ON GENERAL ERGODIC THEOREMS II

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1. Introduction. E. Hopf [6] has established a pointwise ergodic theorem which asserts the convergence almost everywhere of averages $\frac{1}{n}\sum_{j=0}^{n-1} T^j f$ where T is an operator defined by a Markov process with an invariant distribution and where f is an integrable function. Recently this theorem has been extended to one for more general operator by N. Dunford and J. T. Schwartz [4]. We shall here observe the convergence almost everywhere of averages $\sum_{j=0}^{n-1} T^j f \Big/ \sum_{j=0}^{n-1} T^j g$ where T is a linear positive operator with some restrictions and where f and g are integrable and g is positive almost everywhere.

2. Notations and preliminaries. Let $(X, \mathfrak{F}, \mu)$ be a finite measure space such that X is a set and $\mathfrak{F}$ a σ-field consisting of subsets of X and μ a non-negative countably additive set function defined on $\mathfrak{F}$ and $\mu(X) < +\infty$.

Throughout this paper, "measurable", "almost all (almost everywhere)" and "integrable" mean "$\mathfrak{F}$-measurable", "μ-almost all (μ-almost everywhere)" and "μ-integrable", respectively, and every function under consideration is real-valued.

We denote by $L_1(A)$ the Lebesgue space of measurable integrable functions f defined on $A \in \mathfrak{F}$, the norm being

$$|f|_1 = \int_A |f(x)|\mu(dx),$$

and by $L_\infty(A)$ the Lebesgue space of measurable essentially bounded functions f defined on $A \in \mathfrak{F}$, the norm being

$$|f|_\infty = \operatorname*{ess\,sup}_{x \in A} |f(x)|.$$

If $A = X$, we drop "X" in $L_1(X)$ and $L_\infty(X)$ and write L_1 and L_∞.

Let f and g be measurable and $A \in \mathfrak{F}$. If $f(x) \geqq g(x)$ for almost all $x \in A$, we write "$f \geqq g$ in A". Further, "$f > g$ in A" and "$f = g$ in A" are defined in like manner. If $A = X$, we drop the term "in X".

Let T be a linear operator of L_p into itself where $p = 1$ or ∞. If T is a continuous operator, the operator norm of T is defined as usual and denoted by $|T|_p$. The operator T is called positive provided that $Tf \geqq 0$ for every $f \in L_p$ with $f \geqq 0$. A set $A \in \mathfrak{F}$ is called T-invariant provided that

$$T(f \cdot e_A) = Tf \quad \text{in } A$$

for every $f \in L_p$, where e_A denotes the charactaristic function of A.

For a set A we consider now the contraction of T related to A. The contraction T_A is defined by

$$T_A f = e_A \cdot T(f \cdot e_A)$$

for $f \in L_p$. Then, for every T-invariant set A, it is a simple matter to show that

$$T_A^j f = T^j f \quad \text{in } A$$

for every $f \in L_p$ and for $j = 0, 1, 2, \ldots$. This means that every T-invariant set can be considered as a new whole space as far as the operator T is concerned.

We denote by $Tf(x)$ a value of Tf at a place $x \in X$ and by T_n the sum of operators $\sum_{j=0}^{n-1} T^j$, that is,

$$T_n f = \sum_{j=0}^{n-1} T^j f$$

for every $f \in L_p$.

We shall then state the maximal ergodic theorem which plays a fundamental rôle to prove the ergodic theorem.

THEOREM 2.1. *Let T be a linear positive operator of L_1 into itself with $|T|_1 \leqq 1$. For any functions $f \in L_1$ and $g \in L_1$ with $g > 0$ and for any real numbers α and β, let*

$$A^*(\alpha) = \left\{ x ; \sup_{1 \leqq n < \infty} \frac{T_n f(x)}{T_n g(x)} \geqq \alpha \right\},$$

$$A_*(\beta) = \left\{ x ; \inf_{1 \leqq n < \infty} \frac{T_n f(x)}{T_n g(x)} \leqq \beta \right\}.$$

Then

$$\alpha \int_{A^*(\alpha)} g(x)\, \mu(dx) \leqq \int_{A^*(\alpha)} f(x)\, \mu(dx),$$

(2.1)

$$\beta \int_{A_*(\beta)} g(x)\, \mu(dx) \geqq \int_{A_*(\beta)} f(x)\, \mu(dx).$$

If a set A is T-invariant, the sets $A^*(\alpha)$ and $A_*(\beta)$ in (2.1) are replaced by the sets $A \cap A^*(\alpha)$ and $A \cap A_*(\beta)$, respectively.

