AN ASPECT OF LOCAL PROPERTY OF $|R, \log n, 1|$ SUMMABILITY OF FOURIER SERIES

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1. 1. DEFINITION. Let S_n denote the n-th partial sum of the series Σa_n. We write

$$R_n = \left\{ S_1 + \frac{1}{2} S_2 + \dots\dots + \frac{1}{n} S_n \right\} \Big/ \log n.$$

Then the series Σa_n is said to be *absolutely summable* $(R, \log n, 1)$ or *summable* $|R, \log n, 1|$ if the sequence $\{R_n\}$ is of bounded variation, that is to say, the infinite series

$$\sum |R_n - R_{n+1}|$$

is convergent.

It has been pointed out by Bosanquet* that for the case $\lambda_n = \log n$, this definition is equivalent to the definition of the summability $|R, \lambda_n, 1|$ used by Mohanty [5], λ_n being a monotonic increasing sequence tending to infinity with n.

1. 2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series of $f(t)$ can be taken to be zero, so that

(1. 2. 1) $$f(t) \sim \Sigma (a_n \cos nt + b_n \sin nt) = \Sigma A_n(t),$$

and

(1. 2. 2) $$\int_{-\pi}^{\pi} f(t)\, dt = 0.$$

We write

$$\varphi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \}.$$

1. 3. It has been proved independently by Izumi [3] and Mohanty [5] that summability $|R, \log n, 1|$ of a Fourier series is not a local property of the generating function. The question, naturally arises as to what conditions

* L. S. Bosanquet, Mathematical Review, 12 (1951), 254, see review of the paper of Izumi [3].

should be satisfied by the general terms of a Fourier series at a point such that its summability $|R, \log n, 1|$ may depend only upon the behaviour of the generating function in the immediate neighbourhood of the point considered. The first answer to a question of this character is due to Izumi [3] who proved that if

$$A_n(x) = O((\log n)^{-2}),$$

then the summability $|R, \log n, 1|$ of the Fourier series $\Sigma A_n(t)$, at $t = x$, is a local property. More recently Mohanty and Izumi [6] have improved upon this result and established the following theorem:

THEOREM A. *If*

$$\sum \frac{|A_n(x)|}{n} \log\log n < \infty, \tag{1. 3. 1}$$

then the summability $|R, \log n, 1|$ *of* $\Sigma A_n(x)$ *depends only upon a local condition.*

It is known [5, 7] that if Σa_n is summable $|R, \lambda_n, k|$, $k > 0$, then $\Sigma a_n/\lambda_n^k$ is summable $|R, e^{\lambda_n}, k|$. Hence it follows that if Σa_n is summable $|R, \log n, 1|$, then $\Sigma a_n/\log n$ is summable $|R, n, 1|$ i. e. summable $|C, 1|$, [2]. Therefore, by a well-known result of Kogbetliantz [4], it follows that

$$\Sigma |a_n|/\{n \log n\} < \infty.$$

Thus it follows that the summability $|R, \log n, 1|$ of the Fourier series necessarily implies that

$$\Sigma |A_n(x)|/\{n \log n\} < \infty. \tag{1. 3. 2}$$

In this paper we establish a theorem, more general than theorem A, inasmuch as we assume, instead of the condition (1. 3. 1) the less stringent condition (1. 3. 2) which is seen to be also the *necessary* condition of the $|R, \log n, 1|$ summability of the corresponding Fourier series.

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2. 1. We prove the following theorem.

THEOREM. *If*

$$\Sigma |A_n(x)|/\{n \log n\} < \infty,$$

then the $|R, \log n, 1|$ *summability of* $\Sigma A_n(t)$ *depends only on the behaviour of the generating function* $f(t)$ *in the immediate neighbourhood of the point* $t = x$.

2. 2. We require the following lemma for the proof of the theorem.

LEMMA. *If the series*

$$\sum_{n=1}^{\infty} |S_n|/\{n \log (n+1)\}$$

is convergent then the sequence $\{S_n\}$ *is summable* $|R, \log n, 1|$.

PROOF.

$$R_n - R_{n+1} = \frac{1}{\log n}\sum_{\nu=1}^{n}\frac{S_\nu}{\nu} - \frac{1}{\log(n+1)}\sum_{\nu=1}^{n+1}\frac{S_\nu}{\nu}$$

$$= \Delta\left(\frac{1}{\log n}\right)\sum_{\nu=1}^{n}\frac{S_\nu}{\nu} - \frac{1}{\log(n+1)}\frac{S_{n+1}}{n+1},$$

where

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

Therefore

$$\sum_{n=2}^{m} |R_n - R_{n+1}|$$

$$\leqq A + \sum_{n=2}^{m}\Delta\left(\frac{1}{\log n}\right)\sum_{\nu=2}^{n}\frac{|S_\nu|}{\nu} + \sum_{n=2}^{m}\frac{|S_{n+1}|}{(n+1)\log(n+1)}$$

$$= A + \sum_{\nu=2}^{m}\frac{|S_\nu|}{\nu}\sum_{n=\nu}^{m}\Delta\left(\frac{1}{\log n}\right) + \sum_{n=2}^{m}\frac{|S_{n+1}|}{(n+1)\log(n+1)}$$

$$= A + O\left(\sum_{\nu=2}^{m}\frac{|S_\nu|}{\nu\log\nu}\right)$$

$$= A + O\left(\sum_{\nu=1}^{m}\frac{|S_\nu|}{\nu\log(\nu+1)}\right).$$

This completes the proof of the lemma.

