CAUCHY INTEGRAL FOR FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. A. Weil [L'intégrale de Cauchy et les fonctions de plusieurs variables, Math. Annalen, 111(1935), 178–182] has produced an integral representation for a holomorphic function f valid in a special kind of region. He integrates over an analytic chain of 2 dimensions (in the case of 2 variables). This chain has singularities on its boundary; but on the other hand it does lie on the boundary of the region in which the representation is valid.

Our object here is to integrate Weil's integrand over an analytic chain with no singularities, allowing ourselves to use a chain in a neighborhood of the boundary. The resulting theorem is perhaps less elegant than Weil's, but it is adequate for most applications. Moreover, the proof is much simpler since it does not require an appeal to a theorem on the triangulation of analytic loci.

The result is stated (for 2 variables) in 4.3 below. The extension to n variables can be obtained by comparing Weil's proof of his theorem for $n = 2$ with ours, and then considering his statement for the general case.

In a mimeographed note reporting on the Séminaire H. Cartan, 1951–52, VI, entitled "Intégrale d'André Weil" (14–1–1952), Michel Hervé discusses a different method (integration over an open set) of surmounting the difficulties presented by the singularities on the boundary. Another point, namely the question of the existence of the functions X_i, Y_i (see 3.1, below) is treated by K. Oka [Sur les fonctions analytiques de plusieurs variables, V. L'intégrale de Cauchy, Jap. Journ. of Math., 17(1941), 523–531] and (indepently) by H. Hefer [Zur Funktionentheorie mehrerer Veränderlichen; Math. Annalen, 122(1950), 276–278.]

2. Some Geometry. Let C^2 be the class of pairs $z = (x,y)$ of complex numbers x,y. Let $\mathfrak{D}$ be an open set in C^2. Let $P_1, \ldots, P_n$ be (complex valued) functions holomorphic on $\mathfrak{D}$. Let F_i, D_i $(i = 1, \ldots, n)$ be closed and open (respectively) subsets of the complex plane C with $F_i \subset D_i$. Let Φ_i be the set in $\mathfrak{D}$ where $P_i(z) \in F_i$; and let Δ_i be similarly defined with D_i. Suppose that

2.1 $\quad \Delta \equiv \Delta_1 \cap \ldots \cap \Delta_n$ *has* Δ^- *compact and inside* $\mathfrak{D}$.

With this understood, we shall prove the following.

2.2 *There exists a 4-dimensional polyhedron k lying inside Δ, containing $\Phi \equiv \Phi_1 \cap \ldots \cap \Phi_n$, and whose boundary ∂k can be written as*

2.3 $$\partial k = g_1 + \ldots + g_n$$

where

2.4 $$g_i \subset \Gamma_i \cap \Delta \quad \textit{where} \quad \Gamma_i = \Delta_i - \Phi_i.$$

PROOF. Let B be the frontier of Δ. One can find a positive d such that for any point z of B, the d-neighborhood of z relative to Δ lies entirely in some Γ_i (i depending on z). Now dissect C^2 into 4-cells of diameter less than d. Let k be the sum of those which lie in Δ. Let g be a 2-cell occurring on the boundary of k. It is a face of two 4-cells, one in k and the other not. Hence the other meets B whence it and g lie in some Γ_i. We can now collect the terms of ∂k into a sum $g_1 + \dots + g_n$ where g_i contains only 3-cells in Γ_i, q. e. d.

3. Some analytic forms. Besides the functions P_i of sec. 2, suppose there are functions X_i, Y_i holomorphic on $\mathfrak{D} \times \mathfrak{D}$ such that for z, z_0 in $\mathfrak{D}$ ($z = (x, y)$ etc.)

3.1 $$P_i(z) - P_i(z_0) = (x - x_0)X_i(z, z_0) + (y - y_0)Y_i(z, z_0).$$

(If the P_i are holomorphic on sets $A \times B$ containing $\mathfrak{D}$, then such functions as in 3.1 can be easily found.)

Let f be any function holomorphic on $\mathfrak{D}$.

Consider the (analytic) differential 2-forms

3.2 $$q_{ij} = \frac{f(z) \begin{vmatrix} X_i(z, z_0) & X_j(z, z_0) \\ Y_i(z, z_0) & Y_j(z, z_0) \end{vmatrix}}{8\pi^2 (P_i(z) - P_i(z_0))(P_j(z) - P_j(z_0))} dx\, dy$$

wherein z_0 is a parameter. These have the properties:

3.3 $$q_{ij} = -q_{ji}, \qquad q_{ij} + q_{jk} + q_{ki} = 0.$$

3.4 *If* z_0 *(belongs to* $\mathfrak{D}$ *but) is not on* $\Gamma \equiv \Gamma_1 \cup \dots \cup \Gamma_n$ *then* q_{ij} *is holomorphic on* $\Gamma_i \cap \Gamma_j$.

