AFFINE CONNECTIONS IN ALMOST PRODUCT MANIFOLDS WITH SOME STRUCTURES

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The study of affine connections in a manifold with a system of distributions has been made by several authors [8, 9, 10, 11, 12].[1] A. G. Walker [8] and T. J. Willmore [10] gave an affine connection in the large with respect to which the distributions are parallel and which is symmetric if the system is integrable. In the present paper we shall give a general method to obtain all affine connections making the distributions parallel.

M. Obata [6] solved on a common principle the problem of finding all affine connections on manifolds with almost complex, quaternion or Hermitian structure which leave invariant the given structures. His results will be utilized throughout our study of affine connections in Riemannian, almost complex, almost Hermitian or quaternion almost product manifolds.

We shall be concerned only with manifolds of class C^∞ satisfying the second axiom of countability. In such a manifold there always exists a Riemannian metric [7].

1. Almost product manifolds. By an *almost product structure* in a differentiable manifold M we shall mean a system of distributions $D_1, \ldots\ldots, D_m$ such that the tangent space at every point is the direct sum of their elements of contact at the point. The system of distributions will then be said to form a complete system [8]. A tangent vector u at a point x decomposes to give $u = u_1 + \ldots\ldots + u_m$, where $u_\rho(\rho = 1, \ldots\ldots, m)$ belongs to the plane of D_ρ at x. The projection tensors $\underset{1}{a}, \ldots\ldots, \underset{m}{a}$ are given by $\underset{\rho}{a}u = u_\rho$ at every point. They satisfy the identities

$$(1.1) \qquad \underset{\rho}{a}^2 = \underset{\rho}{a}, \quad \underset{\rho}{a}\,\underset{\sigma}{a} = 0 \ (\rho \neq \sigma), \quad \sum \underset{\rho}{a}{}^{2)} = I,$$

I being the identity transformation of the tangent space. Conversely, a set of tensors $\underset{\rho}{a}$ satisfying (1.1) determines an almost product structure.

We shall give a method to get all affine connections with respect to

1) Numbers in brackets refer to the bibliography at the end of the paper.

2) In the present paper Σ always stands for $\underset{\rho}{\Sigma}$ unless otherwise stated.

which every D_ρ is parallel. Conditions for D_ρ to be parallel with respect to an affine connection $L = (L^i_{jk})$ are [8]

(1.2) $$\underset{\rho}{\bar{a}^i_p}\, \underset{\rho}{a^p_{j|k}} = 0,$$

or equivalently

(1.3) $$\underset{\rho}{a^i_{p|k}}\, \underset{\rho}{a^p_j} = 0,$$

where $\underset{\rho}{\bar{a}} = I - \underset{\rho}{a}$ and a vertical line denotes the covariant differentiation with respect to L. If D and D' are any two disjoint parallel distributions, then their sum $D + D'$ is also parallel. Applying the condition (1.2) to D_ρ and $D^\rho = \sum_{\sigma \neq \rho} D_\sigma$, we have

$$\underset{\rho}{\bar{a}^i_p}\, \underset{\rho}{a^p_{j|k}} = 0 \quad \text{and} \quad \underset{\rho}{a^i_p}\, \underset{\rho}{\bar{a}^p_{j|k}} = 0.$$

Since $\underset{\rho}{a} + \underset{\rho}{\bar{a}} = I$, we easily obtain

(1.4) $$\underset{\rho}{a^i_{j|k}} = 0.$$ [3)]

Thus (1.4) is a necessary and sufficient condition for every D_ρ to be parallel with respect to L.

We first choose any affine connection $\Gamma = (\Gamma^i_{jk})$ defined over M. Such a connection always exists in the manifold. Then L is of the form

(1.5) $$L = \Gamma + T,$$

where $T = (T^i_{jk})$ is a tensor which remains to be determined. If a comma denotes covariant differentiation with respect to Γ, then we have the identity

(1.6) $$\underset{\rho}{a^p_{j|k}} = \underset{\rho}{a^p_{j,k}} + \underset{\rho}{a^q_j}\, T^p_{qk} - \underset{\rho}{a^p_q}\, T^q_{jk}.$$

The conditions for D_ρ to be parallel with respect to L are now given [8] by

(1.7) $$\underset{\rho}{\bar{a}^i_p}\, \underset{\rho}{a^q_j}\, T^p_{qk} = -\underset{\rho}{\bar{a}^i_p}\, \underset{\rho}{a^p_{j,k}} = -\underset{\rho}{a^i_{p,k}}\, \underset{\rho}{a^p_j}.$$

To get the general solution of (1.7) we prove the following

LEMMA 1.1. *Let* $\underset{\rho}{f}$ $(\rho = 1, \ldots\ldots, m)$ *be linear transformations of a vector space* V *satisfying the identities*

(1.8) $$\underset{\rho}{f}^2 = \underset{\rho}{f}, \quad \underset{\rho}{f}\underset{\sigma}{f} = 0 \ (\rho \neq \sigma).$$

Then in order that a system of equations

(1.9) $$\underset{\rho}{f}\, T = \underset{\rho}{A}$$

with unknown T *admit a solution, it is necessary and sufficient that we*

3) The condition (1.4) is due to Dr. T. Nagano.

have

$$\underset{\rho}{f}\,\underset{\rho}{A} = \underset{\rho}{A}, \quad \underset{\rho}{f}\,\underset{\sigma}{A} = 0 \ (\rho \neq \sigma). \tag{1.10}$$

The general solution of (1.9) *is of the form*

$$T = \sum \underset{\rho}{A} + A - \sum \underset{\rho}{f} A, \tag{1.11}$$

A being an arbitrary vector in V.

PROOF. If (1.9) has a solution, then $\underset{\rho}{f}\,\underset{\rho}{A} = \underset{\rho}{f}{}^2 T = \underset{\rho}{f}\, T = \underset{\rho}{A}$, $\underset{\rho}{f}\,\underset{\sigma}{A} = \underset{\rho}{f}\,\underset{\sigma}{f}\, T = 0$ $(\rho \neq \sigma)$. Conversely if we have $\underset{\rho}{f}\,\underset{\rho}{A} = \underset{\rho}{A}$, $\underset{\rho}{f}\,\underset{\sigma}{A} = 0$ $(\rho \neq \sigma)$, then $T = \sum \underset{\rho}{A}$ is a solution of (1.9). The general solution of (1.9) is $T = \sum \underset{\rho}{A} + B$, where B is any vector satisfying $\underset{\rho}{f} B = 0$. If B is of the form $B = A - \sum \underset{\rho}{f} A$, then we have $\underset{\rho}{f} B = 0$. If conversely $\underset{\rho}{f} B = 0$, then, putting B in place of A, we have $B = B - \sum \underset{\rho}{f} B$, which proves the lemma.

