ON LINEAR LIE ALGEBRAS II

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1. Introduction. In a recent paper [1] some properties of subalgebras $\mathfrak{g}$ of a general linear Lie algebra $\mathfrak{gl}(R^n)$ on an n-dimensional linear space R^n over the field of real numbers were studied. When a base S composed of a set of n linearly independent vectors e_λ $(\lambda = 1, \ldots\ldots, n)$ is taken in R^n, $\mathfrak{g}$ is represented by a set $\mathfrak{K}(\mathfrak{g}, S)$ of matrices of degree n. If there is no possibility of confusion, it may be denoted by $\mathfrak{K}$ for short. $\mathfrak{K}$ is a linear space, hence, if $K_1 \in \mathfrak{K}$, $K_2 \in \mathfrak{K}$, $\lambda_1 \in R$, $\lambda_2 \in R$, then $\lambda_1 K_1 + \lambda_2 K_2 \in \mathfrak{K}$. Moreover we have $[K_1, K_2] \in \mathfrak{K}$ for bracket operation. If we take another base $\widetilde{S}(\widetilde{e}_\lambda)$ such that $\widetilde{S} = AS$, $\widetilde{e}_\lambda = A^{\alpha}_{\cdot\lambda}\, e_\alpha$, then $\mathfrak{g}$ is represented by $\widetilde{\mathfrak{K}}$ where $\widetilde{\mathfrak{K}} = A^{-1}\,\mathfrak{K}\,A$. In short we have $\mathfrak{K}(\mathfrak{g}, AS) = A^{-1}\,\mathfrak{K}\,(\mathfrak{g}, S)\,A$. The set of matrices V which satisfy $K^{\lambda}_{\cdot\mu}\, V^{\mu}_{\cdot\lambda} = 0$ for all matrices $K^{\lambda}_{\cdot\mu}$ of $\mathfrak{K}$ is denoted by $\mathfrak{V}(\mathfrak{g}, S)$ or by $\mathfrak{V}$ for short.[1] $\mathfrak{V}$ is a linear space and is transformed as follows

$$\widetilde{\mathfrak{V}} \equiv \mathfrak{V}(\mathfrak{g}, AS) = A^{-1}\,\mathfrak{V}(\mathfrak{g}, S) A.$$

If $K \in \mathfrak{K}$ and $V \in \mathfrak{V}$, then $[K, V] \in \mathfrak{V}$, which fact may be expressed by $[\mathfrak{K}, \mathfrak{V}] \subset \mathfrak{V}$.

If $\dim\, \mathfrak{g} = r$, then $\dim\, \mathfrak{V} = n^2 - r$. If a base S of R^n and a base M of $\mathfrak{V}(\mathfrak{g}, S)$ composed of the matrices $\underset{A}{V}(A = 1, \ldots\ldots, n^2 - r)$ are taken suitably, then the matrices $\underset{A}{V}$ assume special forms and we get the notion of d series. This is a sequence of natural numbers $d_1, \ldots\ldots, d_{P+1}$ satisfying $d_1 + \ldots\ldots + d_{P+1} = n$.

Unless otherwise specified indices are used as follows,

$$\alpha,\ \beta,\ \gamma,\ \lambda,\ \mu,\ \nu = 1, \ldots\ldots, n,$$

$$h,\ i,\ j,\ k = 1, \ldots\ldots, n - n_1,$$

$$t,\ u,\ v,\ x,\ y,\ z = n - n_1 + 1, \ldots\ldots, n,$$

$$S,\ T,\ U = 1, \ldots\ldots, P \text{ or } P + 1,$$

$$h_T,\ i_T,\ j_T,\ k_T = n - n_{T-1} + 1, \ldots\ldots, n - n_T,$$

$$t_T,\ u_T,\ v_T,\ x_T,\ y_T,\ z_T = n - n_T + 1, \ldots\ldots, n,$$

1) Matrices are denoted by letters such as M, K, V or by the elements $M^{\lambda}_{\cdot\mu}$, $K^{\lambda}_{\cdot\mu}$, $V^{\lambda}_{\cdot\mu}$. Parentheses may be used but they are omitted especially when the elements are complicated expressions.

where $$n_{T-1} - n_T = d_T,\ n_0 = n,\ n_{P+1} = 0.$$

We adopt the summation convention with respect to indices in small letters, so that for example

$$M^{t_T}_{\cdot i_T}$$

is a sum of d_T diagonal elements of M. By an n_T-submatrix of a matrix $M^{\lambda}_{\cdot\mu}$ is meant the following matrix of degree n_T

$$M^{t_T}_{\cdot u_T}.$$

Then the results obtained in [1] are as follows.