2. 3. Proof of the theorem. We have

$$S_n(x) = \sum_{\nu=1}^{n} A_\nu(x)$$

$$= \frac{1}{2\pi}\int_0^\pi \varphi(u)\frac{\sin\left(n+\frac{1}{2}\right)u}{\sin u/2}\,du$$

$$= \frac{1}{2\pi}\left\{\int_0^\eta \varphi(u)\frac{\sin\frac{u}{2}}{\sin^2\frac{\eta}{2}}\sin\left(n+\frac{1}{2}\right)u\,du\right.$$

$$+ \int_\eta^\pi \varphi(u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin u/2}\, du \Bigg\}$$

$$+ \frac{1}{2\pi} \int_0^\eta \varphi(u) \left\{1 - \left(\frac{\sin u/2}{\sin \eta/2}\right)^2\right\} \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin u/2}\, du$$

$$= \frac{1}{2\pi}[P_n + Q_n], \text{ say.} \tag{2. 3. 1}$$

The sequence $\{S_n(x)\}$ will be summable $|R, \log n, 1|$ if the sequences $\{P_n\}$ and $\{Q_n\}$ are summable $|R, \log n, 1|$. We observe that, for positive η, however small but fixed, the summability $|R, \log n, 1|$ of the sequence $\{Q_n\}$ depends only upon the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point x, defined by $(x - \eta, x + \eta)$. Hence to prove the theorem it is sufficient to show that the sequence $\{P_n\}$ is summable $|R, \log n, 1|$ under the hypothesis of the theorem. By virtue of the lemma, this will be satisfied if we prove that

$$\Sigma\, |P_n(x)|/\{n \log (n+1)\} < \infty. \tag{2. 3. 2}$$

We now proceed to prove (2. 3. 2). Let us define a function $\xi(u)$, as follows.

$$\xi(u) = \begin{cases} \left(\sin \dfrac{\eta}{2}\right)^{-2} \sin \dfrac{u}{2} & (0 \leqq u \leqq \eta) \\ \left(\sin \dfrac{u}{2}\right)^{-1} & (\eta \leqq u \leqq \pi). \end{cases}$$

Then, for $0 \leqq u \leqq \pi$, $\xi(u)$ is of bounded variation and continuous, with $\xi(+0) = 0$. Also $\xi'(u)$ is bounded and $\xi'(u)$ is integrable (L). Now, since $\xi(u)$ is of bounded variation in $(0, \pi)$, by a well known result* we have, setting

$$A_{-\nu}(x) = A_\nu(x) = A_\nu,$$

$$P_n = \frac{1}{2} A_0 \int_0^\pi \xi(u) \sin\left(n + \frac{1}{2}\right) u\, du$$

$$+ \sum_{\nu=1}^\infty A_\nu \int^\pi \xi(u) \cos \nu u \sin\left(n + \frac{1}{2}\right) u\, du$$

$$= \frac{1}{2} \sum_{\nu=-\infty}^\infty A_\nu \int_0^\pi \xi(u) \sin\left(n - \nu + \frac{1}{2}\right) u\, du$$

* See Hobson [1], page 567.

$$= -\frac{1}{2}\sum_{\nu=-\infty}^{\infty} A_\nu \left[\xi(u) \frac{\cos\left(n-\nu+\frac{1}{2}\right)u}{n-\nu+\frac{1}{2}} \right]_0^\pi$$

$$+ \frac{1}{2}\sum_{\nu=-\infty}^{\infty} A_\nu \int_0^\pi \xi'(u) \frac{\cos\left(n-\nu+\frac{1}{2}\right)u}{n-\nu+\frac{1}{2}}\, du$$

$$= \frac{1}{2}\sum{}' A_\nu \int_0^\pi \xi'(u) \frac{\cos\left(n-\nu+\frac{1}{2}\right)u}{n-\nu+\frac{1}{2}}\, du \; + O(|A_n|),$$

where Σ' denotes summation extending over $-\infty < \nu \leqq n-1$ and $(n+1) \leqq \nu < \infty$. Let

$$\mu = \min(|n-\nu|^{-1}, \eta).$$

Then we have

$$P_n = \frac{1}{2}\sum{}' A_\nu \left(\int_0^\mu + \int_\mu^\pi\right) \xi'(u) \frac{\cos\left(n-\nu+\frac{1}{2}\right)u}{n-\nu+\frac{1}{2}}\, du + O(|A_n|)$$