The system of equations (1.7) may be considered as an example of (1.9) in which identities (1.8) and conditions (1.10) [8] are easily seen to be satisfied. Therefore the general solution of (1.7) is now seen to be

$$\begin{aligned} T^i_{jk} &= -\sum \underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j + A^i_{jk} - \sum \underset{\rho}{\bar{a}}{}^i_p\, \underset{\rho}{a}{}^q_j\, A^p_{qk} \\ &= -\sum \underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j + \sum_{\rho,\sigma} \underset{\sigma}{a}{}^i_p\, \underset{\rho}{a}{}^q_j\, A^p_{qk} - \sum \underset{\rho}{\bar{a}}{}^i_p\, \underset{\rho}{a}{}^q_j\, A^p_{qk} \\ &= -\sum \underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j + \sum \underset{\rho}{a}{}^i_p\, \underset{\rho}{a}{}^q_j\, A^p_{qk}, \end{aligned} \tag{1.12}$$

$A = (A^i_{jk})$ being an arbitrary tensor. Thus we have

THEOREM 1.1. *Let Γ be an arbitrary but fixed affine connection in an almost product manifold. Then in order that every D_ρ be parallel with respect to an affine connection L it is necessary and sufficient that L be of the form*

$$L^i_{jk} = \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j + \sum \underset{\rho}{a}{}^i_p\, \underset{\rho}{a}{}^q_j\, A^p_{qk}, \tag{1.13}$$

A being an arbitrary tensor.

For an affine connection Γ we define an affine connection $L(\Gamma)$ by

$$L^i_{jk}(\Gamma) = \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j. \tag{1.14}$$

Then by Theorem 1.1 every D_ρ is parallel with respect to $L(\Gamma)$. If Γ makes every D_ρ parallel, then we have $\underset{\rho}{a}{}^i_{p,k}\, \underset{\rho}{a}{}^p_j = 0$ and hence $L(\Gamma) = \Gamma$. Thus we have

THEOREM 1.2. *In an almost product manifold every D_ρ is parallel with respect to an affine connection L if and only if there exists an affine connection Γ such that $L = L(\Gamma)$.*

For an affine connection Γ and a tensor $A = (A^i_{jk})$ we have by (1.6)

$$
\begin{aligned}
L^i_{jk}(\Gamma + A) &= \Gamma^i_{jk} + A^i_{jk} - \sum (\underset{\rho}{a}{}^i_{p,k} + \underset{\rho}{a}{}^q_p A^i_{qk} - \underset{\rho}{a}{}^i_q A^q_{pk}) \underset{\rho}{a}{}^p_j \\
&= \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j + A^i_{jk} - \sum (A^i_{pk} - \underset{\rho}{a}{}^i_q A^q_{pk}) \underset{\rho}{a}{}^p_j \qquad (1.15) \\
&= L^i_{jk}(\Gamma) + A^i_{jk} - \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk} \\
&= L^i_{jk}(\Gamma) + \sum \underset{\rho}{a}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk}.
\end{aligned}
$$

Thus we have

COROLLARY 1. *Let Γ be an affine connection with respect to which every D_ρ is parallel. Then, $\Gamma + T$, T being a tensor field, is also an affine connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$T^i_{jk} = \sum \underset{\rho}{a}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk}. \tag{1.16}$$

A. G. Walker [8] showed that in an almost product manifold the connection L given by

$$L = \Gamma + \sum T(\underset{\rho}{\bar{a}}, \Gamma) \tag{1.17}$$

with symmetric affine connection Γ has the property that it makes every D_ρ parallel and is symmetric if the system $\{D_\rho\}$ is integrable, where

$$T^i_{jk}(a, \Gamma) = - a^i_{p,j} a^p_k - a^i_{p,k} a^p_j + a^i_{p,q} a^p_k a^q_j. \tag{1.18}$$

We have

$$
\begin{aligned}
T^i_{jk}(\underset{\rho}{\bar{a}}, \Gamma) &= T^i_{jk}(I - \underset{\rho}{a}, \Gamma) \qquad (1.19)\\
&= \underset{\rho}{a}{}^i_{j,k} - \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j + \underset{\rho}{a}{}^i_{k,p} \underset{\rho}{a}{}^p_j - \underset{\rho}{a}{}^i_{p,q} \underset{\rho}{a}{}^p_k \underset{\rho}{a}{}^q_j
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\Gamma^i_{jk} + \sum T_{jk}(\underset{\rho}{\bar{a}}, \Gamma) &= \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j + \sum \underset{\rho}{a}{}^i_{k,p} \underset{\rho}{a}{}^p_j - \sum \underset{\rho}{a}{}^i_{p,q} \underset{\rho}{a}{}^p_k \underset{\rho}{a}{}^q_j \qquad (1.20)\\
&= L^i_{jk}(\Gamma) + \sum \underset{\rho}{a}{}^i_{k,p} \underset{\rho}{a}{}^p_j - \sum \underset{\rho}{a}{}^i_{p,q} \underset{\rho}{a}{}^p_k \underset{\rho}{a}{}^q_j.
\end{aligned}
$$

The connection $L = \Gamma + \sum T(\underset{\rho}{\bar{a}}, \Gamma)$ is obtained as $L(\Gamma + A)$ with

$$A^i_{jk} = \sum_\rho \underset{\rho}{a}^i_{k,p}\, \underset{\rho}{a}^p_j. \tag{1.21}$$

In fact we have by (1.15)

$$\begin{aligned} L^i_{jk}(\Gamma + A) &= L^i_{jk}(\Gamma) + \sum \underset{\sigma}{a}^i_p\, \underset{\sigma}{a}^q_j\, A^p_{qk} \\ &= L^i_{jk}(\Gamma) + \sum_{\sigma,\rho} \underset{\sigma}{a}^i_p\, \underset{\sigma}{a}^q_j\, \underset{\rho}{a}^p_{k,r}\, \underset{\rho}{a}^r_q \\ &= L^i_{jk}(\Gamma) + \sum \underset{\rho}{a}^i_p\, \underset{\rho}{a}^p_{k,q}\, \underset{\rho}{a}^q_j \\ &= L^i_{jk}(\Gamma) + \sum \underset{\rho}{a}^i_{k,p}\, \underset{\rho}{a}^p_j - \sum \underset{\rho}{a}^i_{p,q}\, \underset{\rho}{a}^p_k\, \underset{\rho}{a}^q_j. \end{aligned}$$

