*-algebra, acting on the Hilbert space $L^2(X, F, \mu)$, having a faithful normal trace $E(\cdot)$. Conversely, if a commutative W^*-algebra M has a faithful normal trace τ, then Gelfand-Neumark's respresentation theorem shows that there is a probability measure space on which the algebra M is isomorphically, preserving the value of the trace, represented to the algebra of all bounded random variables of the space, i. e., $\tau(a) = E(a^\#)$, $a^\#$ denotes the representation of a. Since the W^*-sub-algebra A generated by an hermitian element a and 1 in M consists of all Baire functions of a, the above notion of independence can be transfered into any commutative W^*-algebra M with a faithful normal trace τ as follows: Two hermitian elements a and b are called *mutually independent* if $\tau(uv) = \tau(u)\tau(v)$ for every $u \in A$, $v \in B$, where A and B are W^*-sub-algebras generated by $\{a, 1\}$ and $\{b, 1\}$ respectively.

Under these considerations we set the following definition for C^*-algebras.

DEFINITION. Let A be a C^*-algebra, and $A_i (i = 1, 2)$ be algebraically independent sub-algebras. Then A_i are called *mutually independent* if the product functional $\sigma \odot \tau$ is continuous on A and the totality of its continuous extension $\sigma \times \tau$ is complete on the C^*-sub-algebra generated by $A_i (i = 1, 2)$.

Now, our aimed proposition is the following:

PROPOSITION 1. *Let A be a C^*-algebra, $A_i (i = 1, 2)$ be C^*-sub-algebras of A, and if A is generated by A_i, then the following two statements are mutually equivalent:*

(1) *$A_i (i = 1, 2)$ are mutually independent,*

(2) *A is isomorphic to the direct product $A_1 \times_\alpha A_2$ of A_i.*

PROOF. Implication (1)$\to$(2). It is sufficient to prove that the isomorphism $\sum_i a_i b_i \longleftrightarrow \sum_i a_i \times b_i$ between A_0 and $A_1 \odot A_2$ described in (I) is isometric.

Since A_i are mutually independent and generate A, the totality of $\sigma \times \tau$, where σ and τ are pure states of A_1 and A_2 respectively, is complete on A, and we have by [4]

$$\|\sum_i a_i b_i\|^2 = \sup_{\sigma\times\tau} \frac{[\sigma \times \tau]((\Sigma_j a_j' b_j')\,(\Sigma_i a_i b_i)\,(\Sigma_i a_i b_i)^*\,(\Sigma_j a_j' b_j')^*)}{[\sigma \times \tau]\ ((\Sigma_j a_j' b_j')\,(\Sigma_j a_j')b_j')^*)}$$
$$= \alpha\Big(\sum_i a_i \times b_i\Big)^2.$$

Implication (2) → (1). Since $A \simeq A_1 \times_\alpha A_2$, then $A_1 \simeq A_1 \times 1, A_2 \simeq 1 \times A_2$; hence A_1 and A_2 are algebraically independent, and finally the continuity of the product functional is clear from the definition of norm in $A_1 \times {}_\alpha A_2$. [cf. 4] Q. E. D.

For infinitely many sub-algebras we can prove the following

PROPOSITION 2. *Let A and A_i $(i \in I)$ be C^*-algebras with units, then A is isomorphic to the infinite product of A_i $(i \in I)$ in the sense of Takeda* [3], *if and only if there exists a mapping Φ from the set union $\bigcup_{i\in I} A_i$ of A_i into A such that*

(1) *Restriction ϕ_i of Φ on A_i is the principal isomorphism of A_i into A,*
(2) *Any finite set of images $\{\phi_{i_k}(A_{i_k}) \mid k = 1, .., n\}$ are mutually independent,*
(3) *$\{\phi_i(A_i) \mid i \in I\}$ generate A as C^*-algebra.*

PROOF. For any finite subset of indices $\gamma = \{i_1, \dots, i_n\}$, let A^γ be the C^*-sub-algebra of A generated by $\{\phi_{i_k}(A_{i_k}) \mid k = 1, .., n\}$, then A^γ is isomorphic to $A_{i_1} \times \dots \times A_{i_n}$ by the above Prop. 1. Then A is isomorphic to the infinite direct product of A_i by [3: Definitions 1, 2].

2. In this section, we wish to answer partly for the question described in the preceding section by proving MacDowell's theorem for the complex case: that is, in an abelian case to obtain the topological condition under which product functional becomes continuous. For this purpose, we begin with the

DEFINITION. Let A be a commutative C^*-algebra with the unit, and B, C be two C^*-sub-algebras of A which contain units. Then they are called *additively related* sub-algebras if and only if for every pair (b, c) of $b \in B$ and $c \in C$, there exists a scalar $\theta = \theta(b, c) \in I = [0, 2\pi]$ such that $\|b + e^{i\theta}c\| = \|b\| + \|c\|$.

Then we have the following

THEOREM. *Let A be a commutative C^*-algebra with unit, and let B, C be two C^*-sub-algebras of A. Suppose that A is generated by B and C, then the following three statements are mutually equivalent:*

(1) *A is isomorphic to the direct product $B \times_\alpha C$,*
(2) *B and C are mutually independent,*
(3) *B and C are additively related.*

REMARK. R. MacDowell had proved the equivalency of (1) and (3) for real case.

PROOF. Since A, B and C are commutative C^*-algebras with the unit, let $C(\Omega)$, $C(\Gamma)$ and $C(\Delta)$ be their representations on the rings of all continuous complex-valued functions on compact Hausdorff spaces respectively.