3.5 *If* z_0 *(belongs to* $\mathfrak{D}$ *but) lies outside* $\Gamma \cup \Delta_i$ *then* q_{ij} *is holomorphic on* $\Delta \cap \Gamma_j$.

These properties are easily established (see Weil). For the latter two, it is merely a matter of seeing when $P_i(z) \neq P_i(z_0)$, etc.

4. The integral. Suppose we have any 4-chain k satisfying the conditions of 2.2 (even though not constructed as in the proof). Let h_{ij} be the sum of those 2-cells that appear in both ∂g_i, ∂g_j, but with the sign as in the former. (For example let $k = a \times b$ where a, b are square 2-cells in the respective planes. Then $\partial k = \partial a \times b + a \times \partial b$. Suppose the first is g_1 and the second is g_2. Then $\partial g_1 = -\partial a \times \partial b$, $g_2 = \partial a \times \partial b$. Hence $h_{12} = g_1$. The appearance of the $-$ sign in ∂g_1 warns us that the calculus of combinatorial topology has to be taken seriously here.)

We define no h_{ii}. Then

4.1 $$\sum_j h_{ij} = \partial g_i, \quad h_{ij} = -h_{ji}.$$

For a given k and choice of $g_1, \dots, g_n$ we define

4.2 $$J(z_0) = \sum_{i,j} \int_{h_{ij}} q_{ij} \qquad \text{(Cauchy-Weil integral).}$$

(The i, j term is the same as the j, i term, by 4.1 and 3.3. This explains the 1/2 in

$$-\frac{1}{2}\frac{1}{(2\pi i)^2} = \frac{1}{8\pi^2}.$$

The $-$ is to take care of the $-$ in the example above.)

The integral 4.2 exists for z_0 as in 3.4 and evidently is a holomorphic function of z_0. For other values of z_0, it may exist as an *improper* integral.

4.3 THEOREM. *For z_0 in $\mathfrak{D}$ but not in $\Gamma \cup \Delta$ (see 2.1), $J(z_0) = 0$. For z_0 interior to Φ (see 2.2), $J(z_0) = f(z_0)$.*

PROOF. We treat first (as does Weil) the case z_0 not in $\Gamma \cup \Delta$, following (*mutatis mutandis*) the method of Weil. Suppose z_0 is not in some Δ_k, as must be for z not in Δ. Since $q_{ij} = q_{kj} - q_{ki}$

$$J(z_0) = \sum_{i,j} \int_{h_{ij}} q_{kj} - \sum_{i,j} \int_{h_{ij}} q_{ki} = \sum_{j} \int_{\partial g_j} q_{kj} - \sum_{i} \int_{-\partial g_i} q_{ki}$$

where we have used 4.1. By 3.5, q_{kj} is holomorphic on g_j. By the Cauchy-Poincaré integral theorem, a form taken around the boundary of a chain on which the form is holomorphic, gives 0. (One may prove this by observing that the exterior differential dq_{kj}, as a form in R^4, is 0, and using the Green-Cartan integral theorem: $\int_{\partial g} q = \int_{g} dq$.) This proves the first half of 4.3. We do not use this half (as Weil does) to establish the second half, as this $J(z_0)$ is not precisely a "fonction additive de domaine," but we do defer the proof until sec. 5.

4.4 LEMMA. *Let z_0 be in $\mathfrak{D}$ but not in Γ. Then the value of 4.2 is independent of how (for a given k) we choose the expression 2.3 satisfying 2.4.*

It will suffice to show that if

$$k = g_1 + g_2 + g_3 + \dots + g_n = g_1' + g_2' + g_3 + \dots + g_n$$

then 4.2 is the same for both. It will have to be that

4.5 $$g_1 - g_1' = g_2' - g_2 = g \subset \Gamma_1 \cap \Gamma_2.$$

Let h_1, h_2, h_k be the parts of ∂g which it has in common with $\partial g_1'$, ∂g_2, and ∂g_k ($k = 3, 4, \dots$). Then these latter three are of form $a - h_1$, $b - h_2$, $c - h_k$ where a, b, c do not share anything with ∂g. They may be ignored when we want to determine the *change* in h_{12}. Doing so, we obtain

$$\partial g_1 = h_2 + \dots + h_k + \dots \qquad \partial g_1' = -h_1$$

$$\partial g_2 = -h_2 \qquad \partial g_2' = h_1 + h_3 + \ldots + h_k + \ldots$$
$$\partial g_k = -h_k \qquad \partial g_k' = -h_k.$$

Hence (still ignoring a, b, c), $h_{12} = h_2, h'_{12} = -h_1$ so $h'_{12} - h_{12} = -h_1 - h_2$. Next, $h_{2k} = 0$, $h'_{2k} = h_k$, so $h'_{2k} - h_{2k} = +h_k$. Finally, $h_{k1} = -h_k$, $h'_{k1} = 0$, so $h'_{k1} - h_{k1} = h_k$. The change in $J(_{k2})$ is thus twice

$$\int_{-h_1-h_2} q_{12} + \sum_{k>2} \int_{h_k} (q_{2k} + q_{k1})$$
$$= \int_{-h_1-h_2} q_{12} + \sum_{k>2} \int_{h_k} (-q_{12}) \qquad \text{(by 3.3)}$$
$$= -\int_{g} q_{12} \qquad \text{(by 4.5)}$$
$$= 0 \text{ (q. e. d.)}$$

This enables us to prove the following.