The assumption for T in Theorem 2.1 is somewhat weaker than that in the maximal ergodic theorem in [6]. The first part of Theorem 2.1 is immediately deduced from Lemma 3.2 in [4] which is a slight generalisation of Theorem 7.1 in [6][1]. The second part is easily seen from the fact that every

1) See Appendix.

T-invariant set can be considered as a whole space as far as T is concerned.
We shall next state the decomposition theorem.

LEMMA 2.2. *Let T be a linear positive operator of L_1 into itself with $|T|_1 \leqq 1$. Then the space X splits into two disjoint measurable sets C and D with the properties*:

$$\sum_{j=0}^{\infty} T^j f = +\infty \text{ in } C \qquad \text{for every } f \in L_1 \text{ with } f > 0; \tag{2.2}$$

$$\sum_{j=0}^{\infty} T^j f < +\infty \text{ in } D \qquad \text{for every } f \in L_1 \text{ with } f \geqq 0. \tag{2.3}$$

Further the set D is T-invariant.

The sets C and D are called the conservative and dissipative part of X, respectively. The assumption for T in Lemma 2.2 is somewhat weaker than that in [6]. The decomposition of X into C and D is proved from Theorem 2.1 by the same way as in the proof of Theorem 8.1 in [6]. Further we can prove, by the same way as in the proof of Lemma 8.2 in [6], that $Tf = 0$ in D for every $f \in L_1$ such that $f = 0$ in D. Hence $T(f - f \cdot e_D) = 0$ in D for every $f \in L_1$ where e_D denotes the characteristic function of D, so that D is a T-invariant set.

3. Ergodic theorem. Let T be an operator with the properties:

(i) T is a linear positive operator of L_1 into itself;

(ii) $$|T|_1 \leqq 1;$$

(iii) $$T1 > 0.$$

If we set here

$$Uf = \frac{Tf}{T1}$$

for every $f \in L_1$, then U is a linear positive operator such that U maps the functions in L_1 to the measurable functions and maps L_∞ into itself and $U1 = 1$.

Further, assume that U satisfies the properties:

(iv) $Uf = f$ for every $f \in L_\infty$ such that $Uf \geqq f$;

(v) $U(f \cdot T^j 1) = Uf \cdot UT^j 1$ for every $f \in L_\infty$ and $j = 0, 1, 2, \ldots$.

The assumptions (iv) and (v) for T are artificial in view of operator theory, but they are of some significance in connection to a Markov process and to a measurable point transformation. Hopf [6] formulated the ergodic theorem for a Markov process with an invariant distribution μ in terms of the operator T with the properties:

(3.1) T is a linear positive operator of L_1 into itself;

$$\int_X Tf(x)\,\mu(dx) = \int_X f(x)\,\mu(dx) \qquad \text{for every } f \in L_1; \tag{3.2}$$

$$T1 = 1. \tag{3.3}$$

Then (i), (ii) and (iii) follow from (3.1), (3.2) and (3.3), respectively. Further, $U = T$ by virtue of (3.3), so that (iv) and (v) are shown by (3.2) and (3.3), respectively.

We shall next consider the case of a measurable incompressible point transformation. Let φ be a single-valued point transformation of X into itself. The transformation φ is called measurable if φ and its inverse φ^{-1} send the sets in $\mathfrak{F}$ to the sets in $\mathfrak{F}$ and the sets of measure zero to the sets of measure zero. Then the set function $\mu(\varphi A')$ of a variable A' is a measure as long as $A' \in \varphi^{-1}\mathfrak{F} = \{\varphi^{-1}A; A \in \mathfrak{F}\}$, and $\mu(\varphi A')$ is absolutely continuous with respect to μ on $\varphi^{-1}\mathfrak{F}$ and conversely. Hence by the Radon-Nikodym theorem there exists a $\varphi^{-1}\mathfrak{F}$-measurable function $w > 0$ such that

$$\mu(\varphi A') = \int_{A'} w(x)\mu(dx)$$

for every $A' \in \varphi^{-1}\mathfrak{F}$. Then it is a simple matter to show that

