A NOTE ON CONTRACTION SEMI-GROUPS OF OPERATORS

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1. Let $\Sigma = \{T(\xi);\ 0 \leq \xi < \infty\}$ be a one-parameter semi-group of operators from an abstract (L)-space X into itself satisfying the following conditions:

(a) For each $\xi > 0$, $T(\xi)$ is a contraction (transition) operator[1].

(b) $T(\xi + \eta) = T(\xi)T(\eta)$ for each ξ, $\eta \geqq 0$ and $T(0) = I$.

(c) $\lim\limits_{\xi \downarrow 0} T(\xi)x = x$ for each $x \in X$.

Such a semi-group is called a contraction (transition) semi-group of operators. We say that $\Sigma' = \{T'(\xi);\ 0 \leqq \xi < \infty\}$ dominates $\Sigma = \{T(\xi);\ 0 \leqq \xi < \infty\}$ if

$$T'(\xi)x \geqq T(\xi)x$$

for each $x \geqq 0$ and $\xi > 0$.

We shall deal with the problem on the generation of contraction semi-groups dominating a given contraction semi-group. This problem has been discussed by G. E. H. Reuter[2].

2. We shall define a linear functional $(e,\cdot)$ by

$$(2.\ 1) \qquad (e\,, x) = \| x^+ \| - \| x^- \| \qquad \text{for each } x \in X.$$

An elementary argument shows that $(e, \cdot)$ is a positive linear functional and $|(e,\ x)| \leqq \|x\|$ for each $x \in X$.

The following theorem is due to Reuter and is a variant of the Hille-Yosida theorem which will be convenient for our purposes.

THEOREM 1. *A linear operator A with an dense domain $D(A)$ generates a contraction (transition) semi-group if and only if*

(i) $\quad (e,\ Ax) \leqq 0\,(=0)$ *for $x \geqq 0$ in $D(A)$,*

(ii) *for each $\lambda > 0$ and $x \in X$, the equation*

$$\lambda y - Ay = x$$

has a unique solution $y = R(\lambda\,;\ A)x \in D(A)$ and $R(\lambda\,;\ A)x \geqq 0$ for $x \geqq 0$.

We shall first prove the following

1) A positive linear operator T on X is called a contraction (transition) operator if $\|Tx\| \leqq \|x\|$ ($\|Tx\| = \|x\|$) for $x \geqq 0$.

2) A note on contraction semi- groups, Math. Scand., vol. 3, 1955.

THEOREM 2. *Let A generate a contraction semi-group Σ and let B be a linear operator with domain $D(B) \supset D(A)$. Then $A = A + B$ will generate a contraction (transition) semi-group Σ' which dominates Σ if and only if*

(i') $\quad Bx \geqq 0$ $\qquad$ *for $x \geqq 0$ in $D(A)$,*

(ii') $\quad (e, Bx) \leqq -(e, Ax)\ (= -(e, Ax))$ $\qquad$ *for $x \geqq 0$ in $D(A)$,*

(iii') *any one of the following*;

(a') $\quad (I - BR(\lambda; A))[X] = X$ $\qquad$ *for each $\lambda > 0$,*

(b') $\quad \sum_{n=0}^{\infty} \| [BR(\lambda; A)]^n y\| < \infty$ *for each $\lambda > 0$ and $y > 0$.*

PROOF. The necessity of (ii) follows from Theorem 1, and that of (i') follows from

$$(A + B)x = \lim_{\xi \downarrow 0} \frac{T'(\xi) - I}{\xi} x \geqq \lim_{\xi \downarrow 0} \frac{T(\xi) - I}{\xi} x = Ax$$

for $x \geqq 0$ in $D(A)$. Since

(2. 1) $\qquad (\lambda - A')R(\lambda; A)x = (I - BR(\lambda; A))x \qquad$ for each $x \in X$,

$(I - BR(\lambda; A))[X] = (\lambda - A')R(\lambda; A)[X] = (\lambda - A')[D(A)] = (\lambda - A')[D(A')] = X$. Thus we obtain the property (iii'-a').

Conversely, if (i'), (ii') and (iii'-a') hold, then we have that the inverse $(I - BR(\lambda; A))^{-1}$ exists for each $\lambda > 0$ and that $(I - BR(\lambda; A))^{-1}$ is a positive linear operator with domain X.