If a base S of R^n is chosen suitably, then $\mathfrak{B}$ is decomposed as follows ([1] § 6)

$$\left.\begin{aligned}\mathfrak{B} &= \mathfrak{B}_1 + \cdots\cdots + \mathfrak{B}_T + \mathfrak{B}_{\hat{T}},\\ \mathfrak{B}_{\hat{T}} &= \mathfrak{B}_{T+1} + \mathfrak{B}_{(T\hat{+}1)}\end{aligned}\right\}\quad (1 \leqq T \leqq P),$$

where the matrices V of $\mathfrak{B}_{\hat{T}}$ satisfy $V^{\lambda}_{\cdot\mu} = 0$ for

$$\begin{aligned}
&\lambda = 1, \ldots\ldots, n; && \mu = 1, \ldots\ldots, n - n_1,\\
&\lambda = n - n_1 + 1, \ldots\ldots, n; && \mu = n - n_1 + 1, \ldots\ldots, n - n_2,\\
&\cdots\cdots \quad \cdots\cdots && \cdots\cdots \quad \cdots\cdots\\
&\lambda = n - n_{T-1} + 1, \ldots\ldots, n; && \mu = n - n_{T-1} + 1, \ldots\ldots, n - n_T.
\end{aligned}$$

Moreover they satisfy ([1] (48))

$$(1.1)_T \qquad V^{t_T}_{\cdot\mu} = 0$$

if $d_T \geqq 2$. $\mathfrak{B}_T$ is spanned by $d_T(n - d_1 - \cdots\cdots - d_T)$ or $d_T(n - d_1 - \cdots\cdots - d_T) + 1$ linearly independent matrices

$$(1.2)_T \qquad \overset{k_T}{\underset{z_T}{V}}$$

or

$$(1.2')_T \qquad \underset{T}{V},\quad \overset{k_T}{\underset{z_T}{V}} \qquad\qquad (\underset{1}{V} \text{ means } \overset{1}{\underset{1}{V}} \text{ of [1] § 5})$$

whose property will be shown at once. If we get $(1.2)_T$, then this fact is indicated by $T \in \mathrm{C}\,1$, while if we get $(1.2')_T$ this fact is indicated by $T \in \mathrm{C}\,2$. Since we have $\mathfrak{B}_{(T\hat{-}1)} = \mathfrak{B}_T + \mathfrak{B}_{\hat{T}}$, the matrices $(1.2)_T$, $(1.2')_T$ must belong to $\mathfrak{B}_{(T\hat{-}1)}$. Now, there are five possible cases in each step, ([1] § 5)

$$(\mathrm{i})_T: \qquad d_T = 1, \qquad T \in \mathrm{C}\,1,$$

$$
\begin{array}{llll}
(\mathrm{ii})_T: & d_T \geqq 2, & T \in \mathrm{C}\,1, \\
(\mathrm{iii})_T: & d_T = 1, & T \in \mathrm{C}\,2, \\
(\mathrm{iv})_T: & d_T = 2, & T \in \mathrm{C}\,2, \\
(\mathrm{v})_T: & d_T \geqq 3, & T \in \mathrm{C}\,2.
\end{array}
$$

If we get $(\mathrm{i})_T$ or $(\mathrm{ii})_T$, then we have

$$
(1.\ 3)_T \qquad \overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot\mu} = 0, \qquad \overset{k_T}{\underset{z_T}{V}}{}^{x_T}{}_{\cdot\mu} = \delta^{x_T}_{z_T}\ \delta^{k_T}_{\mu}.
$$

If we get $(\mathrm{iii})_T$, then[2)]

$$
(1.\ 4)_T \qquad
\begin{array}{ll}
\underset{T}{V}{}^{i_T}{}_{\cdot j_T} = \delta^{i_T}_{j_T}, & \underset{T}{V}{}^{x_T}{}_{\cdot j_T} = 0, \\
\overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot j_T} = 0, & \overset{k_T}{\underset{z_T}{V}}{}^{x_T}{}_{\cdot j_T} = \delta^{x_T}_{z_T}\ \delta^{k_T}_{j_T}.
\end{array}
$$

If we get $(\mathrm{iv})_T$, then

$$
(1.\ 5)_T \quad
\begin{cases}
\underset{T}{V}{}^{i_T}{}_{\cdot\mu} = \delta^{i_T}_{\mu}, \quad \underset{T}{V}{}^{x_T}{}_{\cdot j_T} = 0, \\
\overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot j_T} = 0, \quad \overset{k_T}{\underset{z_T}{V}}{}^{x_T}{}_{\cdot y_T} = 0, \quad \overset{k_T}{\underset{z_T}{V}}{}^{x_T}{}_{\cdot j_T} = \delta^{x_T}_{z_T}\ \delta^{k_T}_{j_T}, \\
\overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot y_T} = -\overset{i_T}{\underset{z_T}{V}}{}^{k_T}{}_{\cdot y_T} = \overset{i_T}{\underset{y_T}{V}}{}^{k_T}{}_{\cdot z_T} = -\overset{k_T}{\underset{y_T}{V}}{}^{i_T}{}_{\cdot z_T}.
\end{cases}
$$