$$= P_1 + P_2 + O(|A_n|), \text{ say.}$$

Thus we have

$$P_1 = O(1)\sum{}' \frac{|A_\nu|}{(n-\nu)^2};$$

and

$$P_2 = \frac{1}{2}\sum{}' A_\nu \left[\xi'(u) \frac{\sin\left(n-\nu+\frac{1}{2}\right)u}{\left(n-\nu+\frac{1}{2}\right)^2} \right]_{\mu,\ \eta+0}^{\eta-0,\ \pi}$$

$$-\frac{1}{2}\sum{}' A_\nu \int_\mu^\pi \xi''(u) \frac{\sin\left(n-\nu+\frac{1}{2}\right)u}{\left(n-\nu+\frac{1}{2}\right)^2}\, du,$$

where integration by parts is taken separately over the ranges $(\mu, \eta-0)$ and $(\eta+0, \pi)$. Thus we have

$$P_2 = O(1) \sum' \frac{|A_\nu|}{(n-\nu)^2}\ .$$

Hence

$$P_n = O(1) \sum' \frac{|A_\nu|}{(n-\nu)^2} + O(|A_n|)$$

$$= O(1)\Bigl(\sum_{\nu=-\infty}^{0} + \sum_{\nu=1}^{n-1} + \sum_{\nu=n+m+1}^{\infty} + \sum_{\nu=n+1}^{n+m}\Bigr)\frac{|A_\nu|}{(n-\nu)^2} + O(|A_n|)$$

$$= O(1)[M_1 + M_2 + M_3 + M_4 + |A_n|], \text{ say.}$$

Now in order to prove (2. 3. 2), it is sufficient to show that

(2. 3. 3) $$\Sigma M_r/\{n \log (n+1)\} < \infty, \quad r = 1, 2, 3, 4,$$

since, by hypothesis

$$\Sigma\ |A_n|/\{n \log n\} < \infty.$$

Let $0 < \delta < 1$, then we have

$$\sum_{n=1}^{m} \{n \log (n+1)\}^{-1} M_1 \leqq \sum_{n=1}^{m} n^{-2+\delta} \sum_{\nu=0}^{\infty} \frac{|A_\nu|}{|\nu+1|^{1+\delta}}$$

$$= O(1) \sum_{n=1}^{m} n^{-2+\delta} = O(1),$$

as $m \to \infty$.

Again

$$\sum_{n=2}^{m} \{n \log (n+1)\}^{-1} M_2 = \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|A_{n-\nu}|}{n \log (n+1)}$$

$$\leqq \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|A_{n-\nu}|}{(n-\nu) \log (n-\nu+1)}$$

$$= O(1);$$

and

$$\sum_{n=1}^{m} \{n \log (n+1)\}^{-1} M_3 = o(1) \Bigl\{ \sum_{n=1}^{m} \{n \log (n+1)\}^{-1} \sum_{\nu=n+m+1}^{\infty} (\nu - n)^{-2} \Bigr\}$$

$$= \frac{o(1)}{m+1} \sum_{n=1}^{m} \frac{1}{n \log (n+1)} = o(1);$$

as $m \to \infty$.

Lastly

$$\sum_{n=1}^{m} \{n \log (n+1)\}^{-1} M_4 = \sum_{n=1}^{m} \{n \log (n+1)\}^{-1} \sum_{\nu=1}^{m} \nu^{-2}\, |A_{\nu+n}|$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{m} \nu^{-2} \sum_{n=1}^{m} \{n \log (n+1)\}^{-1} \; |A_{\nu+n}| \\
&= \sum_{\nu=1}^{m} \nu^{-2} \left[\sum_{n=1}^{\nu} + \sum_{n=\nu+1}^{m} \right] \\
&= \Sigma_1 + \Sigma_2, \text{ say.}
\end{aligned}
$$

Now

$$
\begin{aligned}
\Sigma_1 &= O\Big(\sum_{\nu=1}^{m} \nu^{-2} \log \nu \Big) \\
&= O(1);
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_2 &= \sum_{\nu=1}^{m} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|A_{\nu+n}|}{(\nu+n) \log (\nu+n)} \left\{ \frac{(\nu+n) \log (\nu+n)}{n \log (n+1)} \right\} \\
&= O(1),
\end{aligned}
$$

as $m \to \infty$, since

$$
\begin{aligned}
\frac{(n+\nu) \log (n+\nu)}{n \log (n+1)} &= \left(1 + \frac{\nu}{n}\right) \left\{ 1 + \frac{\log\left(1 + \dfrac{\nu-1}{n+1}\right)}{\log (n+1)} \right\} \\
&= O(1)
\end{aligned}
$$

for $n \geqq \nu + 1$.

Thus we have established (2. 3. 3) and thereby (2. 3. 2). This completes the proof of the theorem.

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