In fact, if $(I - BR(\lambda; A))x = 0$, then (i') and (ii') together with the equation $AR(\lambda; A) = \lambda R(\lambda; A) - I$ imply that

$$\begin{aligned} \|x\| &\leqq \|BR(\lambda; A)|x|\| \leqq -(e, AR(\lambda; A)|x|) \\ &= \|x\| - \|\lambda R(\lambda; A)|x|\|. \end{aligned}$$

Hence we get $\lambda R(\lambda; A)|x| = 0$, so that $|x| = (\lambda - A)R(\lambda; A)|x| = 0$. Then the inverse $(I - BR(\lambda; A))^{-1}$ exists for each $\lambda > 0$ and its domain is the whole space X from (iii -a). If $y = (I - BR(\lambda; A))^{-1} x$ ($x \geqq 0$), then $y - BR(\lambda; A)y = x \geqq 0$. Hence $y^- \leqq (BR(\lambda; A)y)^- \leqq BR(\lambda; A)y^-$, so that

$$\begin{aligned} \|y^-\| &\leqq \|BR(\lambda; A)y^-\| \leqq -(e, AR(\lambda; A)y^-) \\ &= \|y^-\| - \|\lambda R(\lambda; A)y^-\|. \end{aligned}$$

This shows that $\lambda R(\lambda; A)y^- = 0$. Therefore $y^- = (\lambda - A)R(\lambda; A)y^- = 0$ and this concludes that $(I - BR(\lambda; A))^{-1}$ is a positive operator.

We define $R(\lambda; A')$ by

(2. 2) $$R(\lambda; A') = R(\lambda; A)(I - BR(\lambda; A))^{-1},$$

then $R(\lambda; A')$ is a positive linear operator with domain X. It follows directly from (2.2) that

(2. 3) $$(\lambda - A')R(\lambda ; A')x = x \qquad \text{for each } \lambda > 0 \text{ and } x \in X.$$

The range of $R(\lambda ; A')$ is precisely $D(A)$ since the range of $(I - BR(\lambda ; A))^{-1}$ is X. Thus for each $x \in D(A)$ there exists an element y such that $x = R(\lambda ; A')y$. By (2. 3),

(2. 4) $$R(\lambda ; A')(\lambda - A')x = R(\lambda ; A')(\lambda - A')R(\lambda ; A')y = R(\lambda ; A')y = x$$

for each $x \in D(A)$. This shows that A' satisfiies the condition (ii) in Theorem 1. Furthermore it is obvious from (ii′) that

$$(e, A'x) \leqq 0 \,(= 0)$$

for each $x \geqq 0$ in $D(A)$. Thus it follows from Theorem 1 that A' generates a contraction (transition) semi-group $\Sigma' = \{T'(\xi);\ 0 \leqq \xi < \infty\}$. It is seen at once dy (2. 2) that $R(\lambda ; A')x \geqq R(\lambda ; A)x$ for $x \geqq 0$ and $\lambda > 0$, so that the formula

$$T(\xi)x = \lim_{\lambda \to \infty} \{\exp(-\lambda\xi) \sum_{n=0}^{\infty} (\lambda^2\xi)^n [R(\lambda ; A)]^n / n!\}\, x$$

shows that

$$T'(\xi)x \geqq T(\xi)x$$

for each $\xi \geqq 0$ and $x \geqq 0$.

Since $(I - BR(\lambda ; A))^{-1}$ is a positive linear operator, we have

$$x = (I - BR(\lambda ; A))^{-1} y \geqq 0$$

for each $y \geqq 0$. Hence

(2. 5) $$x = y + BR(\lambda ; A)y + \cdots\cdots + [BR(\lambda ; A)]^{n-1} y + [BR(\lambda ; A)]^n x$$

and

(2. 6) $$x \geqq BR(\lambda ; A)x \geqq [BR(\lambda ; A)]^2 x \geqq \cdots\cdots \geqq 0.$$

Then there exists the limit $x_0 = \lim_{n \to \infty} [BR(\lambda ; A)]^n x$ and $x_0 = BR(\lambda ; A)x_0$, so that $x_0 = 0$. Therefore, by (2. 5), we get

$$x = \sum_{n=0}^{\infty} [BR(\lambda ; A)]^n y$$

and a fortiori

$$\sum_{n=0}^{\infty} \| [BR(\lambda ; A)]^n y \| < \infty.$$

Suppose that b(iii′) holds. Then the series $\sum_{n=0}^{\infty} [BR(\lambda ; A)]^n y$ converges for each $y \in X$ and is equal to $(I - BR(\lambda\,;\,A))^{-1}\,y$, so that (iii$'$-a$'$) holds. This concludes the proof of Theorem 2.