The torsion tensor $t^i_{jk} = L^i_{[jk]}$ of Walker's connection $L = \Gamma + \sum_\rho T(\underset{\rho}{\bar{a}}, \Gamma)$ is independent of the particular symmetric connection Γ chosen [10]. If we set

$$\underset{\rho}{\sigma}^i_{jk} = T^i_{[jk]}(\underset{\rho}{\bar{a}}, \Gamma), \tag{1.22}$$

then we have

$$\underset{\rho}{\sigma}^i_{jk} = \underset{\rho}{\bar{a}}^q_{[j}\, \underset{\rho}{\bar{a}}^p_{k]}\, \partial_p\, \underset{\rho}{a}^i_q. \tag{1.23}$$

For two vector fields $u = (u^i)$ and $v = (v^i)$ we denote by $\underset{\rho}{\sigma}\,(u, v)$ the vector field $\underset{\rho}{\sigma}^i_{jk}\, u^j v^k$. Then we have

$$2\underset{\rho}{\sigma}(u, v) = \underset{\rho}{a}[\underset{\rho}{\bar{a}}\, u, \underset{\rho}{\bar{a}}\, v]. \tag{1.24}$$

Consequently the vanishing of a set of tensors $\underset{\rho}{\sigma}^i_{jk}$ is a necessary and sufficient condition for the system of distributions to be integrable.

The tensor $\underset{\rho}{\sigma}$ and the Nijenhuis tensor of $\underset{\rho}{a}$ are related in the following way. The Nijenhuis tensor $N(a)$ of a tensor $a = (a^i_j)$ is given [4, 5] by

$$N^i_{jk}(a) = 2(a^p_{[j}\, \partial_{|p|}\, a^i_{k]} - a^i_p\, \partial_{[j}\, a^p_{k]}). \tag{1.25}$$

For vector fields u and v we denote by $N(a)(u, v)$ the vector field $N^i_{jk}(a)$ $u^j v^k$. Then we have [4, 5]

$$N(a)(u, v) = -a[a\, u, v] - a[u, a\, v] + a^2[u, v] + [a\, u, a\, v]. \tag{1.26}$$

Simple calculation shows

$$2\underset{\rho}{\sigma}(u, v) = \underset{\rho}{a}\, N(\underset{\rho}{a})(u, v) \tag{1.27}$$

and hence we have

$$t^i_{jk} = \sum_\rho \underset{\rho}{\sigma}^i_{jk} = \frac{1}{2} \sum_\rho \underset{\rho}{a}^i_p\, N^p_{jk}(\underset{\rho}{a}). \tag{1.28}$$

For the case $\rho = 1, 2$, we have [5] $N(\underset{1}{a}) = N(\underset{2}{a})$ and hence

$$t^i_{jk} = \frac{1}{2}(\underset{1}{a}{}^i_p N^p_{jk}(\underset{1}{a}) + \underset{2}{a}{}^i_p N^p_{jk}(\underset{2}{a})) \tag{1.29}$$
$$= \frac{1}{2} N^i_{jk}(\underset{1}{a}).$$

2. Riemannian almost product manifolds. A Riemannian metric g on an almost product manifold is said to be an *almost product metric* if the distributions D_ρ are orthogonal to each other with respect to the metric g, i. e. if, at every point x, $g_{ij}u^i v^j = 0$ for all vectors u in D_ρ and v in $D_\sigma (\rho \neq \sigma)$. This condition for orthogonality is equivalent to

$$g_{pq}\, \underset{\rho}{a}{}^p_i\, \underset{\sigma}{a}{}^q_j = 0 \quad (\rho \neq \sigma). \tag{2.1}$$

An almost product manifold with an almost product metric is said to be a *Riemannian almost product manifold.*

Let h be an arbitrary Riemannian metric in an almost product manifold. We define a tensor g by

$$g_{ij} = \sum h_{pq}\, \underset{\rho}{a}{}^p_i\, \underset{\rho}{a}{}^q_j. \tag{2.2}$$

If we denote by $(u, v)_g$ and $(u, v)_h$ the inner products of the tangent space at every point defined by g and h respectively, then the definition (2. 2) can be written as

$$(u, v)_g = (u_1, v_1)_h + \cdots\cdots + (u_m, v_m)_h, \tag{2.3}$$

where $u = u_1 + \cdots\cdots + u_m$ with $u_\rho \in D_\rho$ and $v = v_1 + \cdots\cdots + v_m$ with $v_\rho \in D_\rho$. From (2. 2) or (2. 3) one sees at once that g is an almost product metric. The metric g defined by (2. 2) from h will be denoted by $g(h)$. If h is an almost product metric, then $g(h) = h$. Thus we have

THEOREM 2. 1. *In an almost product manifold there always exists an almost product metric. In an almost product manifold a metric g is an almost product metric if and only if there exists a metric h such that $g = g(h)$.*

Let g be an almost product metric. Then we have

$$(u, v)_g = (u_1, v_1)_g + \cdots\cdots + (u_m, v_m)_g. \tag{2.4}$$

Therefore we have

$$(u, \underset{\rho}{a}\, v)_g = (u, v_\rho)_g = (u_\rho, v_\rho)_g = (v_\rho, u_\rho)_g = (v, u_\rho)_g = (v, \underset{\rho}{a} u)_g.$$

This means that for the tensor $\underset{\rho}{a}{}_{ij} = g_{ip}\, \underset{\rho}{a}{}^p_j$ we have $\underset{\rho}{a}{}_{ij} = \underset{\rho}{a}{}_{ji}$. Let Γ be a metric connection with respect to g, i. e. $g_{ij,k} = 0$. Then an affine connection

$\Gamma + Q$ is metric if and only if

$$Q^i_{jk} + g^{ib} g_{ja} Q^a_{bk} = 0. \tag{2.5}$$

We shall now show that the connection $L(\Gamma)$ of (1.14) is also a metric connection with respect to g. In fact we have

$$\begin{aligned}\sum_\rho \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + g^{ib} g_{ja} \sum_\rho \underset{\rho}{a^a_{p,k}} \underset{\rho}{a^p_b} &= \sum_\rho \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + \sum_\rho \underset{\rho}{a_{jp\,k}} \underset{\rho}{a^{pi}} \\ &= \sum_\rho \underset{\rho}{a^i_{p\,k}} \underset{\rho}{a^p_j} + \sum_\rho \underset{\rho}{a^i_{j,k}} - \sum_\rho \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} \\ &= \delta^i_{j,k} \\ &= 0.\end{aligned}$$