Implication (3)$\to$(1). We use the following terminologies (MacDowell [1]): We call a subset T of a Banach space S a *T-set* of S if T is maximal with respect to the property $b_1, \dots, b_n \in T \to \|\sum_i b_i\| = \sum_i \|b_i\|$. Moreover if T is a T-set of S, then we call $\bigcup_{\theta\in I} e^{i\theta}\, T$ a circular T-set. If the space $S = C(X)$ (the space of all continuous complex valued functions on a compact Hausdorff space X) then every T-set T of S has the form $\{f \in S \mid f(x_0) = e^{i\theta_0}\|f\|\}$ for some point x_0 of X and scalar $\theta_0 \in I$, and conversely every such subset is a T-set of S.

Now let γ and δ be arbitrary pure states of B and C respectively, then $T_B(\gamma) = \{b \in B \mid b(\gamma) = \|b\|\}$, $T_C(\delta) = \{c \in C \mid c(\delta)\| = \|c\|\}$ are T-sets of B and C respectively. We can prove that there exists a circular T-set of A which contains the set $\bigcup_{\theta'\in I} e^{i\theta'}\, T_B(\gamma) \cup \bigcup_{\theta''\in I} e^{i\theta''} T_C(\delta)$. Indeed, if not, for every $\theta_0 \in I$ the set $T_B(\gamma) \cup e^{i\theta_0} T_C(\delta)$ can be contained in no T-set of A, hence there exist $b_{\theta_0} \in T_B(\gamma)$, $c_{\theta_0} \in T_C(\delta)$ such that $\|b_{\theta_0} + e^{i\theta_0} c_{\theta_0}\| < \|b_{\theta_0}\| + \|c_{0\theta}\|$. Hence, there exists a neighborhood $I(\theta_0)$ of θ_0 such that $\|b_{\theta_0} + e^{i\theta} c_{0\theta}\| < \|b_{\theta_0}\| + \|c_{\theta_0}\|$ for $\theta \in I(\theta_0)$; and finally I can be covered by a finite number of $I(\theta)$: $I \subset \bigcup_{k=1}^{n} I(\theta_k)$. Put $b = \sum_{i=1}^{n} b_{\theta_i}$ and $c = \sum_{i=1}^{n} c_{\theta_i}$ then $\|b + e^{i\theta} c\| < \|b\| + \|c\|$ for all θ, which contradicts to the additive relatedness of B and C.

Now, let the $\bigcup_{\theta} e^{i\theta} T_A(\omega)$ be a circular T-set of A which contains the $\bigcup_{\theta'\in I} e^{i\theta'} T_B(\gamma) \cup \bigcup_{\theta''\in I} e^{i\theta''} T_C(\delta)$, then we can show $T_B(\gamma) = T_A(\omega) \cap B$ and $T_C(\delta) = T_A(\omega) \cap C$. Indeed, if $b_1', b_2' \in T_B(\gamma)$ then $b_1'(\omega) = e^{i\theta_1}\|b_1'\|$ and $b_2'(\omega) = e^{i\theta_2}\|b_2'\|$ for some $\theta_1, \theta_2 \in I$; hence $b_1 = b_1'/\|b_1'\|$ and $b_2 = b_2'/\|b_2'\|$ belong to $T_B(\gamma)$ and $b_1(\omega) = e^{i\theta_1}$ and $b_2(\omega) = e^{i\theta_2}$. Since b_1 and b_2 belongs to $T_B(\gamma)$, $b_1 + b_2$ belongs to $T_B(\gamma)$; hence $b_1 + b_2 \in e^{i\theta} T_A(\omega)$ for some $\theta \in I$, that is $(b_1 + b_2)(\omega) = e^{i\theta}\|b_1 + b_2\| = 2e^{i\theta}$. Hence $e^{i\theta_1} + e^{i\theta_2} = 2e^{i\theta}$, therefore $\theta_1 = \theta_2 = \theta$, and since $1 \in T_B(\gamma)$ belongs to $T_A(\omega)$, $\theta_1 = \theta_2 = \theta = 0$; hence $T_B(\gamma) \subset T_A(\omega)$. Similarly $T_C(\delta) \subset T_A(\omega)$. Then clearly the pure state ω of A is a common extension of γ and δ, hence $\gamma \odot \delta$ can be extended continuously, that is B and C are mutually independent.

Implication (2)$\to$(1). We have proved this in Prop. 1.

Implication (1)$\to$(2). Let b and c be arbitrary elements of B and C respectively, then there exist $\theta', \theta'' \in I$, $\gamma \in \Gamma$ and $\delta \in \Delta$ such that $b(\gamma) = e^{i\theta'}\|b\|$, $c(\delta) = e^{i\theta''}\|c\|$. Now, if we define $\theta = \theta' - \theta''$ (mod 2π), $\theta \in I$ then $\|b\| + \|c\| \geqq \|b + e^{i\theta} c\| \geqq |(b + e^{i\theta} c)(\gamma, \delta)| = |b(\gamma) + e^{i\theta} c(\delta)| = |e^{i\theta'}\|b\| + e^{i(\theta+\theta'')}\|c\|| = \|b\| + \|c\|$, that is, B and C are additively related.

BIBLIOGRAPHY

[1] R. MACDOWELL, Banach spaces and algebras of continuous functions, Proc. Amer. Math. Soc., 6(1955), 67-78.
[2] M. NAKAMURA, On the direct product of finite factors, Tôhoku Math. Journ., 6(1954), 205-207.
[3] Z. TAKEDA, Inductive limit and infinite direct product of operator algebras, Tôhoku Math. Journ., 7(1955), 67-86.
[4] T. TURUMARU, On the direct product of operator algebras, I, Tôhoku Math. Journ., 4(1952), 242-251.

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