COROLLARY 1. *Let A generate a contraction semi-group Σ, and let B be a linear operator with domain $D(B) \supset D(A)$. Further assume that there exist real numbers $\lambda_0 > 0$ and $\varepsilon_0 > 0$ such that $\| \lambda_0 R(\lambda_0\,;\,A)x \| \geqq \varepsilon_0 \| x \|$ for all $x \geqq 0$. Then $A = A + B$ will generate a contraction (transition) semi-group Σ' which dominates Σ if and only if the conditions* (i$'$) *and* (ii$'$) *in Theorem* 2 *hold.*

PROOF. The necessity is obvious. We shall now prove the sufficiency. From (i$'$) and (ii$'$),

$$(2.\ 7) \qquad \| BR(\lambda\,;\,A)x \| \leqq \| x \| - \| \lambda R(\lambda\,;\,A)x \| \qquad \text{for } x \geqq 0.$$

Let us put

$$\varepsilon_\lambda = \inf_{\|x\|=1, x>0} \| \lambda R(\lambda\,;\,A)x \| \qquad (\lambda > 0).$$

If $\varepsilon_\lambda = 0$, then there exists a sequence $\{x_n\,;\, \| x_n \| = 1 \text{ and } x_n > 0\}$ such that $\lambda R(\lambda\,;\,A)x_n \to 0$. Then we have $\lim\limits_{n\to\infty} \lambda_0 R(\lambda_0\,;\,A)x_n = 0$ by the resolvent equation

$$(2.\ 8) \qquad R(\lambda_0\,;\,A) - R(\lambda\,;\,A) = -(\lambda_0 - \lambda)R(\lambda_0\,;\,A)R(\lambda\,;\,A).$$

From this contradiction we conclude that $\varepsilon_\lambda > 0$ for each $\lambda > 0$. Therefore we get, by (2. 7),

$$\| BR(\lambda\,;\,A)x \| \leqq (1 - \varepsilon_\lambda) \| x \|$$

for each $x \geqq 0$, so that (iii$'$-b$'$) in Theorem 2 holds.

COROLLARY 2. *Let $\Sigma = \{T(\xi)\,;\, 0 \leqq \xi < \infty\}$ be uniformly continuous at $\xi = 0$ (if and only if A is a bounded linear operator), and let B be a linear operator with domain $D(B) = X$. Then $A' = A + B$ will generate a contraction (transition) semi-group Σ' which dominates Σ if and only if the conditions* (i$'$) *and* (ii$'$) *in Theorem* 2 *hold.*

PROOF. *Since*

$$\| x \| \leqq (\lambda + \| A \|) \| R(\lambda\,;\,A)x \| \qquad \text{for each } x \in X,$$

this corollary follows from Corollary 1.

COROLLARY 3. *Let A generate a contraction semi-group Σ, and let B be a linear operator with domain $D(B) \supset D(A)$ satisfying the conditions* (i$'$) *and* (ii) *in Theorem* 2. *Further assume that $BR(\lambda_0\,;\,A)$ is completely continuous for some $\lambda_0 > 0$. Then $A = A + B$ generates a contraction (transition) semi-group Σ' which dominates Σ.*

PROOF. From the resolvent equation (2. 8),

$$BR(\lambda\,;\,A) = BR(\lambda_0\,;\,A) + (\lambda_0 - \lambda)BR(\lambda_0\,;\,A)R(\lambda\,;\,A).$$

Then $BR(\lambda\,;\,A)$ is completely continuous for each $\lambda > 0$ since the product of a bounded linear operator and a completely continuous linear operator is completely continuous and since the sum of two completely continuous linear operators is again completely continuous. Since 1 is not eigen-value of $BR(\lambda\,;\,A)$, we have by the theorem of F.Riesz that $(I - BR(\lambda\,;\,A))[X] = X$, so that (iii′-a′) in Theorem 2 holds. Hence this corollary follows from Theorem 2.

3. It is seen at once by using the identity

$$\|x + y\| = \|x\| + \|y\| \qquad (x \geqq 0,\ y \geqq 0)$$

that if a contraction semi-group Σ' dominates a transition semi-group Σ, then $\Sigma' = \Sigma$. Thus if Σ is a transition semi-group, no distinct contraction semi-group dominates Σ.

We now suppose that Σ is a contraction but not transition semigroup. The following theorem is due to Reuter.