$$\int_X f(\varphi x)w(x)\,\mu(dx) = \int_X f(x)\,\mu(dx) \tag{3.4}$$

for every $f \in L_1$. Now we define the operator T induced by φ upon setting

$$Tf(x) = f(\varphi x)w(x)$$

for every $f \in L_1$. Then $T1 = w > 0$, and $Uf(x) = f(\varphi x)$ for every $f \in L_1$, and further $U(f\cdot g) = Uf \cdot Ug$ for every $f \in L_\infty$ and every $g \in L_1$. From this and (3.4) it is shown that T satisfies (i), (ii), (iii) and (v). Further, let φ be now incompressible, that is, if $A \in \mathfrak{F}$ and $\varphi^{-1}A \supset A$ then $\mu(\varphi^{-1}A - A) = 0$ or, equivalently, if $A \in \mathfrak{F}$ and $A \cap T^n A = 0$ for $n = \pm 1, \pm 2, \ldots.$ then $\mu(A) = 0$. Then we shall prove that (iv) holds. Suppose now that $Uf \geqq f$. Then, for every real α, it holds that $\varphi^{-1}\{x; f(x) > \alpha\} = \{x; Uf(x) > \alpha\} \supset \{x; f(x) > \alpha\}$. Hence, by the incompressibility of φ, it follows that $\{x; Uf(x) > \alpha\} = \{x; f(x) > \alpha\}$ execept a set of measure zero. From this we can easily show that $Uf = f$. Thus T satisfies (iv).

Under these considerations we state an ergodic theorem which contains the Hopf ergodic theorem for a Markov process with an invariant distribution and the Hurewicz ergodic theorem without invariant measure [2] (cf. [7], [5], [9]).

THEOREM 3.1. *Let T be an operator with the properties* (i) ~ (v). *Then, for every $f \in L_1$ and every $g \in L_1$ with $g > 0$, the sequence of averages*

$$\frac{T_n f(x)}{T_n g(x)}$$

converges for almost all $x \in X$. For the limit function h it holds that

$$\int_A h(x)g(x)\,\mu(dx) = \int_A f(x)\,\mu(dx) \tag{3.5}$$

for every T-invariant subset A of C where C denotes the conservative part of X with respect to T. [2)]

4. Proof of Theorem 3.1. Throughout this section let T be an operator with the properties in Theorem 3.1, that is, T satisfies (i) $\sim$ (v).

From the definition (in Lemma 2.2) of the dissipative part D of X it follows that $\sum_{j=0}^{\infty} T^j|f| < +\infty$ in D for every $f \in L_1$ and that $\sum_{j=0}^{\infty} T^j g < +\infty$ in D for every $g \in L_1$ with $g > 0$. Hence *the sequence of averages $T_n f/T_n g$ converges almost everywhere in D.* The conservative part C is the vital part as far as the ergodic theory is concerned, and the essential part of the proof of Theorem 3.1 is to prove the convergence in C of averages $T_n f/T_n g$ and to prove (3.5).

We note a fact which will be used often in the sequel without references.

"*If a sequence of functions $f_n \in L_1$ is monotone increasing or decreasing and tends to a function $f \in L_1$ almost everywhere, then* $\lim_n Tf_n(x) = Tf(x)$ *almost everywhere and, a fortiori,* $\lim_n Uf_n(x) = Uf(x)$ *almost everywhere.*"

LEMMA 4.1. *A set A is T-invariant if and only if $Ue_A = e_A$.*

PROOF. Assume that $Ue_A = e_A$. In order to show the T-invariance of A it suffices to prove that $Tf = T(f\cdot e_A)$ in A for every f with $0 \leqq f \leqq 1$. Since $0 \leqq f - f\cdot e_A \leqq e_{A^c}$ where A^c denotes the complement of A, $0 \leqq U(f - f\cdot e_A) \leqq Ue_{A^c} = U1 - Ue_A = 1 - e_A = e_{A^c}$, so that $U(f - f\cdot e_A) = 0$ in A. Hence $Tf = T(f\cdot e_A)$ in A.

Next assume that A is T-invariant. Then $Te_A = T1$ in A. Hence $Ue_A = 1$ in A, so that $Ue_A \geqq e_A$. Thus, by the property (iv) of T, $Ue_A = e_A$.

q. e. d.

It is easily seen, directly from the definition of T-invariance or by use of Lemma 4.1, that *the intersection, the union, the complement and the limit of T-invariant sets are all T-invariant.* This result will be used in the sequel without references.

LEMMA 4.2. *The conservative part C of X is a T-invariant set.*

PROOF. By Lemma 2.2, the dissipative part D is a T-invariant set. Since C is the complement of D, C is T-invariant. q. e. d.

Hence the conservative part C can be considered as a whole space. Especially we note here that the properties (i) $\sim$ (v) remain true even if X, T, U, L_1 and L_∞ in the descriptions of the properties are replaced by C, T_C, U_C, $L_1(C)$ and $L_\infty(C)$, respectively. Then such properties contracted to C will

2) It will be shown in Lemma 4.2 that C is a T-invariant set. We note here that if T is the operator induced by a measurable, incompressible, one-to-one point transformation then $C = X$ except a set of meaure zero [7], [5], but in general it is not true [10].

be referred by the same numbers (i) ~ (v), and T and U will be used instead of T_C and U_C without confusion.