For any affine connection Γ, the affine connection defined by

$$\Lambda\Gamma^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} g^{ia} g_{ja,k} \tag{2.6}$$

is a metric connection with respect to g [2,6]. If Γ is a metric connection with respect to g, then $\Lambda\Gamma = \Gamma$. Consequently, for any affine connection Γ, the affine connection $L(\Lambda\Gamma)$ is a metric connection with respect to g which makes every D_ρ parallel. If Γ is a metric connection making every D_ρ parallel, then $L(\Lambda\Gamma) = \Gamma$. Thus we have

THEOREM 2.2. *In a Riemannian almost product manifold, in order that an affine connection L be a metric connection with respect to g making every D_ρ parallel it is necessary and sufficient that there exists an affine connection Γ such that $L = L(\Lambda\Gamma)$.*

From (1.15) and (2.6) we have

$$L^i_{jk}(\Lambda\Gamma) = \Gamma^i_{jk} - \sum_\rho \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + \frac{1}{2} \sum_\rho \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} g^{pa} g_{qa,k}. \tag{2.7}$$

If Γ is a metric connection making every D_ρ parallel, then we have [6]

$$\begin{aligned}\Lambda(\Gamma^i_{jk} + A^i_{jk}) &= \Gamma^i_{jk} + \Lambda_1 A^i_{jk} \\ &= \Gamma^i_{jk} + \frac{1}{2}(A^i_{jk} - g^{ib} g_{ja} A^a_{bk}),\end{aligned} \tag{2.8}$$

and hence from (1.15)

$$L(\Lambda(\Gamma^i_{jk} + A^i_{jk})) = \Gamma^i_{jk} + \frac{1}{2} \sum_\rho \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} (A^p_{qk} - g^{pb} g_{qa} A^a_{bk}). \tag{2.9}$$

Thus we have

THEOREM 2.3. *Let Γ be a metric connection making every D_ρ parallel in a Riemannian almost product manifold. Then $L = \Gamma + B$ is also a metric connection making every D_ρ parallel if and only if there exists* *a tensor field A such that*

$$B^i_{jk} = \frac{1}{2} \sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} (A^p_{qk} - g^{pb} g_{qa} A^a_{bk}). \tag{2.10}$$

If Γ is chosen to be the Christoffel connection given by g and $\rho = 1, 2$, then $L(\Gamma)$ is identical with the connection given by A. G. walker [9] in a Riemannian almost product manifold for $\rho = 1, 2$.

3. Almost complex almost product manifolds. An almost product manifold with an almost complex structure $\phi = (\phi^i_j)$ is said to be an *almost complex almost product manifold* if we have $\underset{\rho}{a}\phi = \phi \underset{\rho}{a}$.

Let Γ be a ϕ-connection, i. e., $\phi^i_{j,k} = 0$, in an almost complex manifold. Then an affine connection $L = \Gamma + A$ is a ϕ-connection if and only if $\Phi_2 A = 0$ [6], where

$$\Phi_2 A^i_{jk} = \frac{1}{2}(A^i_{jk} + \phi^i_a \phi^b_j A^a_{bk}). \tag{3.1}$$

If Γ is ϕ-connection in an almost complex almost product manifold, then the connection $L(\Gamma)$ is also a ϕ-connection. In fact we have

$$\begin{aligned} \sum \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + \phi^i_a \phi^b_j \sum \underset{\rho}{a^a_{p,k}} \underset{\rho}{a^p_b} &= \sum \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + \phi^a_p \phi^p_b \sum \underset{\rho}{a^i_{a,k}} \underset{\rho}{a^b_j} \\ &= \sum \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} - \sum \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} \\ &= 0. \end{aligned}$$

For any affine connection Γ in an almost complex manifold, the affine connection $\Phi\Gamma$ defined by

$$\Phi \Gamma^i_{jk} = \Gamma^i_{jk} - \frac{1}{2} \phi^i_a \phi^a_{j,k} \tag{3.2}$$

is a ϕ-connection [6]. If Γ is a ϕ-connection, $\Phi\Gamma = \Gamma$. It follows that, for any affine connection Γ in an almost complex almost product manifold, the affine connection $L(\Phi\Gamma)$ is a ϕ-connection making every D_ρ parallel. If Γ is a ϕ-connection making every D_ρ parallel, then $L(\Phi\Gamma) = \Gamma$. Therefore we have proved

THEOREM 3.1. *In an almost complex almost product manifold, an affine connection L is a ϕ-connection making every D_ρ parallel if and only if there exists an affine connection Γ such that $L = L(\Phi\Gamma)$.*

We have from (1.15) and (3.2)

$$L^i_{jk}(\Phi\Gamma) = \Gamma^i_{jk} - \sum \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} - \frac{1}{2}\sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} \phi^p_a \phi^a_{q,k}. \tag{3.3}$$

If Γ is a ϕ-connection making every D_ρ parallel, then we have [6]

$$(3.4) \qquad \Phi(\Gamma^i_{jk} + A^i_{jk}) = \Gamma^i_{jk} + \frac{1}{2}(A^i_{jk} - \phi^i_a \phi^b_j A^a_{bk})$$

and therefore from (1. 15)

$$(3.5) \qquad L(\Phi(\Gamma^i_{jk} + A^i_{jk})) = \Gamma^i_{jk} + \frac{1}{2}\Big(\sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} A^p_{qk} - \sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} \phi^p_a \phi^b_q A^a_{bk}\Big).$$

Thus we have

THEOREM 3. 2. *Let Γ be a ϕ-connection making every D_ρ parallel in an almost complex almost product manifold. Then $L = \Gamma + B$ is also a ϕ-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$(3.6) \qquad B^i_{jk} = \frac{1}{2}\Big(\sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} A^p_{qk} - \sum \underset{\rho}{a^i_p} \underset{\rho}{a^q_j} \phi^p_a \phi^b_q A^a_{bk}\Big).$$

Theorem 3. 2 may also be proved in the following way. The conditions that B must satisfy for $L = \Gamma + B$ to be a ϕ-connection making every D_ρ parallel are [6]

$$\Phi_2 B^i_{jk} = \frac{1}{2}(B^i_{jk} + \phi^i_a \phi^b_j B^a_{bk}) = 0.$$

and from (1. 7)

$$\underset{\rho}{\bar{a}^i_p} \underset{\rho}{a^q_j} B^p_{qk} = 0.$$

Theorem 3. 2 can be derived from the following

LEMMA 3. 1. *Let f_r $(r = 1, \ldots\ldots, t)$ de linear transformations of a vector space V satisfying the identities*

$$(3.7) \qquad f_r^2 = f_r, \quad f_r f_s = f_s f_r.$$

Then the general solution of a system of equations

$$(3.8) \qquad f_r B = 0$$

is of the form

$$(3.9) \qquad B = (I - f_1) \ldots\ldots (I - f_t)A,$$

A being an arbitrary vector in V.