THEOREM 3. *Let Σ be a contraction semi-group, generated by A. Then the operator A_c defined by*

$$A_c x = Ax - (e,\ Ax)c,\ x \in D(A)\ (\textit{with } c \geqq 0 \textit{ and } \|c\| \leqq 1),$$

generates a contraction semi-group Σ_c dominating Σ. Also $\Sigma_{c1} \neq \Sigma_{c2}$ if $c_1 \neq c_2$, and Σ_c is a transition semi-group if and only if $\|c\| = 1$.

PROOF. Let us put

$$Bx = -(e,\ Ax)c \qquad \text{for } x \in D(A),$$

where $c \geqq 0$ and $0 \leqq \|c\| \leqq 1$. It is obvious that the assumptions in Corollary 3 hold. Hence $A_c = A + B$ generates a contraction semi-group Σ_c which dominates Σ.

Since Σ was assumed to be not a transition semi-group, Theorem 1 shows that

$$(e,\ Ax_0) < 0 \qquad \text{for some } x_0 > 0 \text{ in } D(A). \tag{3. 1}$$

Now

$$(e,\ A_c x_0) = (e,\ Ax_0)(1 - \|c\|),$$

so (3. 1) implies that Σ_c is a transition semi-group if and only if $\|c\| = 1$. If $c_1 \neq c_2$, then $A_{c1} \neq A_{c2}$, so that $\Sigma_{c1} \neq \Sigma_{c2}$.

LEMMA. *Let A generate a contraction (but not transition) semi-geoup Σ, and let B be a linear operator with domain $D(B) \supset D(A)$ such that $Bx = 0$ for each $x \in E$, where $E \equiv \{x \in D(A)\,;\,(e\,,Ax) = 0\}$. If $A = A + B$ generates a contraction semi-group which dominates Σ, then there exists a* *non-negative element c_B with $\|c_B\| \leqq 1$ such that*

$$Bx = -(e, Ax)c_B$$

for all $x \in D(A)$.

PROOF. Since Σ was assumed to be not a transition semi-group, Theorem 1 shows that $(e, Ax_0) < 0$ for some $x_0 > 0$ in $D(A)$. Let us put

$$\alpha(x) = \frac{(e, Ax)}{(e, Ax_0)}$$

for each $x \in D(A)$. It is obvious that $x - \alpha(x)x_0 \in E$. Hence

$$Bx = B(x - \alpha(x)x_0) + \alpha(x)Bx_0 = \alpha(x)Bx_0 = -(e, Ax)c_B,$$

where $c_B = -Bx_0/(e, Ax_0)$. By Theorem 2, $Bx \geqq 0$ for $x \geqq 0$ in $D(A)$ and $\|Bx\| \leqq -(e, Ax)$ for $x \geqq 0$ in $D(A)$. Hence $c_B \geqq 0$ and $\|c_B\| \leqq 1$. Thus the lemma is proved.

This lemma and Theorem 3 show that if $E = E^+ - E^+$, where $E^+ \equiv \{x \geqq 0 \text{ and } x \in E\}$, then contraction semi-groups dominating Σ are always the type Σ_c in Theorem 3.

In fact, if $A + B$ generates a contraction semi-group Σ' which dominates Σ, then it follows from Theorem 2 that $\|Bx\| = 0$ for each $x \in E^+$. Hence $Bx = 0$ for all $x \in E = E^+ - E^+$, so that we have by the lemma,

$$(A + B)x = Ax - (e, Ax)c, \ x \in D(A) \text{ (with } c \geqq 0 \text{ and } \|c\| \leqq 1).$$

Therefore Σ' is the type Σ_c in Theorem 3.

4. In this section we shall deal with the space (l) and we shall assume that Σ is uniformly continuous at $\xi = 0$. ($\|T(\xi) - I\| \to 0$ as $\xi \downarrow 0$). We first prove the following

THEOREM 4. *Let A generate a contraction (but not transition) semi-group Σ, and let $E \supsetneqq E^+ - E^+$. Then there exist contraction semi-groups which dominate Σ and which are different from the type Σ_c in Theorem 3.*

PROOF. Since A is a bounded linear operator, there exists a non-negative element $(a_1, a_2, a_3 \ldots\ldots) \in (l^\infty)$ such that

$$-(e, Ax) = \sum_{i=1}^{\infty} a_i x_i$$

for all $x = (x_1, x_2, x_3, \ldots\ldots) \in (l)$. The set $N \equiv \{i\,;\, a_i > 0\} \neq \phi$ since Σ is not transition. In this case

$$E^+ = \{x = (x_1, x_2, x_3 \ldots\ldots) \geqq 0\,;\, x_i = 0 \text{ for all } i \in N\}$$

and

$$E^+ - E^+ = \{x = (x_1, x_2, x_3 \ldots\ldots) \in (l)\,;\, x_i = 0 \text{ for all } i \in N\}.$$