A function $f \in L_\infty(C)$ is called U-invariant provided that $Uf = f$ in C. Then we may state the analogue to Theorem 9.1 in [6].

LEMMA 4.3. *If a function $h \in L_\infty(C)$ is U-invariant, then, for every real α, $\{x \in C;\ h(x) > \alpha\}$ is T-invariant.* [3]

PROOF. If $f \in L_\infty(C)$ is U-invariant, $|f| = |Uf| \leqq U|f|$ in C. Then, by (iv), $|f| = U|f|$ in C and hence $f^+ = Uf^+$ in C[4].

Suppose now that $h \in L_\infty(C)$ is U-invariant. Let $A = \{x \in C;\ h(x) > \alpha\}$. $[n(h-\alpha)]^+$ and $[n(h-\alpha)-1]^+$ are U-invariant, and the sequence $\{[n(h(x)-\alpha)]^+ - [n(h(x)-\alpha)-1]^+\}$ is monotone increasing and tends Then the functions to $e_A(x)$ for almost all $x \in C$ as $n \to +\infty$, so that $Ue_A = e_A$ in C. Hence, by Lemma 4.1, the set A is T-invariant. q. e. d.

LEMMA 4.4 *For every $f \in L_\infty(C)$ and for every real α and β, the sets $\left\{x \in C;\ \limsup_n \frac{T_n f(x)}{T_n 1(x)} > \alpha\right\}$ and $\left\{x \in C;\ \liminf_n \frac{T_n f(x)}{T_n 1(x)} < \beta\right\}$ are T-invariant.*

PROOF. If we set

$$h(x) = \limsup_n \frac{T_n f(x)}{T_n 1(x)},\quad h_n(x) = \sup_{k \geqq n} \frac{T_k f(x)}{T_k 1(x)},\quad n = 1, 2, \dots .$$

for $f \in L_\infty(C)$, then $h \in L_\infty(C)$ and $h_n \in L_\infty(C)$, so that Uh and Uh_n's are well defined. By repeated uses of (v) and by (iii) we have

$$Uh_n \geqq U\left(\frac{T_n f}{T_n 1}\right) = \frac{UT_n f}{UT_n 1} = \frac{\sum_{j=0}^{n-1} UT^j f}{\sum_{j=0}^{n-1} UT^j 1}$$

$$= \frac{\sum_{j=0}^{n-1} T1 \cdot UT^j f}{\sum_{j=0}^{n-1} T1 \cdot UT^j 1} = \frac{T_{n+1} f - f}{T_{n+1} 1 - 1} \qquad \text{in } C.$$

Since $h_n(x)$ is monotone decreasing and tends to $h(x)$ for almost all $x \in C$ and $\sum_{j=0}^{\infty} T^j 1 = +\infty$ in C, it follows that

$$Uh(x) = \lim_n Uh_n(x) \geqq \limsup_n \frac{T_{n+1} f(x) - f(x)}{T_{n+1} 1(x) - 1}$$

$$= \limsup_n \frac{T_n f(x)}{T_n 1(x)} = h(x) \qquad \text{in } C.$$

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3) The converse of Lemma 4.3 is also valid, that is, if every $\{x \in C;\ h(x) > \alpha\}$ is T-invariant, h is a U-invariant function. However this fact is not used in this paper.

4) The symbol f^+ denotes the positive part of f, that is, $f^+ = \max(f, 0)$.

Hence, by the property (iv), h is a U-invariant function, so that, for every real α, $\{x \in C;\ h(x) > \alpha\}$ is T-invariant by virtue of Lemma 4.3.

Since $\left\{x \in C;\ \liminf_n \frac{T_nf(x)}{T_n1(x)} < \beta\right\} = \left\{x \in C;\ \limsup_n \frac{T_n(-f)(x)}{T_n1(x)} > (-\beta)\right\}$, the T-invariance of $\left\{x \in C;\ \liminf_n \frac{T_nf(x)}{T_n1(x)} < \beta\right\}$ follows from the fact proved above. q. e. d.

It is convenient in the sequel to prove here Theorem 3.1 assuming the T-invariance of sets $\left\{x \in C;\ \limsup_n \frac{T_nf(x)}{T_ng(x)} > \alpha\right\}$ and $\left\{x \in C;\ \liminf_n \frac{T_nf(x)}{T_ng(x)} < \beta\right\}$.