PROOF. If B is a solution of $f_r B = 0$, then $(I - f_r)B = B$ and hence $B = (I - f_1) \ldots\ldots (I - f_t)B$. From $f_r^2 = f_r$ and $f_r f_s = f_s f_r$ we have $f_r(I - f_r) = 0$ and $(I - f_r)(I - f_s) = (I - f_s)(I - f_r)$. It follows that $f_r(I - f_1) \ldots (I - f_t) A = 0$, which proves the lemma.

If we set $f_\rho B^i_{jk} = \underset{\rho}{\bar{a}^i_p} \underset{\rho}{a^q_j} B^p_{qk}$ $(\rho = 1, \ldots\ldots, m)$ and $f_{m+1} B^i_{jk} = \Phi_2 B^i_{jk}$, then identities $f_r^2 = f_r$ and $f_r f_s = f_s f_r$ are easily seen and we have $(I - f_1) \ldots\ldots$

$(I - f_m) = I - \Sigma f_\rho$.

4. Almost Hermitian almost product manifolds. An almost complex almost product manifold with an almost Hermitian metric g with respect to ϕ is said to be an *almost Hermitian almost product manifold* if the metric g is an almost product metric.

Let h be an arbitrary Riemannian metric in an almost complex almost product manifold. Then the tensor $\Phi_4 h$ given by

$$(4.1)\qquad \Phi_4 h_{ij} = \frac{1}{2}(h_{ij} + \phi_i^a \phi_j^b h_{ab})$$

is an almost Hermitian metric with respect to ϕ [1, 3, 6]. If h is an almost Hermitian metric with respect to ϕ, then $\Phi_4 h = h$. From (2.2) we have for $g(\Phi_4 h)$

$$(4.2)\qquad g_{ij}(\Phi_4 h) = \frac{1}{2}\sum (h_{pq} + \phi_p^a \phi_q^b h_{ab})\, \underset{\rho}{a}{}_i^p\, \underset{\rho}{a}{}_j^q.$$

If we denote by $(u, v)_h$ and $(u, v)_g$ the inner products defined by h and $g(\Phi_4 h)$ respectively, then (4.2) is written as

$$(4.3)\qquad (u, v)_g = \frac{1}{2}((u_1, v_1)_h + \dots + (u_m, v_m)_h + (\phi u_1, \phi v_1)_h + \dots\dots + (\phi u_m, \phi v_m)_h).$$

Since $\underset{\rho}{a}\phi = \phi \underset{\rho}{a}$, we see at once from (4.2) or (4.3) that $g(\Phi_4 h)$ is an almost product metric and almost Hermitian with respect to ϕ. If h is an almost product metric and almost Hermitian with respect to ϕ, then $g(\Phi_4 h) = h$. Thus we have

THEOREM 4.1. *In an almost complex almost product manifold there always exists an almost product metric. In an almost complex almost product manifold a metric g is an almost Hermitian almost product metric if and only if there exists a metric h such that $g = g(\Phi_4 h)$.*

In an almost Hermitian manifold, for any affine connection Γ the connection $\Phi\Lambda\Gamma$ is a metric ϕ-connection [6], where

$$(4.4)\qquad \Phi\Lambda\Gamma_{jk}^i = \Lambda\Phi\Gamma_{jk}^i = \Gamma_{jk}^i + \frac{1}{4}(g^{ia} g_{ja,k} - \phi_a^i \phi_{j,k}^a - \phi^{ai}\phi_{ja,k}),$$

$$\phi_{ij} = \phi^a{}_i\, g_{aj}, \quad \phi^{ij} = g^{ib}\phi_b^j.$$

It follows from §§ 2 and 3 that, in an almost Hermitian almost product manifold, for any affine connection Γ the connection $L(\Phi\Lambda\Gamma)$ is a metric ϕ-connection making every D_ρ parallel. If Γ is a metric ϕ-connection making every D_ρ parallel, then $L(\Phi\Lambda\Gamma) = L(\Phi\Gamma) = L(\Gamma) = \Gamma$. Thus we have

THEOREM 4. 2. *In an almost Hermitian almost product manifold an affine connection L is a metric ϕ-connection making every D_ρ parallel, if and only if there exists an affine connection Γ such that $L = L(\Phi\Lambda\Gamma)$.*

From (4. 4) and (1. 15) we have

$$(4.5)\qquad L^i_{jk}(\Phi\Lambda\Gamma) = \Gamma^i_{jk} - \sum_\rho \underset{\rho}{a^i_{p,k}} \underset{\rho}{a^p_j} + \frac{1}{4}\sum \underset{\rho}{a^i_p}\underset{\rho}{a^q_j}(g^{pa}g_{qa,k} - \phi^p_a\phi^a_{q,k} - \phi^{ap}\phi_{qa,k}).$$

Since we have [6]

$$(4.6)\qquad \Lambda(\Gamma + A) = \Lambda\Gamma + \Lambda_1 A,$$
$$\Phi(\Gamma + A) = \Phi\Gamma + \Phi_1 A,$$

and hence

$$\Phi\Lambda(\Gamma + A) = \Phi\Lambda\Gamma + \Phi_1\Lambda_1 A,$$

where

$$\Lambda_1 A^i_{jk} = \frac{1}{2}(A^i_{jk} - g^{ib}\, g_{ja}\, A^a_{bk}),$$

$$\Phi_1 A^i_{jk} = \frac{1}{2}(A^i_{jk} - \phi^i_a\, \phi^b_j\, A^a_{bk}),$$

and

$$\Phi_1\Lambda_1 A^i_{jk} = \Lambda_1\Phi_1 A^i_{jk} = \frac{1}{4}(A^i_{jk} - \phi^i_a\,\phi^b_j\, A^a_{bk} - g^{ib}\, g_{aj}\, A^a_{bk} + \phi^{ib}\,\phi_{aj}\, A^a_{bk}),$$

we get

$$(4.7)\qquad L^i_{jk}(\Phi\Lambda(\Gamma + A)) = L^i_{jk}(\Phi\Lambda\Gamma) + \frac{1}{4}\sum \underset{\rho}{a^i_p}\, \underset{\rho}{a^q_j}$$
$$\times (A^p_{qk} - \phi^p_t\,\phi^b_q\, A^a_{bk} - g^{pb}\, g_{aq}\, A^a_{bk} + \phi^{pb}\,\phi_{aq}\, A^a_{k}).$$

