ON A THEOREM OF LINDELÖF CONCERNING PRIME ENDS

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A short proof of the following well known theorem of Lindelöf has been given by Tsuji [2].

THEOREM. *Let D be a bounded simply-connected domain, and let $w=f(z)$ map D conformally on $|w|<1$. If $\{z_n\}$ is a sequence of points of D such that the sequence $w_n=f(z_n)$ converges to a point α of $|w|=1$ in a Stolz angle, $|\arg(\alpha-w)|<\frac{1}{2}\pi-\delta$, then every limit-point of the sequence $\{z_n\}$ is a principal point*[1] *of the prime end of D which corresponds to α.*

In this note we show how the proof of the theorem may be simplified still further by using a very elementary topological argument.

Tsuji proves the theorem by combining the following results.

LEMMA A. *Let Dba bounded simply-connected domain, and let $w=f(z)$ map D conformally on $|w|<1$. Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_{n+1}\leqq\frac{1}{2}\rho_n<1$, and let S_n be the domain $\frac{1}{2}\rho_n<|1-w|<\rho_n$, $|w|<1$. Then we can find an increasing sequence $\{n_\nu\}$ of positive integers, and a chain $\{q_\nu\}$ of cross-cuts of D associated with the prime end of D which corresponds to $w=1$, such that the image of q_ν in $|w|<1$ is an arc of a circle with centre $w=1$ lying in S_{n_ν}.*

LEMMA B. *Let D be a bounded simply-connected domain, $F(D)$ its frontier, let $w=f(z)$ map D conformally on $|w|<1$, and let $z=\psi(w)$ be the inverse of f. Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_{n+1}\leqq\frac{1}{2}\rho_n<1$, and let T_n be the domain $\frac{1}{2}\rho_n<|w-1|<\rho_n$, $|w|<1$, $|\arg(1-w)|<\frac{1}{2}\pi-\delta<\frac{1}{2}\pi$. Then we can find an increasing sequence $\{n_\nu\}$ of positive integers such that the values of $\psi(w)$ in $\overline{T}_{n_\nu}$ converge to a point a of $F(D)$.*

The deduction of the theorem from these two lemmas is immediate. For we may suppose that $\alpha=1$, and that $\{z_n\}$ converges to a point a of $F(D)$. We can then find a chain of cross-cuts $\{q_\nu\}$, associated with the prime end of D which corresponds to $w=1$, converging to the point a, and this is the required result.

1) For the definition of this and other terms belonging to the theory of prime ends, see Carathéodory [1].

We have nothing new to add concerning the proof of Lemma A. In place of Tsuji's proof of Lemma B (which uses Lemma A), we have, however, the following argument.

We may evidently suppose ρ_0 so small that each of the domains U_n defined by the relations

$$\frac{1}{4}\rho_n < |1-w| < 2\rho_n, \quad |\arg(1-w)| < \frac{1}{2}\pi - \frac{1}{2}\delta$$

is completely contained in $|w| < 1$. Let

$$\psi_n(w) = \psi\{1 + \rho_0\rho_n^{-1}(w-1)\}.$$

The functions ψ_n are regular and uniformly bounded in U_0, and the values taken by ψ_n in T_0 and U_0 are the values taken by ψ in T_n and U_n respectively. By Montel's theorem, we can select a subsequence of the ψ_n which converges uniformly in any closed set in U_0, and, in particular, in $\overline{T}_0$, to a regular ϕ. If ϕ is not constant in U_0, the values taken by ϕ in U_0 form an open set. This set contains the area of some complete circle, and this is impossible, since it is contained in $F(D)$. Hence ϕ is constant in U_0, and this proves the lemma.

REFERENCES

[1] C. CARATHÉODORY, Über die Begrenzung einfach zusammenhangender Gebiete, Math. Ann., 73 (1913), 323–370.

[2] M. TSUJI, On the theorems of Carathéodory and Lindelöf in the theory of conformal representation, Jap. J. Math., 7 (1930), 91–99.

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