LEMMA 4.5. *Let $f \in L_1(C)$, $g \in L_1(C)$ and $g > 0$ in C. Assume that, for every real α and β, the sets $\left\{x \in C;\ \limsup_n \frac{T_nf(x)}{T_ng(x)} > \alpha\right\}$ and $\left\{x \in C;\ \liminf_n \frac{T_nf(x)}{T_ng(x)} < \beta\right\}$ are T-invariant. Then the sequence of averages $T_nf(x)/T_ng(x)$ converges for almost all $x \in C$. For the limit function h it holds that*

$$\int_A h(x)g(x)\,\mu(dx) = \int_A f(x)\,\mu(dx) \tag{4.1}$$

for every T-invariant subset A of C.

PROOF. For every real α and β with $\alpha > \beta$ we set

$$A_{\alpha\beta} = \left\{x \in C;\ \limsup_n \frac{T_nf(x)}{T_ng(x)} > \alpha > \beta > \liminf_n \frac{T_nf(x)}{T_ng(x)}\right\},$$

then every $A_{\alpha\beta}$ is T-invariant. Hence we can take $A_{\alpha\beta}$ as a T-invariant set in Theorem 2.1 and $A_{\alpha\beta} \cap A^*(\alpha) = A_{\alpha\beta}$, $A_{\alpha\beta} \cap A_*(\beta) = A_{\alpha\beta}$, so that by Theorem 2.1 we obtain that

$$\alpha \int_{A_{\alpha\beta}} g(x)\,\mu(dx) \leqq \int_{A_{\alpha\beta}} f(x)\,\mu(dx) \leqq \beta \int_{A_{\alpha\beta}} g(x)\,\mu(dx).$$

Since $\alpha > \beta$, $\int_{A_{\alpha\beta}} g(x)\,\mu(dx) = 0$, and then since $g > 0$, $\mu(A_{\alpha\beta}) = 0$.

On the other hand we set

$$A(+\infty) = \left\{x \in C;\ \sup_{1 \leqq n < \infty} \frac{T_nf(x)}{T_ng(x)} = +\infty\right\}.$$

Then, for every positive α, $A(+\infty) \subset C \cap A^*(\alpha)$, so that by Theorem 2.1 we obtain that

$$\int_{A(+\infty)} g(x)\,\mu(dx) \leqq \int_{C\cap A^*(\alpha)} g(x)\,\mu(dx) \leqq \frac{1}{\alpha}\int_{C\cap A^*(\alpha)} f(x)\,\mu(dx) \leqq \frac{1}{\alpha}\int_C |f(x)|\,\mu(dx).$$

Hence we have $\int_{A(+\infty)} g(x)\,\mu(dx) = 0$ upon letting tend to $+\infty$, and hence $\mu(A(+\infty)) = 0$. Similarly we obtain $\mu(A(-\infty)) = 0$ where

$$A(-\infty) = \left\{x \in C;\ \inf_{1\leqq n<\infty} \frac{T_nf(x)}{T_ng(x)} = -\infty\right\}.$$

Now let B denote the set where $\{T_nf(x)/T_ng(x)\}$ diverges. Then

$$B \subset \bigcup_{\substack{\alpha>\beta \\ \alpha,\beta:\text{rational}}} A_{\alpha\beta} \cup A(+\infty) \cup A(-\infty).$$

Since $A_{\alpha\beta}$, $A(+\infty)$ and $A(-\infty)$ are all of measure zero, we have $\mu(B) = 0$, so that the sequence $\{T_nf(x)/T_ng(x)\}$ converges for almost all $x \in C$.

Let h be the limit function of the sequence $\{T_nf(x)/T_ng(x)\}$ and A a T-invariant subset of C. If we set

$$A_k = \left\{x \in A;\ \frac{k}{n} \leqq h(x) < \frac{k+1}{n}\right\}, \qquad -n^2 \leqq k < n^2,$$

then every A_k is a T-invariant set and $A_k \cap A^*\left(\frac{k}{n}\right) = A_k$, $A_k \cap A_*\left(\frac{k+1}{n}\right) = A_k$. Thus by Theorem 2.1 we have that

$$\sum_{k=-n^2}^{n^2-1} \int_{A_k} \frac{k}{n} \cdot g(x)\,\mu(dx) \leqq \sum_{k=-n^2}^{n^2-1} \int_{A_k} f(x)\,\mu(dx) \leqq \sum_{k=-n^2}^{n^2-1} \int_{A_k} \frac{k+1}{n} \cdot g(x)\,\mu(dx).$$

Since A_k's are mutually disjoint and $\bigcup_{k=-n^2}^{n^2-1} A_k \to A$ as $n \to +\infty$,

$$\left| \int_{\bigcup_k A_k} h(x) g(x)\,\mu(dx) - \int_{\bigcup_k A_k} f(x)\,\mu(dx) \right| \leqq \frac{1}{n} \int_{\bigcup_k A_k} g(x)\,\mu(dx) \leqq \frac{1}{n} \int_C g(x)\,\mu(dx).$$

Thus we get (4.1) upon letting n tend to $+\infty$. q. e. d.