Thus we have

THEOREM 4. 3. *Let Γ be a metric ϕ-connection making every D_ρ parallel in an almost Hermitian almost product manifold. Then $L = \Gamma + B$ is also a metric ϕ-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$(4.8)\qquad B^i_{jk} = \frac{1}{4}\sum \underset{\rho}{a^i_p}\, \underset{\rho}{a^q_j}\,(A^p_{qk} - \phi^p_t\,\phi^b_q\, A^a_{bk} - g^{pb}\, g_{aq}\, A^a_{bk} + \phi^{pb}\phi_{aq}A^a_{bk}).$$

5. Quaternion almost product manifolds. An almost product manifold with a quaternion structure (ϕ, ψ) [6] is called a *quaternion almost product manifold* if the manifold is an almost complex almost product manifold with respect to both ϕ and ψ.

In a quaternion manifold, for any affine connection Γ the connection $\Phi\Psi\Gamma$ is a (ϕ, ψ)-connection [6], where

$$\Phi\Psi\Gamma^i_{jk} = \Gamma^i_{jk} - \frac{1}{4}(\phi^i_a \phi^a_{j,k} + \psi^i_a \psi^a_{j,k} + \kappa^i_a \kappa^a_{j,k}). \tag{5.1}$$

Therefore from § 3 in a quaternion almost product manifold the connection $L(\Phi\Psi\Gamma)$ for any affine connection Γ is a (ϕ, ψ)-connection with respect to which every D_ρ is parallel. If Γ is a (ϕ, ψ)-connection making every D_ρ parallel, then $L(\Phi\Psi\Gamma) = \Gamma$. Thus we have

THEOREM 5.1. *In a quaternion almost product manifold an affine connection L is a (ϕ, ψ)-connection making every D_ρ parallel, if and only if there exists an affine connection Γ such that $L = L(\Phi\Psi\Gamma)$.*

From (5.1) and (1.15) we have

$$L^i_{jk}(\Phi\Psi\Gamma) = \Gamma^i_{jk} - \sum_\rho \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j - \frac{1}{4}\sum_\rho \underset{\rho}{a}{}^i_p \underset{\rho}{a}{}^q_j (\phi^p_a \phi^a_{q,k} + \psi^p_a \psi^a_{q,k} + \kappa^p_a \kappa^a_{q,k}). \tag{5.2}$$

Since we have [6]

$$\Phi\Psi(\Gamma + A) = \Phi\Psi\Gamma + \Phi_1\Psi_1 A, \tag{5.3}$$

where $\Phi_1\Psi_1 A^i_{jk} = \frac{1}{4}(A^i_{jk} - \phi^i_a \phi^b_j A^a_{bk} - \psi^i_a \psi^b_j A^a_{bk} - \kappa^i_a \kappa^b_j A^a_{bk})$, we get from (1.15)

$$\begin{aligned} L^i_{jk}(\Phi\Psi(\Gamma + A)) = L^i_{jk}(\Phi\Psi\Gamma) + \frac{1}{4}\sum_\rho \underset{\rho}{a}{}^i_p \underset{\rho}{a}{}^q_j \\ \times (A^p_{qk} - \phi^p_a \phi^b_q A^a_{bk} - \psi^p_a \psi^b_q A^a_{bk} - \kappa^p_a \kappa^b_q A^a_{bk}). \end{aligned} \tag{5.4}$$

Thus we have

THEOREM 5.2. *Let Γ be a (ϕ, ψ)-connection making every D_ρ parallel in a quaternion almost product manifold. Then $L = \Gamma + B$ is also (ϕ, ψ)-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$B^i_{jk} = \frac{1}{4}\sum_\rho \underset{\rho}{a}{}^i_p \underset{\rho}{a}{}^q_j (A^p_{qk} - \phi^p_a \phi^b_q A^a_{bk} - \psi^p_a \psi^b_q A^a_{bk} - \kappa^p_a \kappa^b_q A^a_{bk}). \tag{5.5}$$

An almost Hermitian manifold with quaternion structure [6] is by definition a quaternion manifold with a Riemannian metric which is almost Hermitian with respect to both ϕ and ψ. By an *almost Hermitian almost product manifold with quaternion structure* we mean a quaternion almost product manifold with an almost product metric which is almost Hermitian with respect to both ϕ and ψ.

In a quaternion manifold for an arbitrary Riemannian metric h the tensor $g = \Phi_4\Psi_4 h$ is an almost Hermitian metric with respect to both ϕ and ψ [6], where

(5.6) $$\Psi_4 h_{ij} = \frac{1}{2}(h_{ij} + \psi_i^a \psi_j^b h_{ab}),$$

and hence

$$\Phi_4\Psi_4 h_{ij} = \Psi_4\Phi_4 h_{ij} = \frac{1}{4}(h_{ij} + \phi_i^a \phi_j^b h_{ab} + \psi_i^a \psi_j^b h_{ab} + \kappa_i^a \kappa_j^b h_{ab}).$$

It follows from § 4 that in a quaternion almost product manifold the almost product metric $g(\Phi_4\Psi_4 h)$ for any Riemannian metric h is almost Hermitian with respect to both ϕ and ψ. If h is an almost product metric and almost Hermitian with respect to both ϕ and ψ in a quaternion almost product manifold, then $g(\Phi_4\Psi_4 h) = h$. Thus we have

THEOREM 5.3. *In a quaternion almost product manifold there always exists an almost product metric which is almost Hermitian with respect to both ϕ and ψ. In a quaternion almost product manifold a metric g is an almost product metric which is almost Hermitian with respect to both ϕ and ψ, if and only if there exists a metric h such that $g = g(\Phi_4\Psi_4 h)$.*

In an almost Hermitian manifold with quaternion structure the connection $\Phi\Psi\Lambda\Gamma$ defined for any affine connection Γ is a metric (ϕ, ψ)-connection [6]. If Γ is a metric (ϕ, ψ)-connection, then $\Phi\Psi\Lambda\Gamma = \Gamma$. Therefore, in an almost Hermitian almost product manifold with quaternion structure, the connection $L(\Phi\Psi\Lambda\Gamma)$ defined for any affine connection Γ is a metric (ϕ, ψ)-connection making every D_ρ parallel. If Γ is a metric (ϕ, ψ)-connection making every D_ρ parallel, then $L(\Phi\Psi\Lambda\Gamma) = \Gamma$. Thus we have

