

## ON THE DIRECT PRODUCT OF $W^*$ -FACTORS

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The problem in what conditions we can represent a  $W^*$ -factor as the direct product of its two subfactors has been considered by Nakamura [3], in which he set the assumption of finite type. In the present note we generalize his result and get the necessary and sufficient condition in general case, together with some applications.

Let  $M$  be a  $W^*$ -algebra, then  $M$  is the conjugate space of  $M_*$ , the space of all  $\sigma$ -weakly continuous linear functionals on  $M$ . We define the operator  $L_a$  and  $R_a$  on  $M_*$  for  $a \in M$  such as  $\langle x, L_a \varphi \rangle = \langle ax, \varphi \rangle$  and  $\langle x, R_a \varphi \rangle = \langle xa, \varphi \rangle$  for all  $\varphi \in M_*$  and  $x \in M$ . We call a subspace  $V$  of  $M_*$  invariant if it is invariant under the operators  $L_a, R_a$  for all  $a \in M$ . If  $V$  is an invariant subspace of  $M_*$ , then  $V^\circ$ , the polar of  $V$  in  $M$ , is a  $\sigma$ -weakly closed ideal of  $M$ .

Next, let  $M$  be a  $W^*$ -factor,  $M_1, M_2$  subfactors of  $M$  which commute each other elementwise and  $\varphi, \psi$  linear functionals on  $M_1$  and  $M_2$  respectively, then we define the product functional  $\varphi \times \psi$  of  $\varphi$  and  $\psi$  on  $M_0$ , sub  $^*$ -algebra of  $M$  generated algebraically by  $M_1$  and  $M_2$ , as follows

$$\langle \sum_{i=1}^n x_i y_i, \varphi \times \psi \rangle = \sum_{i=1}^n \langle x_i, \varphi \rangle \langle y_i, \psi \rangle.$$

If  $\varphi \times \psi$  is  $\sigma$ -weakly continuous on  $M_0$ , then it is uniquely extended on  $\overline{M_0}$ ,  $\sigma$ -weak closure of  $M_0$ , by the  $\sigma$ -weak continuity: we denote again this extended functional by  $\varphi \times \psi$ . Moreover, we mean  $V \times W$  the space of all finite linear combinations of  $\varphi \times \psi$  ( $\varphi \in V, \psi \in W$ ) where  $V$  and  $W$  are subspaces of  $M_{1*}$  and  $M_{2*}$  respectively.

**THEOREM 1.** *A  $W^*$ -factor  $M$  is isomorphic to the direct product of its two subfactors  $M_1$  and  $M_2$  if and only if the following conditions 1-3 are satisfied:*

1.  $M_1$  and  $M_2$  commute each other elementwise,
2.  $M_1$  and  $M_2$  generate  $M$ ,
3. there exist some  $\sigma$ -weakly continuous linear functionals  $\varphi_0$  and  $\psi_0$  on  $M_1$  and  $M_2$ , respectively such that the product functional  $\varphi_0 \times \psi_0$  of  $\varphi_0$  and  $\psi_0$  on  $M_0$  which is algebraically generated sub  $^*$ -algebra of  $M$  by  $M_1$  and  $M_2$  is  $\sigma$ -weakly continuous.

**PROOF.** At first, we notice that the space of all  $\sigma$ -weakly continuous linear functionals on a  $W^*$ -factor has no non-trivial invariant closed subspace.

Since  $(L_a \varphi_0) \times (L_b \psi_0) = L_{ab}(\varphi_0 \times \psi_0)$  on  $M_0$  for all  $a \in M_1$  and  $b \in M_2$ , we have  $(L_a \varphi_0) \times (L_b \psi_0) \in M_*$ . Similarly,  $(R_a \varphi_0) \times (R_b \psi_0) \in M_*$ . Thus, if we

put  $V = \left[ \sum_{i=1}^n L_{a_i} R_{b_i} \varphi_0; a_i, b_i \in M_1 \right]$  and  $W = \left[ \sum_{i=1}^n L_{a_i} R_{b_i} \psi_0; a_i, b_i \in M_2 \right]$

we have  $V \times W \subset M_*$ . Since  $V$  and  $W$  are invariant subspaces of  $M_{1*}$  and  $M_{2*}$ ,  $V$  and  $W$  are strongly dense in  $M_{1*}$  and  $M_{2*}$ , respectively. Thus,

$$M_{1*} \times M_{2*} \subset M_*$$

Since  $M_{1*} \times M_{2*}$  is an  $M_0$ -invariant subspace of  $M_*$  and  $M_0$  is  $\sigma(M, M_*)$ -dense in  $M$ ,  $M_{1*} \times M_{2*}$  is strongly dense in  $M_*$ . Therefore,  $\sigma(M, M_*)$ -topology and  $\sigma(M, M_{1*} \times M_{2*})$ -topology coincide on the unit sphere of  $M$ .