Let $M(A)$ denote the space consisting of all measurable functions f defined on the set $A \in \mathfrak{F}$, the quasi-norm being

$$|f|_M = \int_A \frac{|f(x)|}{1+|f(x)|}\,\mu(dx).$$

LEMMA 4.6. *Let every S_n $(n = 1, 2, \ldots.)$ be a linear continuous operator of $L_1(A)$ into $M(A)$. Assume that*

(4.2) $\sup_{1\leqq n<\infty} |S_nf(x)| < +\infty$ *in A for every $f \in L_1(A)$;*

(4.3) *for every f in a dense set of $L_1(A)$, the sequence $\{S_nf(x)\}$ converges for almost all $x \in A$.*

Then, for every $f \in L_1(A)$, the sequence $\{S_nf(x)\}$ converges for almost all $x \in A$.

This lemma is due to S. Banach [1] (cf. [8], [3]).

LEMMA 4.7. *For every $f \in L_1(C)$ the sequence of averages $T_n f(x)/T_n 1(x)$ converges for almost all $x \in C$. If $f \in L_\infty(C)$, the limit function h satisfies that*

$$\int_C h(x)\,\mu(dx) = \int_C f(x)\,\mu(dx).$$

If $f \in L_1(C)$ and $f > 0$ in C, the limit function is also > 0 in C.

PROOF. If $f \in L_\infty(C)$, it follows from Lemma 4.4 that, for every real α and β, the sets $\left\{ x \in C;\ \limsup_n \frac{T_n f(x)}{T_n 1(x)} > \alpha \right\}$ and $\left\{ x \in C;\ \liminf_n \frac{T_n f(x)}{T_n 1(x)} < \beta \right\}$ are T-invariant. Hence, by Lemma 4.5, the sequence $\{T_n f(x)/T_n 1(x)\}$ converges for almost all $x \in C$ and the limit function h satisfies that

$$\int_C h(x)\,\mu(dx) = \int_C f(x)\,\mu(dx).$$

Let $S_n (n = 1, 2, \ldots.)$ be an operator defined by

$$S_n f = \frac{T_n f}{T_n 1}$$

for $f \in L_1(C)$. Since $|S_n f|_M \leqq |S_n f|_1 \leqq n|f|_1$ for every $f \in L_1(C)$, S_n is a linear continuous operator of $L_1(C)$ into $M(C)$. It was already proved that, for every $f \in L_\infty(C)$, $\{S_n f(x)\}$ converges for almost all $x \in C$. Here we note that $L_\infty(C)$ is dense in $L_1(C)$. Further, as was shown in the proof of Lemma 4.5, it holds that, for every $f \in L_1(C)$, $\sup_{1 \leqq n < \infty} |S_n f(x)| < +\infty$ in C. Thus S_n's satisfy (4.2) and (4.3) in Lemma 4.6. Hence by Lemma 4.6 we conclude that, for every $f \in L_1(C)$, the sequence of averages $S_n f(x)$ ($= T_n f(x)/T_n 1(x)$) converges for almost all $x \in C$.

It remains to prove that if $f \in L_1(C)$ and $f > 0$ in C, the limit function h of $\{T_n f(x)/T_n 1(x)\}$ is > 0 in C. If we set

$$A(0) = \{x \in C;\ h(x) = 0\},$$

then by Theorem 2.1 we have that

$$\int_{A(0)} f(x)\,\mu(dx) \leqq \int_{C \cap A_*(0)} f(x)\,\mu(dx) \leqq 0 \cdot \mu(C \cap A_*(0)) = 0.$$

Since $f > 0$ in C, $\mu(A(0)) = 0$, as was to be proved. q. e. d.

From Lemma 4.7 it follows that, *for every $f \in L_1(C)$ and every $g \in L_1(C)$ with $g > 0$ in C, the sequence of averages $T_n f(x)/T_n g(x)$ converges for almost all $x \in C$.* In fact, $\lim_n \frac{T_n g(x)}{T_n 1(x)} > 0$ in C and hence the limit

$$\lim_n \frac{T_n f(x)}{T_n g(x)} = \frac{\lim_n \frac{T_n f(x)}{T_n 1(x)}}{\lim_n \frac{T_n g(x)}{T_n 1(x)}}$$

exists and is finite for almost all $x \in C$.