If we get $(\mathrm{v})_T$, then

$$
(1.\ 6)_T \qquad
\begin{array}{ll}
\underset{T}{V}{}^{i_T}{}_{\cdot\mu} = \delta^{i_T}_{\mu}, & \underset{T}{V}{}^{x_T}{}_{\cdot j_T} = 0, \\
\overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot\mu} = 0, & \overset{k_T}{\underset{z_T}{V}}{}^{x_T}{}_{\cdot\mu} = \delta^{x_T}_{z_T}\ \delta^{k_T}_{\mu}.
\end{array}
$$

Moreover we have for $(\mathrm{iv})_T$

$$
(1.\ 7)_T \qquad \overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot x_T}\ \underset{T}{V}{}^{z_T}{}_{\cdot y_T} + \overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot y_T}\ \underset{T}{V}{}^{z_T}{}_{\cdot x_T} = 0,
$$

and

$$
(1.\ 8)_T \qquad \overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot x_T}\ V^{z_T}{}_{\cdot y_T} + \overset{k_T}{\underset{z_T}{V}}{}^{i_T}{}_{\cdot y_T}\ V^{z_T}{}_{\cdot x_T} = 0
$$

2) Of course in this case indices such as i_T can take only one value $d_1+\ldots\ldots+d_T$.

for all matrices V of $\mathfrak{B}_{\hat{T}}$.

We can construct a base M of $\mathfrak{B}$ such that M contains a base $(1.2)_T$ or $(1.2')_T$ of $\mathfrak{B}_T$ for each $T\,(1 \leqq T \leqq P+1)$ and a base of $\mathfrak{B}_{(P\hat{+}1)}$. Such a base S of R^n and the base M of $\mathfrak{B}$ just obtained compose a base (S, M) of the fourth order. Of course $\mathfrak{B}_{(P\hat{+}1)}$ is nothing else but $\mathfrak{B}'_{P+1}$ in [1].

Addendum. According to [1] § 5, if we get (i) in the first step, this is not an intrinsic property of $\mathfrak{g}$. In order to get rid of this inconveniency we make a rule that the vector $\boldsymbol{v}_1$ is chosen in such a way that we get (iii) whenever this is possible. Hence, if $M_{\hat{1}}''$ in [1] page 173 is not empty, the choice of $\boldsymbol{v}_1$ must be changed so that we get (iii). This is easily seen from the form of the matrix (i)$'$ in [1] page 175. Thus we get (i) only when all matrices V of $\mathfrak{B}_{\hat{1}}$ assume the form $V^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{y}\; V^{y}_{\cdot x}\; \delta^{x}_{\mu}$. Similarly in the T the step $(\text{iii})_T$ is preferred if possible. Hence $(1.1)_T$ is satisfied by the matrices V of $\mathfrak{B}_{\hat{T}}$ not only when $d_T \geqq 2$ but also when we get $(\text{i})_T$.

Let $M^{\lambda}_{\cdot\mu}$ be a matrix of degree n. Its d_T rows obtained by putting

$$\lambda = d_1 + \cdots\cdots + d_{T-1} + 1, \cdots\cdots, d_1 + \cdots\cdots + d_T$$

will be called rows of T. Similarly the d_T columns obtained by putting

$$\mu = d_1 + \cdots\cdots + d_{T-1} + 1, \cdots\cdots, d_1 + \cdots\cdots + d_T$$

will be called columns of T. A submatrix whose elements are the elements of M belonging to rows of S and columns of T simultaneously will be called an $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrix of M. A $\begin{bmatrix} T \\ T \end{bmatrix}$-submatrix is called a T-submatrix for short. A 1-submatrix is a T-submatrix where $T = 1$ and so on. Such a mode of expression is not misleading since an n_T-submatrix will not be called an m-submatrix even if $n_T = m$. An expression $\lambda \in I(T)$ means that λ takes on the values $d_1 + \cdots\cdots + d_{T-1} + 1, \cdots\cdots, d_1 + \cdots\cdots + d_T$ or it belongs to some of the above numbers. Since the notion of a base M of $\mathfrak{B}$ will not be often used hereafter, in a base (S, M) of higher order M may be omitted.

As a d series of a linear Lie algebra $\mathfrak{g}$ is obtained by choosing a base S of R^n suitably, it may happen that another d series is obtained by choosing another base. But it is obvious that to a $\mathfrak{g}$ correspond a set of d series and conversely to a d series correspond a set of $\mathfrak{g}$'s whose possible forms will be studied now.

2. d series containing $d_T = 2$, $T \in C\,2$. If we get $(\text{iv})_T$, we have

$$\overset{\cdot\, k_T}{\underset{z_T}{V}}{}^{\,l_T}_{\cdot\, x_T} \neq 0$$

in general and these must satisfy $(1.7)_T$, $(1.8)_T$, which are rather complicated relations. But such inconveniency is removed by changing the choice of the base S suitably.