Next we define a  $*$ -isomorphism  $\Phi$  from  $M_1 \odot M_2$ , the algebraic direct product of  $M_1$  and  $M_2$ , to  $M_0$  as follows

$$\Phi \left( \sum_{i=1}^n x_i \otimes y_i \right) = \sum_{i=1}^n x_i y_i,$$

where  $\sum_{i=1}^n x_i \otimes y_i$  is an arbitrary element of  $M_1 \odot M_2$ . Since the family of all states of type  $\varphi \times \psi$ , where  $\varphi$  and  $\psi$  are normal states on  $M_1$  and  $M_2$  respectively, is total on  $M$  by our consideration above, we have

$$\begin{aligned} \left\| \sum_{i=1}^n x_i y_i \right\|^2 &= \sup \left\{ \frac{\left\langle \left( \sum_{k=1}^m a_k^* b_k^* \right) \left( \sum_{i=1}^n x_i^* y_i^* \right) \left( \sum_{i=1}^n x_i y_i \right) \left( \sum_{k=1}^m a_k b_k \right), \varphi \times \psi \right\rangle}{\left\langle \left( \sum_{k=1}^m a_k^* b_k^* \right) \left( \sum_{k=1}^m a_k b_k \right), \varphi \times \psi \right\rangle} \right\} \\ &= \left\| \sum_{i=1}^n x_i \otimes y_i \right\|^2 \end{aligned}$$

where  $\varphi, \psi$  are all normal states on  $M_1$  and  $M_2$ , respectively, and  $\sum_{k=1}^m a_k b_k$

is any non-zero element of  $M_0$  for an arbitrary element  $\sum_{i=1}^n x_i \otimes y_i$  of  $M_1 \odot M_2$ .

Hence,  $\Phi$  is an isometry from  $M_1 \odot M_2$  to  $M_0$ . Therefore,  $\Phi$  is uniquely extended to  $M_1 \widehat{\otimes}_\alpha M_2$  by uniform continuity.

Next, we shall show that  $\Phi$  is  $\sigma$ -weakly continuous on the unit sphere of  $M_1 \odot M_2$ . By [6: Theorem 1]  $M_{1*} \odot M_{2*}$  is strongly dense in  $(M_1 \otimes M_2)_*$ , hence  $\sigma(M_1 \otimes M_2, (M_1 \otimes M_2)_*)$ -topology and  $\sigma(M_1 \otimes M_2, M_{1*} \odot M_{2*})$ -topology coincide on the unit sphere of  $M_1 \otimes M_2$ . It is clear that  $\Phi$  is bicontinuous on  $M_1 \odot M_2$  and  $M_0$  with respect to  $\sigma(M_1 \odot M_2, M_{1*} \odot M_{2*})$ -topology and  $\sigma(M_0, M_{1*} \times M_{2*})$ -topology. Therefore,  $\Phi$  is  $\sigma$ -weakly bicontinuous on the unit sphere of  $M_1 \odot M_2$  and  $M_0$ . Hence, by Kaplansky's Density Theorem,  $\Phi$  is uniquely extended to the isomorphism of  $M_1 \widehat{\otimes} M_2$  onto  $M$  by its  $\sigma$ -weak continuity.

Finally the necessity is easily verified from [6: Theorem 1].

**THEOREM 2.** *A  $W^*$ -factor  $M$  is isomorphic to the direct product of its two*

subfactors  $M_1$  and  $M_2$  if and only if the conditions 1, 2 in Theorem 1 and the following condition 3' are satisfied:

3'. there exists a  $\sigma$ -weakly continuous projection of norm one  $\theta_0$  from  $M$  to  $M_1$  (or  $M_2$ ), that is,  $\theta_0(x) = x$  for all  $x \in M_1$  and  $|\theta_0(x)| \leq |x|$  for  $x \in M$ .<sup>1)</sup>

PROOF. By [8: Theorem 1], we have

$$\theta_0(axb) = a\theta_0(x)b \text{ for all } a, b \in M_1 \text{ and } x \in M.$$

Since, by the condition 1, for any  $u \in M_2$ ,  $a\theta_0(u) = \theta_0(au) = \theta_0(ua) = \theta_0(u)a$  for all  $a \in M_1$ ,  $\theta_0(M_2)$  is included in the center of  $M_1$ . As  $M_1$  is a factor, the restriction  $\theta_0$  on  $M_2$  induces a  $\sigma$ -weakly continuous linear functional  $\psi_0$  on  $M_2$ .