Thus for the proof of Theorem 3.1 it remains only to prove (3.5).

LEMMA 4.8. *Let S be the operator defined by*

$$Sf(x) = \lim_n \frac{T_n f(x)}{T_n 1(x)}$$

for every $f \in L_1(C)$. Then S is a linear positive continuous operator of $L_1(C)$ into itself.

PROOF. It is clear that S maps $L_\infty(C)$ and $L_1(C)$ into $L_\infty(C)$ and $M(C)$, respectively, and is linear and positive. By Lemma 4.7 it holds that $|Sf|_1 = |f|_1$ for every $f \in L_\infty(C)$. Hence S, considered as an operator defined on $L_\infty(C)$, has an unique extension to $L_1(C)$, denoted by $\tilde{S}$, such that $\tilde{S}f = Sf$ in C for $f \in L_\infty(C)$ and $\tilde{S}$ is a linear positive operator of $L_1(C)$ into itself with $|\tilde{S}|_1 = 1$. Thus, for the proof of Lemma 4.8 it suffices to prove that $Sf = \tilde{S}f$ in C for every $f \in L_1(C)$.

We define $S_n (n = 1, 2, \ldots.)$ by

$$S_n f = \frac{T_n f}{T_n 1}$$

for $f \in L_1(C)$. Then every S_n is a linear continuous operator of $L_1(C)$ into itself with $|S_n|_1 \leqq n$. We note here that, a fortiori, each one of S_n $(n = 1, 2, \ldots.)$ and $\tilde{S}$ is a linear continuous operator of $L_1(C)$ into $M(C)$.

Let ε be an arbitrary positive number. We set

$B_k = \{f \in L_1(C);\ |S_i f - S_j f|_M \leqq \varepsilon$ for all $i \geqq k$ and all $j \geqq k\}$, $k = 1, 2, \ldots.$.

Since, for every $f \in L_1(C)$, $\{S_n f(x)\}$ converges almost everywhere in C and hence $|S_i f - S_j f|_M \to 0$ as $i, j \to +\infty$, it follows that $L_1(C) = \bigcup_{k=1}^{\infty} B_k$. Further, since every S_n is a continuous operator, every B_k is closed in $L_1(C)$. Hence, by the Baire category theorem, there exists a B_{k_0} of the second category which contains a closed sphere whose center is $f_0 \in L_1(C)$ and radius is $r > 0$, that is, $\{f \in L_1(C);\ |f - f_0|_1 \leqq r\}$. Thus it follows that

$$|(S_i - S_j)f|_M \leqq \varepsilon$$

for $i, j \geqq k_0$ and for $f \in L_1(C)$ with $|f - f_0|_1 \leqq r$, so that

$$|(S_i - S_j)f|_M \leqq |(S_i - S_j)(f + f_0)|_M + |(S_i - S_j)f_0|_M \leqq 2\varepsilon \tag{4.4}$$

for $i, j \geqq k_0$ and for $f \in L_1(C)$ with $|f|_1 \leqq r$. If $f \in L_1(C)$ and $|f|_1 \leqq r/2$, we can choose $g \in L_\infty(C)$ such that

$$|g|_1 \leqq r, \qquad |S_{k_0}(f - g)|_M \leqq \varepsilon, \qquad |\tilde{S}(f - g)|_M \leqq \varepsilon.$$

On the other hand,

$$\begin{aligned} |Sf - \tilde{S}f|_M \leqq{} & |(S_j - S_{k_0})f|_M + |S_{k_0}(f - g)|_M \\ & + |(S_{k_0} - S_i)g|_M + |S_i g - \tilde{S}g|_M + |\tilde{S}(g - f)|_M, \qquad i = 1, 2, \ldots.. \end{aligned}$$

Since $\widetilde{S}g = Sg$ in C and hence $|S_i g - \widetilde{S}g|_M = |S_i g - Sg|_M \to 0$ as $i \to +\infty$ and, by (4.4), $|(S_j - S_{k_0})f|_M \leqq 2\varepsilon$, $|(S_{k_0} - S_i)g|_M \leqq 2\varepsilon$ for $i, j \geqq k_0$, we have

$$|Sf - \widetilde{S}f|_M \leqq 6\varepsilon$$

for $f \in L_1(C)$ with $|f|_1 \leqq r/2$. For every $f \in L_1(C)$ we can choose $g \in L_\infty(C)$ such that $|f - g|_1 \leqq r/2$. Since $\widetilde{S}g = Sg$ in C, we obtain that

$$|Sf - \widetilde{S}f|_M \leqq |S(f-g) - \widetilde{S}(f-g)|_M + |Sg - \widetilde{S}g|_M \leqq 6\varepsilon$$

for every $f \in L_1(C)$. Since ε is arbitrary,

$$Sf = \widetilde{S}f \quad \text{in } C$$

for every $f \in L_1(C)$, as was to be proved.