First we assume that we obtained (iv) in the first step. Let S be a base of R^n of the third order for the moment. Since we have

$$\overset{1}{\underset{x}{V}}{}^{y}_{\cdot 1} = \delta^y_x,$$

if $\widetilde{S}$ is another base of the third order such that $\widetilde{S} = AS$, $A^{\lambda}_{\cdot j} = \delta^{\lambda}_j$, $A^{t}_{\cdot\mu} = \delta^{t}_{\mu}$, then the part of $\overset{1}{\underset{x}{V}}$ in $\mathfrak{V}_1$ is played by the matrices $A^t_{\cdot x} A^{-1} \overset{1}{\underset{t}{V}} A$ in $\widetilde{\mathfrak{V}}_1 = A^{-1} \mathfrak{V}_1 A$. This fact is easily understood from

$$[A^t_{\cdot x} A^{-1} \overset{1}{\underset{t}{V}} A]^y_{\cdot 1} = \delta^y_x$$

and means that $S \to \widetilde{S}$ induces a transformation

$$\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y} \to [A^t_{\cdot x} A^{-1} \overset{1}{\underset{t}{V}} A]^2_{\cdot y}.$$

If we put

$$v_{xy} = [A^t_{\cdot x} A^{-1} \overset{1}{\underset{t}{V}} A]^2_{\cdot y} = \overset{1}{\underset{t}{V}}{}^{2}_{\cdot u} A^t_{\cdot x} A^u_{\cdot y},$$

then we get $v_{xy} = - v_{yx}$ since

$$\overset{1}{\underset{t}{V}}{}^{2}_{\cdot u} = - \overset{1}{\underset{u}{V}}{}^{2}_{\cdot t}.$$

Moreover, if the submatrix $A^x_{\cdot y}$ is chosen suitably, v_{xy} become such that

$$v_{34} = - v_{43} = \ldots\ldots = v_{2m-1\ 2m} = - v_{2m\ 2m-1} = 1,$$

$$\text{other } v_{xy} = 0$$

where $2m - 2$ is the rank of the matrix (v_{xy}). This fact means that, once we get $(\text{iv})_1$, a base S of the third order can be chosen in such a way that the matrices $(1.2')_1$ satisfy

$$(2.1)\quad \begin{cases} \overset{1}{\underset{3}{V}}{}^{2}_{\cdot 4} = - \overset{1}{\underset{4}{V}}{}^{2}_{\cdot 3} = \cdots = \overset{1}{\underset{2m-1}{V}}{}^{2}_{\cdot 2m} = - \overset{1}{\underset{2m}{V}}{}^{2}_{\cdot 2m-1} = 1, \\ \text{other } \overset{1}{\underset{x}{V}}{}^{2}_{\cdot y} = 0. \end{cases}$$

Then, since we have $(1.5)_1$, we find immediately that the following $2(n - 2m)$ matrices belong to $\mathfrak{V}_1$,

$$\overset{1}{\underset{\varphi}{V}}{}^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\varphi}\,\delta^{1}_{\mu}, \quad \overset{2}{\underset{\varphi}{V}}{}^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\varphi}\,\delta^{2}_{\mu}.\text{[3]} \tag{2. 2}$$

Again, as we have $(1.5)_1$ and (2. 1), we can find out $3+4(m-1)$ linearly independent matrices

$$\begin{array}{ll} \delta^{\lambda}_{1}\,\delta^{1}_{\mu} - \delta^{\lambda}_{2}\,\delta^{2}_{\mu}, \quad \delta^{\lambda}_{1}\,\delta^{2}_{\mu}, \quad \delta^{\lambda}_{2}\,\delta^{1}_{\mu}, & \\ \delta^{\lambda}_{1}\,\delta^{3}_{\mu} - \delta^{\lambda}_{4}\,\delta^{2}_{\mu}, & \delta^{\lambda}_{2}\,\delta^{3}_{\mu} + \delta^{\lambda}_{4}\,\delta^{1}_{\mu}, \\ \delta^{\lambda}_{1}\,\delta^{4}_{\mu} + \delta^{\lambda}_{3}\,\delta^{2}_{\mu}, & \delta^{\lambda}_{2}\,\delta^{4}_{\mu} - \delta^{\lambda}_{3}\,\delta^{1}_{\mu}, \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \delta^{\lambda}_{1}\,\delta^{2m-1}_{\mu} - \delta^{\lambda}_{2m}\,\delta^{2}_{\mu}, & \delta^{\lambda}_{2}\,\delta^{2m-1}_{\mu} + \delta^{\lambda}_{2m}\,\delta^{1}_{\mu}, \\ \delta^{\lambda}_{1}\,\delta^{2m}_{\mu} + \delta^{\lambda}_{2m-1}\,\delta^{2}_{\mu}, & \delta^{\lambda}_{2}\,\delta^{2m}_{\mu} - \delta^{\lambda}_{2m-1}\,\delta^{1}_{\mu} \end{array}$$

from $\mathfrak{K}$. Then we get by bracket operation the following $m(2m+1)$ linearly independent matrices which naturally belong to $\mathfrak{K}$,

$$\delta^{\lambda}_{p}\,\delta^{q}_{\mu} - (-1)^{p+q}\,\delta^{\lambda}_{q'}\,\delta^{p'}_{\mu} \tag{2. 3}$$

$$\begin{cases} p, q = 1, \ldots\ldots, 2m, \\ p' = p+1 \text{ if } p \text{ is odd}, \ p' = p-1 \text{ if } p \text{ is even}, \\ q' = q+1 \text{ if } q \text{ is odd}, \ q' = q-1 \text{ if } q \text{ is even}. \end{cases}$$