THEOREM 5.4. *In an almost Hermitian almost product manifold with quaternion structure, in order that an affine connection L be a metric (ϕ, ψ)-connection making every D_ρ parallel, it is necessary and sufficient that there exists an affine connection Γ such that $L = L(\Phi\Psi\Lambda\Gamma)$.*

From (1.15), (2.6) and (5.3) we have

(5.7) $$L^i_{jk}(\Phi\Psi\Lambda\Gamma) = L^i_{jk}(\Phi\Psi\Gamma) + \frac{1}{8}\sum_\rho a^i_p a^q_j (g^{pa} g_{qa,k} - \phi^{ap}\phi^b_q g_{ab,k} - \psi^{ap}\psi^b_q g_{ab,k} - \kappa^{ap}\kappa^b_q g_{ab,k}).$$

From (2.8) and (5.3) we have

(5.8) $$\Phi\Psi\Lambda(\Gamma + A) = \Phi\Psi\Lambda\Gamma + \Phi_1\Psi_1\Lambda_1 A,$$

where

$$\Phi_1\Psi_1\Lambda_1 A^i_{jk} = \frac{1}{8}(A^i_{jk} - \phi^i_a \phi^b_j A^a_{bk} - \psi^i_a \psi^b_j A^a_{bk} - \kappa^i_a \kappa^b_j A^a_{bk} - g^{ib} g_{ja} A^a_{bk} - \phi^{ib}\phi_{ja} A^a_{bk} - \psi^{ib}\psi_{ja} A^a_{bk} - \kappa^{ib}\kappa_{ja} A^a_{bk}).$$

Consequently we get from (1.15)

$$L^i_{jk}(\Phi\Psi\Lambda(\Gamma + A)) = L^i_{jk}(\Phi\Psi\Lambda\Gamma)$$

$$(5.9) \qquad + \frac{1}{8} \sum \underset{\rho}{a^i_p}\, \underset{\rho}{a^q_j}(A^p_{qk} - \phi^p_a \phi^b_q A^a_{bk} - \psi^p_a \psi^b_q A^a_{bk} - \kappa^p_a \kappa^b_q A^a_{bk} - g^{pb} g_{qa} A^a_{bk} - \phi^{pb}\phi_{qa} A^a_{bk} - \psi^{pb}\psi_{qa} A^a_{bk} - \kappa^{pb}\kappa_{qa} A^a_{bk}).$$

Thus we have

THEOREM 5.5 *Let Γ be a metric (ϕ, ψ)-connection making every D_ρ parallel in an almost Hermitian almost product manifold with quaternion structure. Then $L = \Gamma + B$ is also a metric (ϕ, ψ)-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$(5.10) \qquad B^i_{jk} = \frac{1}{8} \sum \underset{\rho}{a^i_p}\, \underset{\rho}{a^q_j}(A^p_{qk} - \phi^p_a \phi^b_q A^a_{bk} - \psi^p_a \psi^b_q A^a_{bk} - \kappa^p_a \kappa^b_q A^a_{bk} - g^{pb} g_{qa} A^a_{bk} - \phi^{pb}\phi_{qa} A^a_{bk} - \psi^{pb}\psi_{qa} A^a_{bk} - \kappa^{pb}\kappa_{qa} A^a_{bk}).$$

6. Manifolds with a system of disjoint distributions. So far we have considered only complete systems of distributions. In this section we shall deal with a manifold having a system of disjoint distributions and give all affine connections making every distribution parallel. If such a manifold has some additional structures, then we shall discuss affine connections making the structures covariant constant.

Suppose that disjoint distributions $D_1, \dots, D_m$ are given over a differentiable manifold M and we wish to find all affine connections L which make them parallel. Let us set $D = \sum D_\rho$. We may choose a distribution $\overline{D}$ so that D and $\overline{D}$ are disjoint and complementary. $\overline{D}$ could, for example, be the orthogonal complement of D with respect to a Riemannian metric which is known to exist. The system $\{D_\rho, \overline{D}\}$ is complete and therefore determines tensors $\underset{\rho}{a}$, $\underset{\rho}{\bar{a}}$ $(\rho = 1, \dots\dots, m)$ over M.

As in § 1, we first choose an affine connection $\Gamma = (\Gamma^i_{jk})$ over M and write the affine connection L in the from

$$(6.1) \qquad L = \Gamma + T.$$

The conditions for D_ρ to be parallel with respect to L are (1.7). Lemma 1.1 applies also to the present case and we get the following general solution for affine connections which make the distributions $D_1, \dots\dots, D_m$ parallel:

$$(6.2) \qquad L^i_{jk} = \Gamma^i_{jk} - \sum \underset{\rho}{a^i_{p,k}}\, \underset{\rho}{a^p_j} + A^i_{jk} - \sum \underset{\rho}{\bar{a}^i_p}\, \underset{\rho}{a^q_j} A^p_{qk},$$

$A = (A^i_{jk})$ being an arbitrary tensor. Thus we have

THEOREM 6.1. *Let Γ be an arbitrary but fixed affine connection in a* *manifold with a system of disjoint distributions $D_1, \dots, D_m$. Then in order that every D_ρ be parallel with respect to an affine connection L, it is necessary and sufficient that L be of the form*

$$L^i_{jk} = \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j + A^i_{jk} - \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk}, \tag{6.3}$$

A being an arbitrary tensor.