Hence $E^+ - E^+$ is a closed set. By the assumption $E \supsetneqq E^+ - E^+$ there exists an element $x' = (x_1', x_2', x_3', \ldots\ldots)$ such that $x' \in E$ and $x \notin E^+ - E^+$. Thus $0 = (e, Ax') = (e, Ax'^+) - (e, Ax'^-)$ and $(e, Ax^+) = (e, Ax^-) < 0$, so that $N_1 \cap N \neq \phi$ and $N_2 \cap N \neq \phi$, where $N_1 \equiv \{i; x_i' > 0\}$ and $N_2 \equiv \{i; x_i' < 0\}$.

Let us put

$$b_i = \begin{cases} a_i & \text{for } i \in N_1 \cap N, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a positive bounded linear functional $f(x)$ by

$$f(x) = \sum_{i=1}^{\infty} b_i x_i.$$

The operator $f(x)c$ with $c \geqq 0$ and $\|c\| \leqq 1$ satisfies that $f(x)c \geqq 0$ for $x \geqq 0$ and $\|f(x)c\| = (e, f(x)c) \leqq -(e, Ax)$ for $x \geqq 0$, so that it follows from Corollary 2 that $Ax + f(x)c$ generates a contraction semi-group Σ dominating Σ. .

On the other hand

$$f(x') = \sum_{i=1}^{\infty} b_i x_i' = \sum_{i \in N_1 \cap N} a_i x_i' = -(e, Ax'^+) > 0,$$

hence $Ax + f(x)c$ is different from the type A_c in Theorem 3. This concludes the proof.

It follows from Theorem 4 the following

COROLLARY 4. *Let A generate a contraction (but not transition) semi-group Σ. Then each contraction semi-group dominating Σ is always of the type Σ_c in Theorem 3 if and only if $E = E^+ - E^+$.*

Finally we shall show that there exists a contraction semi-group Σ such that $E = E^+ - E^+$.

EXAMPLE. Let us put

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots\cdots \\ a_2 & 0 & 0 & \cdots\cdots \\ a_3 & 0 & 0 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{bmatrix},$$

where $\{a_n\}$ is a sequence such that $a_n \geqq 0$ for $n \geqq 2$ and $a_1 < -\sum_{n=2}^{\infty} a_n$. It is obvious that A is a bounded linear operator (l) into itself.

We define

$$T(\xi) = \exp \xi A = \sum_{m=0}^{\infty} \xi^m A^m / m! \qquad (\xi \geqq 0).$$

$\Sigma = \{ T(\xi) ; 0 \leqq \xi < \infty \}$ is a semi-group of operators which is generated from the bounded linear operator A. For each $x = (x_1, x_2, x_3, \ldots\ldots) \in (l)$,

$$T(\xi)\, x = \begin{pmatrix} x_1 e^{a_1 \xi} \\ x_2 - \dfrac{a_2}{a_1} (1 - e^{a_1 \xi})\, x_1 \\ \vdots \\ x_n - \dfrac{a_n}{a_1} (1 - e^{a_1 \xi})\, x_1 \\ \vdots \end{pmatrix}$$

Since $-a_n / a_1 \geqq 0$ for $n \geqq 2$ and $1 - e^{a_1 \xi} > 0$, $T(\xi)$ is a positive bounded linear operator. Furthermore, for each $x = (x_1, x_2, x_3, \ldots\ldots) \geqq 0$,

$$\| T(\xi)x \| = \sum_{n=2}^{\infty} x_n + x_1 e^{a_1 \xi} - \frac{a_2 + \ldots\ldots + a_n + \cdots}{a_1} (1 - e^{a_1 \xi}) x_1$$

$$\leqq \sum_{n=2}^{\infty} x_n + x_1 e^{a_1 \xi} + x_1 (1 - e^{a_1 \xi}) = \| x \|$$

and $\| T(\xi)x \| < \|x\|$ if $x_1 > 0$, so that $\Sigma = \{ T(\xi) ; 0 \leqq \xi < \infty \}$ is a contraction but not transition semi-group.

Now

$$(e, Ax) = x_1 \sum_{n=1}^{\infty} a_n,$$

hence

$$E = \{ x = (x_1, x_2, x_3, \ldots\ldots) ; x_1 = 0 \}.$$

It is obvious that $E = E^+ - E^+$.

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