As the matrices V of $\mathfrak{V}_1$ satisfy $V^{1}_{\cdot\varphi} = V^{2}_{\cdot\varphi} = 0$ by virtue of $(1.5)_1$ and (2. 1), $\mathfrak{K}$ also contains the matrices in the right hand sides of (2. 2). Applying bracket operation to matrices of (2. 2) and (2. 3), we find $2m(n-2m)$ matrices

$$\delta^{\lambda}_{\varphi}\,\delta^{\pi}_{\mu} \tag{2. 4}$$

which naturally belong to $\mathfrak{K}$.

Since the matrices (2. 2) belong to $\mathfrak{V}$ and $[\mathfrak{K}, \mathfrak{V}] \subset \mathfrak{V}$, (2. 4) belong also to $\mathfrak{V}$.

Hence we get

$$V^{\pi}_{\cdot\varphi} = 0 \tag{2. 5}$$

for all matrices V of $\mathfrak{V}$ and

$$K^{\pi}_{\cdot\varphi} = 0 \tag{2. 6}$$

for all matrices K of $\mathfrak{K}$.

Consequently $\mathfrak{V}$ is spanned by the matrices (2. 4) and some matrices V such that

3) In § 2 the following indices are used.

$\pi,\ \rho = 1, \ldots\ldots, 2m,$

$\varphi,\ \chi = 2m+1, \ldots\ldots, n.$

(2. 7) $$V^{\pi}_{\cdot\varphi} = 0,\ V^{\varphi}_{\cdot\pi} = 0,$$

which we shall study now.

First, since $\mathfrak{K}$ contains the matrices (2. 3), these must satisfy

(2. 8) $$V^{q}_{\cdot p} = (-1)^{p+q}\, V^{p'}_{\cdot\ q'}$$

which is a system of $m(2m+1)$ linearly independent equations and contains among others

$$V^{1}_{\cdot 2} = V^{2}_{\cdot 1} = \ldots\ldots = V^{2m-1}_{\cdot\ 2m} = V^{2m}_{\cdot\ 2m-1} = 0,$$
$$V^{1}_{\cdot 1} = V^{2}_{\cdot 2}, \ldots\ldots, V^{2m-1}_{\cdot\ 2m-1} = V^{2m}_{\cdot\ 2m}.$$

On the other hand $\mathfrak{B}$ contains the matrices

$$\delta^{\lambda}_{3}\,\delta^{1}_{\mu} + \delta^{\lambda}_{2}\,\delta^{4}_{\mu},\quad \delta^{\lambda}_{4}\,\delta^{1}_{\mu} - \delta^{\lambda}_{2}\,\delta^{3}_{\mu},$$
$$\delta^{\lambda}_{3}\,\delta^{2}_{\mu} - \delta^{\lambda}_{1}\,\delta^{4}_{\mu},\quad \delta^{\lambda}_{4}\,\delta^{2}_{\mu} + \delta^{\lambda}_{1}\,\delta^{3}_{\mu},$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$\delta^{\lambda}_{2m-1}\,\delta^{1}_{\mu} + \delta^{\lambda}_{2}\,\delta^{2m}_{\mu},\quad \delta^{\lambda}_{2m}\,\delta^{1}_{\mu} - \delta^{\lambda}_{2}\,\delta^{2m-1}_{\mu},$$
$$\delta^{\lambda}_{2m-1}\,\delta^{2}_{\mu} - \delta^{\lambda}_{1}\,\delta^{2m}_{\mu},\quad \delta^{\lambda}_{2m}\,\delta^{2}_{\mu} + \delta^{\lambda}_{1}\,\delta^{2m-1}_{\mu}$$

by virtue of $(1.5)_1$ and (2. 1). Applying bracket operation between these matrices and the matrices of (2. 3) we find that the matrices

(2. 9) $$\delta^{\lambda}_{p}\,\delta^{q}_{\mu} + (-1)^{p+q}\,\delta^{\lambda}_{q'}\,\delta^{p'}_{\mu} \qquad (p, p' \neq q, q')$$

belong to $\mathfrak{B}$. It must be noticed that the pair of numbers p, p' must be different from q, q' in (2. 9). It is also found that $\mathfrak{B}$ contains the matrices

(2. 10) $$\delta^{\lambda}_{1}\,\delta^{1}_{\mu} + \delta^{\lambda}_{2}\,\delta^{2}_{\mu} - \delta^{\lambda}_{2m-1}\,\delta^{2m-1}_{\mu} - \delta^{\lambda}_{2m}\,\delta^{2m}_{\mu}, \ldots\ldots,$$
$$\delta^{\lambda}_{2m-3}\,\delta^{2m-3}_{\mu} + \delta^{\lambda}_{2m-2}\,\delta^{2m-2}_{\mu} - \delta^{\lambda}_{2m-1}\,\delta^{2m-1}_{\mu} - \delta^{\lambda}_{2m}\,\delta^{2m}_{\mu}.$$

There are $m(2m-1)-1$ linearly independent matrices in (2. 9) and (2. 10) altogether, which satisfy (2. 8) obviously.