We associate with an affine connection Γ the affine connection $L(\Gamma)$ in the same way as in § 1:

$$L^i_{jk}(\Gamma) = \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j. \tag{6.4}$$

Then we have

THEOREM 6. 2. *In a manifold with a system of disjoint distributions D_ρ, every D_ρ is parallel with respect to an affine connection L if and only if there exists an affine connection Γ such that $L = L(\Gamma)$.*

Corresponding to (1. 15) we have

$$L^i_{jk}(\Gamma + A) = L^i_{jk}(\Gamma) + A^i_{jk} - \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk}. \tag{6.5}$$

Therefore we get

COROLLARY 1. *Let Γ be an affine connection with respect to which every D_ρ is parallel. Then, $\Gamma + T$, T being a tensor field, is also an affine connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$T^i_{jk} = A^i_{jk} - \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk}. \tag{6.6}$$

We next consider a manifold with a system of disjoint distributions $\{D_\rho\}$ and with an almost complex structure ϕ such that we have, when the system $\{D_\rho\}$ is completed by $\bar{D}$, $\underset{\rho}{a}\phi = \phi \underset{\rho}{a}$. Then the arguments in § 3 apply also to such a manifold and we have

THEOREM 6. 3. *In a manifold as above, an affine connection L is a ϕ-connection making every D_ρ parallel if and only if there exists an affine connection Γ such that $L = L(\Phi\Gamma)$, where in the present case we have*

$$L^i_{jk}(\Phi\Gamma) = \Gamma^i_{jk} - \sum \underset{\rho}{a}{}^i_{p,k} \underset{\rho}{a}{}^p_j - \frac{1}{2} \phi^i_a \phi^a_{j,k} + \frac{1}{2} \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j \phi^p_a \phi^a_{q,k}. \tag{6.7}$$

Corresponding to (3. 5) we have

$$L^i_{jk}(\Phi(\Gamma + A)) = L^i_{jk}(\Phi\Gamma) + \frac{1}{2}(A^i_{jk} - \phi^j_a \phi^b_i A^a_{bk} - \sum \underset{\rho}{\bar{a}}{}^i_p \underset{\rho}{a}{}^q_j A^p_{qk} \tag{6.8}$$

$$+ \sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j \phi^p_a \phi^b_q A^a_{bk}).$$

Therefore we obtain

THEOREM 6.4. *Let a manifold be as in Theorem 6.3 and Γ be a ϕ-connection in the manifold making every D_ρ parallel. Then $L = \Gamma + B$ is also a ϕ-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$B^i_{jk} = \frac{1}{2}(A^i_{jk} - \phi^i_a \phi^b_j A^a_{bk} - \sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j A^p_{qk} + \sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j \phi^p_a \phi^b_q A^a_{bk}). \tag{6.9}$$

We now consider a manifold with a system of disjoint distributions $\{D_\rho\}$ and with a quaternion structure (ϕ, ψ) such that we have, when the system $\{D_\rho\}$ is completed by $\bar{D}$, $\underset{\rho}{a}\, \phi = \phi\, \underset{\rho}{a}$ and $\underset{\rho}{a}\, \psi = \psi\, \underset{\rho}{a}$. Then the arguments in § 5 apply also to such a manifold and we have

THEOREM 6.5. *In a manifold as above, an affine connection L is a (ϕ, ψ)-connection making every D_ρ parallel if and only if there exists an affine connection Γ such that $L = L(\Phi\Psi\Gamma)$, where in the present case we have*

$$L^i_{jk}(\Phi\Psi\Gamma) = \Gamma^i_{jk} - \sum_\rho \underset{\rho}{a}^i_{p,k} \underset{\rho}{a}^p_j - \frac{1}{4}(\phi^i_a \phi^a_{j,k} + \psi^i_a \psi^a_{j,k} + \kappa^i_a \kappa^a_{j,k}) \tag{6.10}$$

$$+ \frac{1}{4}\sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j(\phi^p_a \phi^a_{q,k} + \psi^p_a \psi^a_{q,k} + \kappa^p_a \kappa^a_{q,k}).$$

Corresponding to (5.4) we have

$$L^i_{jk}(\Phi\Psi(\Gamma + A)) = L^i_{jk}(\Phi\Psi\Gamma) + \frac{1}{4}(A^i_{jk} - \phi^i_a\phi^b_j A^a_{bk} - \psi^i_a\psi^b_j A^a_{bk} \tag{6.11}$$

$$- \kappa^i_a\kappa^b_j A^a_{bk}) - \frac{1}{4}\sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j(A^p_{qk} - \phi^p_a\phi^b_q A^a_{bk} - \psi^p_a\psi^b_q A^a_{bk} - \kappa^p_a\kappa^b_q A^a_{bk}).$$

Thus we have

THEOREM 6.6. *Let a manifold be as in Theorem 6.5 and Γ be a (ϕ, ψ)-connection in the manifold making every D_ρ parallel. Then $L = \Gamma + B$ is also a (ϕ, ψ)-connection making every D_ρ parallel if and only if there exists a tensor field A such that*

$$B^i_{jk} = \frac{1}{4}(A^i_{jk} - \phi^i_a\phi^b_j A^a_{bk} - \psi^i_a\psi^b_j A^a_{bk} - \kappa^i_a\kappa^b_j A^a_{bk}) \tag{6.12}$$

$$- \frac{1}{4}\sum_\rho \bar{a}^i_p \underset{\rho}{a}^q_j(A^p_{qk} - \phi^p_a\phi^b_q A^a_{bk} - \psi^p_a\psi^b_q A^a_{bk} - \kappa^p_a\kappa^b_q A^a_{bk}).$$

BIBLIOGRAPHY

[1] A. FRÖLICHER, Zur Differentialgeomerie der komplexen Strukturen, Math. Ann., 129 (1955), 50–95.

[2] A. KAWAGUCHI, Beziehung zwischen einer Metrischen linearen Übertragung und einer nichtmetrischen in einem allgemeinen metrischen Raume, Proc. Amsterdam Acad., 40(1937), 3–8.

[3] A. LICHNEROWICZ, Espaces homogènes Kähleriens, Coll. Int. C.N.R.S. Géom. Diff., Strasbourg 1953, 171–184.

[4] A. NIJENHUIS, X_{n-1}-forming sets of eigenvectors, Indag. Math., 13(1951), 200–211.

[5] __________, Jacobi-type identities for bilinear differential concomitants of certain fields I, II, Indag. Math., 17(1955), 390–397, 398–403.

[6] M. OBATA, Affine connections on manifold with almost complex, quaternion or Hermitian structure, Jap. J. Math., 26(1956), 43–77.

[7] N. E. STEENROD, Topological methods for construction of tensor functions, Ann. of Math., 43(1942), 116–131,

[8] A. G. WALKER, Connexions for parallel distributions in the large, Quart. J. Math. Oxford (2), 6(1955), 301–308.

[9] __________, Connexions for parallel distributions in the large (II), Quart. J. Math. Oxford (2), 9(1958), 221–231.

[10] T. J. WILLMORE, Connexions for systems of parallel distributions, Quart. J. Math. Oxford. (2), 7(1956), 269–276.

[11] K. YANO, On Walker differentiation in almost product or almost complex spaces, Indag. Math., 20(1958), 573–580.

[12] __________, Affine connections in an almost product space, Kodai Math. Sem. Rep., 11(1959), 1–24.

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