4.6 LEMMA. *If the system $P_1, F_1, D_1; \ldots P_n, F_n, D_n$ is altered insofar as some of the sets F_i, D_j are diminished maintaining however the proper inclusion relation, then a new chain k^* can be found such that for z_0 neither in Γ nor in Γ^* (* referring to the altered system), there holds $J^*(z_0) = J(z_0)$.*

PROOF: If merely some F_i is diminished, the old k may be retained. It is clearly enough to consider beyond this only the case in which some D_1 is diminished to D_1^*.

Let us first subdivide k into pieces so small that none of them maps (under P_1) into a set meeting both F_1 and the outside of D_1^*. This gives a chain k_1 and

$$J_k(z_0) = J_{k_1}(z_0).$$

The cells of ∂k_1 which do not meet Φ_1 can all be placed in g_1 without affecting $J_{k_1}(z_0)$, by 4.4. (This rearrangement changes the h_{ij} of course.) Eject from k_1 all cells not in Δ^*, and you have k^*. The faces of these ejected cells enter only into g_1 and g^*. Indeed $\partial(k_1 - k^*) = g_1 - g_1^*$, whence $\partial g_1 = \partial g_1^*$. Therefore this passage from k_1 to k^* affects neither the h_{ij} nor $J(z_0)$.

5. **Proof of 4.3.** In applications, the functions $P_1(z) = x$, $P_2(z) = y$ will usually be present among the $P_1, \ldots, P_n$. If either one is not present, it can be "adjoined" (say P_1) together with the sets F_1, D_1 where F_1 is so large as to contain all the x-values of z on the compact set Δ^- (Δ being based on the unaugmented system). This augmentation does not change Δ, Φ, nor Γ and in fact the (formally) created Γ_1 will be void, so that 4.2 cannot involve the adjoined function.

Therefore let us suppose these P_1, P_2 (as above) are in our system. Let z_0 be interior to Φ (or more generally, as there need be no constant P_i) let $P_i(z_0)$ be interior to F_i for each i. Then for some positive d,

$$|x-x_0|,\ |y-y_0|<d \text{ gives } P_i(z) \text{ in } F_i \text{ for } i=3,4,\ldots..$$

Let us replace D_1, D_2 by d-nbds of x_0, y_0; and replace F_1, F_2 by the points x_0, y_0. By 4.6, the integral retains its old value. But now $\Gamma_i^* \cap \Gamma_j^*$ is void for $i=1,2$; $j=3,4,\ldots..$ Hence 4.2 reduces to

5.1 $$J(z_0)=2\int_{h_{12}} q_{12}=\frac{1}{4\pi^2}\iint_{h_{12}}\frac{f(x,y)\,dx\,dy}{(x-x_0)(y-y_0)}$$

where $h_{12}=\partial g_1$, $g_1+g_2=\partial k^*$, $g_i\subset\Gamma_i^*$ $(i=1,2)$.

For *any* 4-chain k, whose boundary misses (x_0,y_0), split ∂k into terms g_1+g_2 where $x \neq x_0$ on g_1, $y\neq y_0$ on g_2. Define

5.2 $$I(k)=\frac{1}{4\pi^2}\iint_{\partial g_1}\frac{f(x,y)\,dx\,dy}{(x-x_0)(y-y_0)}.$$

Any arbitrariness in the selection of g_1 involves only terms g on which $x\neq x_0$, $y\neq y_0$ and has no effect on 5.2 since

$$\int_{\partial g}=0.$$

Thus 5.2 depends only on k and $I(k_1+k_2)=I(k_1)+I(k_2)$.

Let 4-cells $k_1,k_2,\ldots.$ be added to k^* until the sum is a 4-cube of the form

$$a\times b=k^*+k_1+k_2+\ldots..$$

(see the example in sec. 4), where a,b are solid squares about x_0, y_0 contained in D_1^*, D_2^* respectively. It follows that

$$J(z_0)\ (=I(k^*))=I(a\times b).$$

Now

$$\partial(a\times b)=\partial a\times b+a\times\partial b;$$

and the former has $x\neq x_0$ on it, and its boundary is

$$-\partial a\times\partial b.$$

As a result,

$$I(a\times b)=\frac{1}{(2\pi i)^2}\iint_{\partial a\,\partial b}\frac{f(x,y)}{(x-x_0)(y-y_0)}\,dx\,dy=f(x_0,y_0).$$

Thus 4.3 is completely proved.

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