Now consider a matrix V of $\mathfrak{B}$ such that $V^{\pi}_{\cdot\varphi} = 0$, $V^{\varphi}_{\cdot\pi} = 0$. Since it satisfies (2. 8), we get a matrix V' of $\mathfrak{B}$ such that $V'^{\pi}_{\cdot\rho} = 0$ with the exception of $V'^{2m-1}_{\cdot\ 2m-1}$, $V'^{2m}_{\cdot\ 2m}$ by subtracting from V some linear combination of matrices of (2. 9) and (2. 10). We thus obtain a matrix V of $\mathfrak{B}$ such that $V^{\pi}{}_{\varphi} = V^{\varphi}_{\cdot\pi} = 0$, $V^{\pi}_{\cdot\rho} = v(\delta^{\pi}_{2m-1}\,\delta^{2m-1}_{\rho} + \delta^{\pi}_{2m}\,\delta^{2m}_{\rho})$. It may happen that for any matrix V with such properties v becomes zero.

Gathering the results obtained we can state that the space $\mathfrak{B}$ is spanned by the following three sets of matrices,

(a) the $2m(n-2m) + m(2m-1) - 1$ matrices (2. 4), (2. 9), (2. 10), which satisfy $V^{\lambda}_{\cdot\varphi} = 0$,

(b) a matrix satisfying

$$V^{\pi}_{\cdot\rho} = \delta^{\pi}_{2m-1}\,\delta^{2m-1}_{\rho} + \delta^{\pi}_{2m}\,\delta^{2m}_{\rho} \tag{2. 11}$$

and

$$V^{\pi}_{\cdot\varphi} = V^{\varphi}_{\cdot\pi} = 0,$$

(c) matrices such that

$$V^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\varphi}\,V^{\varphi}_{\cdot\chi}\,\delta^{\chi}_{\mu}. \tag{2. 12}$$

It may happen that (b) disappears.

Because of this property which $\mathfrak{V}$ possesses the number 2 appears successively at least m times in the d series. The matrix (b) is the matrix $\underset{m}{V}$ which is a $\underset{T}{V}$ such that $T = m$. The matrices (c) span $\mathfrak{V}_{\hat{m}}$. Moreover it is clear that we have already completed the m th step by getting such a base S of R^n. It is also clear that the 1-submatrices, ..., the m-submatrices of the matrices of $\mathfrak{V}$ are all scalar matrices.

Now we can begin the next step without being disturbed by such relations as $(1.8)_m$. Suppose that we did not obtain (iv) in the next succeeding steps until we encounter $(\text{iv})_{T+1}$ for some T. Then we apply the whole process considered above to the linear space $\mathfrak{V}_{(T)}$ spanned by the n_T-submatrices of the matrices of $\mathfrak{V}_{\hat{T}}$, considering $\mathfrak{K}_{(T)}$ instead of $\mathfrak{K}$, where $\mathfrak{K}_{(T)}$ is the linear space spanned by the n_T-submatrices of those matrices of $\mathfrak{K}$ in which the elements satisfy

$$K^{\lambda}_{\cdot\mu} = 0$$

$$\text{for}\quad \lambda = n - n_1 + 1, \ldots\ldots, n\,;\ \mu = 1, \ldots\ldots, n - n_1,$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$\lambda = n - n_T + 1, \ldots\ldots, n\,;\ \mu = n - n_{T-1} + 1, \ldots\ldots, n - n_T$$

(see § 3 and [1] § 6). Such process is repeated and at last we get the

THEOREM 1. *Among the bases of the fourth order a base S of R^n can be chosen in such a way that $\mathfrak{V}(\mathfrak{g}, S)$ possesses the following property. The numbers T such that $d_T = 2$, $T \in \mathrm{C}\,2$ appear in clusters where a cluster is a set of m successive natural numbers and is divided in general into l smaller parts, so that the numbers T of a cluster can be written as*

$$T = T_1,\ T_1 + 1, \ldots\ldots, T_1 + m_1 - 1,$$
$$T_2,\ T_2 + 1, \ldots\ldots, T_2 + m_2 - 1,$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$T_l,\ T_l + 1, \ldots\ldots, T_l + m_l - 1$$