Now, if we denote the transpose of  $\theta_0$  by  ${}^t\theta_0$ , then, for  $\varphi \in M_{1*}$ , we have

$$\begin{aligned} \left\langle \sum_{i=1}^n x_i y_i, {}^t\theta_0(\varphi) \right\rangle &= \left\langle \theta_0 \left( \sum_{i=1}^n x_i y_i \right), \varphi \right\rangle = \left\langle \sum_{i=1}^n x_i \theta_0(y_i), \varphi \right\rangle \\ &= \left\langle \sum_{i=1}^n \langle y_i, \psi_0 \rangle x_i, \varphi \right\rangle \\ &= \sum_{i=1}^n \langle x_i, \varphi \rangle \langle y_i, \psi_0 \rangle, \end{aligned}$$

where  $\sum_{i=1}^n x_i y_i$  is an arbitrary element of sub\*-algebra of  $M$  generated algebraically by  $M_1$  and  $M_2$ , i.e.  ${}^t\theta_0(\varphi) = \varphi \times \psi_0$ . By the  $\sigma$ -weak continuity of  $\theta_0$ ,  $\varphi \times \psi_0 \in M_*$ . Therefore  $M \cong M_1 \otimes M_2$  by Theorem 1.

Conversely, let  $\psi_0$  be a normal state on  $M_2$ , we define the linear mapping  $\theta'_0$  of  $M_{1*}$  to  $M^*$  as follows

$$\theta'_0(\varphi) = \varphi \otimes \psi_0 \quad \text{for all } \varphi \in M_{1*}.$$

If we put  ${}^t\theta'_0 = \theta_0$ ,  $\theta_0$  satisfies the condition 3'.

Theorem 2 is a generalization of the result of [3] which proves the theorem in the finite case. For, in this case, there exists always a linear mapping described in 3' as an expectation in [9].

Next, we apply Theorem 2 to the problem of non-normality and show an alternative proof for [1: Theorem 3].

**THEOREM 3.** *Let  $M$  be a  $\text{II}_1$ -factor, then  $M$  is non-normal.*

PROOF. There exists an approximately finite sub-factor  $M_1$  in  $M$ . If  $M_1 = (M_1' \cap M) \cap M$ , then  $M \cong M_1 \otimes (M_1' \cap M)$  by Theorem 2. Since  $M_1$  is non-normal one can easily verify that  $M$  is also non-normal. On the other

1) Prof. M. Nakamura noticed me that the existence condition of a  $\sigma$ -weakly continuous projection of norm one is related to that of a  $\sigma$ -weakly continuous module linear homomorphism from  $M$  to  $M_1$ .

hand  $M_1 \neq (M_1' \cap M)' \cap M$  implies also the non-normality of  $M$  by definition.

Combining the argument in the above proof with that in [5] one can easily verify the following

**COROLLARY.** *A  $W^*$ -factor  $M$  on a Hilbert space  $H$  is type I if and only if there exists a  $\sigma$ -weakly continuous projection of norm one from  $B(H)$ , ring of all bounded operators on  $H$ , to  $M$ .*

## REFERENCES

- [1] B. FUGLEDE AND R. KADISON, On a conjecture of Murray and von Neumann, Proc. Nat. Acad. Sc. U. S. A., 37(1951), 420-425.
- [2] Y. MISONOU, On the direct product of  $W^*$ -algebras, Tôhoku Math. Journ., 6(1954), 189-204.
- [3] M. NAKAMURA, On the direct product of finite factors, Tôhoku Math. Journ., 6(1954), 205-209.
- [4] S. SAKAI, A characterization of  $W^*$ -algebra, Pacific Journ. Math., 6(1956), 763-773.
- [5] S. SAKAI, On topological properties of  $W^*$ -algebras, Proc. Japan Acad., 33(1957), 439-444.
- [6] T. TURUMARU, On the direct product of operator algebras III, Tôhoku Math. Journ., 6(1954), 208-211.
- [7] T. TURUMARU, On the direct product of operator algebra IV, Tôhoku Math. Journ., 8(1956), 281-285.
- [8] J. TOMIYAMA, On the projection of norm one in  $W^*$ -algebras, Proc. Japan Acad., 33(1957), 608-612.
- [9] H. UMEGAKI, Conditional expectation in an operator algebra, Tôhoku Math. Journ., 6(1954), 177-181.
- [10] F. MURRAY AND J. VON NEUMANN, On rings of operators. IV, Ann. Math., 44 (1943), 716-808.

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