LEMMA 4.9. *For every $f \in L_1(C)$ and every $g \in L_1(C)$ with $g > 0$ in C and for every real α and β, the sets $\left\{x \in C;\ \lim_n \frac{T_n f(x)}{T_n g(x)} > \alpha\right\}$ and $\left\{x \in C;\ \lim_n \frac{T_n f(x)}{T_n g(x)} < \beta\right\}$ are T-invariant.*

PROOF. We use the notation S defined in Lemma 4.8. Since

$$\left\{x \in C;\ \lim_n \frac{T_n f(x)}{T_n g(x)} < \beta\right\} = \left\{x \in C;\ \lim_n \frac{T_n(-f)(x)}{T_n g(x)} > (-\beta)\right\}$$

and

$$\left\{x \in C;\ \lim_n \frac{T_n f(x)}{T_n g(x)} > \alpha\right\} = \left\{x \in C;\ S(f - \alpha g)(x) > 0\right\},$$

it suffices to prove that, for every $f \in L_1(C)$, the set $\{x \in C;\ Sf(x) > 0\}$ is T-invariant.

Let $f \in L_1(C)$. If we set $f_n(x) = f(x)$ for $|f(x)| \leqq n$ and $f_n(x) = 0$ for $|f(x)| > n$, then $f_n \in L_\infty(C)$ and $f_n(x)$ tends to $f(x)$ for almost all $x \in C$ as $n \to +\infty$. Since, by Lemma 4.8, S is a linear positive continuous operator of $L_1(C)$ into itself, it is easily seen that $\lim_n Sf_n(x) = Sf(x)$ for almost all $x \in C$. Thus

$$\{x \in C;\ Sf_n(x) > 0\} \to \{x \in C;\ Sf(x) > 0\}$$

as $n \to +\infty$, while every $\{x \in C;\ Sf_n(x) > 0\}$ $(n = 1, 2, \ldots.)$ is T-invariant by virtue of Lemma 4.4. Hence $\{x \in C;\ Sf(x) > 0\}$ is T-invariant. q.e.d.

Then (3.5) follows directly from Lemmas 4.9 and 4.5. Thus Theorem 3.1 is completely proved.

Appendix. We note the proof of the maximal ergodic theorem in [6]. Although the theorem is properly true, his proof contains a minor mistake. His proof used that, in our notations,

$$\left\{x;\ \sup_{1 \leqq n \leqq N} \frac{T_n f(x)}{T_n g(x)} \geqq 0\right\} \to \left\{x;\ \sup_{1 \leqq n < \infty} \frac{T_n f(x)}{T_n g(x)} \geqq 0\right\}$$

as $N \to +\infty$, but it does not necessarily hold. In this connection we sketch the proof of Theorem 2.1. Lemma 3.2 in [4] is slightly modified as follows, as their proof shows.

LEMMA. *Let T be a linear positive operator of L_1 into itself with $|T|_1 \leqq 1$. For every $f \in L_1$ and for every positive integer N, let*

$$E = \left\{x;\ \sup_{1 \leqq n \leqq N} T_n f(x) > 0\right\}.$$

Then

$$\int_E f(x)\,\mu(dx) \geqq 0.$$

PROOF OF THEOREM 2.1. For every real γ we set

$$A_N(\gamma) = \left\{x;\ \sup_{1 \leqq n \leqq N} \frac{T_n f(x)}{T_n g(x)} > \gamma\right\} = \left\{x;\ \sup_{1 \leqq n \leqq N} T_n(f - \gamma g)(x) > 0\right\},$$

$$A_\infty(\gamma) = \left\{x;\ \sup_{1 \leqq n < \infty} \frac{T_n f(x)}{T_n g(x)} > \gamma\right\}.$$

Then, by the lemma stated above,

$$(*) \qquad \int_{A_N(\gamma)} f(x)\,\mu(dx) \geqq \gamma \cdot \int_{A_N(\gamma)} g(x)\,\mu(dx).$$

Since $A_N(\gamma) \to A_\infty(\gamma)$ as $N \to +\infty$ and $A_\infty(\gamma) \to A^*(\alpha)$ as γ increases and tends to α, we obtain the first inequality of (2.1) from $(*)$. The second inequality of (2.1) is deduced from the first inequality upon replacing f and α by $-f$ and $-\beta$, respectively.

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