where $T_1 + m_1 = T_2, \ldots\ldots, T_{l-1} + m_{l-1} = T_l$ and $m_1 + \ldots\ldots + m_l = m$. We *also write for convenience* $T_i + m_i = T_{i+1}$. *The relation between* $\mathfrak{V}(\mathfrak{g}, S)$ *and a cluster is as follows. If* $T = T_i - 1$ *and* $T' = T_{i+1} - 1$ *where* $i = 1, \ldots\ldots, l$, *then* $\mathfrak{V}_{\hat{T}}$ *is spanned by* $\mathfrak{V}_{\hat{T}'}$ *and* $\mathfrak{V}^T_{\cdot}$ *where* $\mathfrak{V}^T_{\cdot}$ *is a linear space spanned by* $2m_i(n_T - 2m_i) + m_i(2m_i - 1) - 1$ *or* $2m_i(n_T - 2m_i) + m_i(2m_i - 1)$ *linearly independent matrices not belonging to* $\mathfrak{V}_{\hat{T}'}$. *The* n_T*-submatrices of these matrices of* $\mathfrak{V}^T_{\cdot}$ *are matrices such as are obtained from the matrices* (a) ((2.4), (2.9), (2.10)) *and* (b) *by changing the range for the indices properly and replacing* n, m *with* n_T, m_i. (1.5), (1.7), (1.8) *need not be taken into account though all results stated in* §1 *are preserved.*

Such a base S is called a *base of order* 5. d series takes the form $(\ldots\ldots, 2 \times m_1, \ldots\ldots, 2 \times m_l, \ldots\ldots)$ where $2 \times m_i$ is a chain of 2's in which 2 is repeated m_i times. We write $T \cap S$ when T and S are members of one and the same part of a cluster. Otherwise we write $T \,\overline{\cap}\, S$.

EXAMPLE. Let n be even, $n = 2m$. The linear Lie algebra $\mathfrak{Sp}(m, R)$ is spanned by the matrices (2.3). We get $l = 1$, $T_1 = 1$. d series is $2 \times m$ and we have the matrix (b). For a linear Lie algebra spanned by $\mathfrak{Sp}(m, R)$ and scalar matrices we also obtain $l = 1$, $T_1 = 1$, and the d series is $2 \times m$, but we have no matrix (b).

3. T-submatrices of $\mathfrak{V}$.

3.1. It is already clear that the 1-submatrices of all matrices of $\mathfrak{V}$ are scalar matrices. We now prove that all T-submatrices where T runs from 1 to $P + 1$ are also scalar matrices if the base S is chosen suitably among the bases of order 5. Let us begin this with 2-submatrices. Since it is clear that the 2-submatrices of the matrices of $\mathfrak{V}_{\hat{1}}$ are scalar matrices, proof is needed only for the matrices of $\mathfrak{V}_1$. We assume $d_2 \geqq 2$, for, if $d_2 = 1$, 2-submatrices are of course scalar matrices.

We considered in [1] § 6 a subspace $\mathfrak{K}_1$ of $\mathfrak{K}$. $\mathfrak{K}_1$ can be defined as a subspace spanned by those matrices K of $\mathfrak{K}$ which satisfy $K^y_{\cdot j} = 0$. A matrix K of $\mathfrak{K}_1$ is obtained by taking a matrix $K^y_{\cdot x}$ of degree n_1 such that

$$K^y_{\cdot x} V^x_{\cdot y} = 0 \qquad (V \in \mathfrak{V}_{\hat{1}}) \tag{3.1}$$

for all matrices V of $\mathfrak{V}_{\hat{1}}$, by taking elements $K^i_{\cdot j}$, $K^k_{\cdot z}$ in such a way that [1] (29) or (30) is satisfied, and by putting $K^y_{\cdot j} = 0$. Then, as the elements $K^k_{\cdot j}$ are arbitrary or at most only $K^i_{\cdot i}$ is determined by the second equation of [1] (30), a linear space $\mathfrak{K}_{(1)}$ spanned by the matrices $K^y_{\cdot x}$ just mentioned is in general homomorphic with $\mathfrak{K}_1$ as a linear Lie algebra,

$$\mathfrak{K}_1 \sim \mathfrak{K}_{(1)}. \tag{3.2}$$

The kernel of homomorphism (3. 2) is given by the matrices K satisfying

$$K^{k}_{\cdot x}=0,\ K^{y}_{\cdot j}=0,\ K^{y}_{\cdot x}=0 \tag{3. 3}$$

or

$$K^{i}_{\cdot i}=0,\ K^{k}_{\cdot x}=0,\ K^{y}_{\cdot j}=0,\ K^{y}_{\cdot x}=0 \tag{3. 3'}$$

according as $1 \in C\,1$ or $1 \in C\,2$. Now we obtain the

LEMMA 1. *A subspace $\mathfrak{K}_1^E$ of $\mathfrak{K}$ spanned by matrices $K^{\lambda}_{\cdot\mu}$ such that*

$$K^{k}_{\cdot j}=0,\ K^{k}_{\cdot z}=0,\ K^{y}_{\cdot j}=0 \qquad (\textit{when } 1 \in C\,1) \tag{3. 4}$$

or

$$\left.\begin{aligned} K^{k}_{\cdot j} &= \Big(-K^{y}_{\cdot x}\overset{1}{\underset{1}{V}}{}^{x}_{\cdot y}/d_1\Big)\,\delta^k_j \\ K^{k}_{\cdot z} &= -\,K^{y}_{\cdot x}\overset{k}{\underset{z}{V}}{}^{x}_{\cdot y},\ K^{y}_{\cdot j}=0 \end{aligned}\right\} \qquad (\textit{when } 1 \in C\,2) \tag{3. 4'}$$

is isomorphic with $\mathfrak{K}_{(1)}$ as a linear Lie algebra.

PROOF. If $1 \in C\,1$ or $d_1 = 1$ this lemma is evidently correct. Suppose $1 \in C\,2$ and $d_1 \geqq 2$. Then the space $\mathfrak{J}_1$ spanned by those matrices K which satisfy (3. 3′) is not 0-dimensional and is an ideal of $\mathfrak{K}_1$. Let a matrix of $\mathfrak{J}_1$ be denoted by K' and a matrix of $\mathfrak{K}_1$ by K. Then we get $K'^{i}_{\cdot k}\,K^{k}_{\cdot x}=0$ where $K'^{i}_{\cdot k}$ are restricted by $K'^{i}_{\cdot i}=0$ only. This proves $K^{k}_{\cdot x}=0$.[4] Hence $\mathfrak{K}_1$ is a direct sum of $\mathfrak{K}_1^E$ and $\mathfrak{J}_1$ where $\mathfrak{K}_1^E$ is a linear Lie algebra. We thus obtain

$$\mathfrak{K}_1^E \cong \mathfrak{K}_1/\mathfrak{J}_1 \cong \mathfrak{K}_{(1)}. \tag{3. 5}$$

An isomorphic mapping f_t of $\mathfrak{K}_1^E$ onto $\mathfrak{K}_{(1)}$ is obtained by taking for each matrix of $\mathfrak{K}_1^E$ its n_1-submatrix. This mapping will be denoted more precisely by $f_t(n_1)$. Its inverse $(f_t(n_1))^{-1}$ is obtained from (3. 4) or (3. 4′).

If we consider the second step in the sense of [1] § 6, we can take as a matrix $K^{y}_{\cdot x}$ satisfying (3. 1) any matrix $K^{y}_{\cdot x}$ such that

$$K^{i_2}_{\cdot i_2}=0,\ K^{k_2}_{\cdot x_2}=0,\ K^{y_2}_{\cdot j_2}=0,\ K^{y_2}_{\cdot x_2}=0. \tag{3. 6}$$

As we have assumed $d_2 \geqq 2$, the elements $K^{k_2}_{\cdot j_2}$ are arbitrary within $K^{i_2}_{\cdot i_2}=0$. A linear space spanned by matrices $K^{\lambda}_{\cdot\mu}$ obtained in accordance with (3. 4) or (3. 4′) from those matrices $K^{y}_{\cdot x}$ which satisfy (3. 6) will be denoted by $\mathfrak{K}_{[2]}$. Then $\mathfrak{K}_{[2]} \subset \mathfrak{K}_1^E$ and $\mathfrak{K}_{[2]} \cong \mathfrak{Sl}(d_2)$. Since $[\mathfrak{Sl}, \mathfrak{Sl}] = \mathfrak{Sl}$ we get $[\mathfrak{K}_{[2]}, \mathfrak{K}_{[2]}] = \mathfrak{K}_{[2]}$, which proves that any 1-submatrix of $\mathfrak{K}_{[2]}$ is a zero matrix.

In the case of $1 \in C\,2$ we find from (3. 4′) that

4) This is also obtained from [1] (30) directly.

$$K^{k_2}_{\cdot j_2}\overset{1}{\underset{1}{V}}{}^{j_2}_{\cdot k_2} = 0 \qquad \text{whenever} \qquad K^{i_2}_{\cdot i_2} = 0.$$

Hence the 2-submatrix of the matrix $\overset{1}{\underset{1}{V}}$ is a scalar matrix.

3.2. Next we prove that when the base S of R^n is chosen suitably the 2-submatrices of matrices $\overset{k}{\underset{z}{V}}$ become scalar matrices. Since these are a zero matrix except the case of $d_1 = 1, 1 \in C\,2$, we consider this case only. We first prove the next lemma where indices are used as follows

$$\kappa, \lambda, \mu, \nu = 1, \ldots\ldots, n,$$
$$h, i, j, k = 1, \ldots\ldots, d_1,$$
$$x, y, z = d_1 + 1, \ldots\ldots, n,$$
$$\zeta, \xi, \eta, \pi, \rho, \sigma, \tau = d_1 + 1, \ldots\ldots, d_1 + d_2,$$
$$\varphi, \chi, \psi = d_1 + d_2 + 1, \ldots\ldots, n.$$

LEMMA 2. *Assume that a $(d_2)^2 - 1$ dimensional linear space $\mathfrak{L}$ spanned by a set of $(d_2)^2$ matrices $\overset{\rho}{\underset{\pi}{K}}$ such that*

$$(3.\ 7)\qquad \begin{cases} \overset{\rho}{\underset{\pi}{K}}{}^{\lambda}{}_{j} = 0, \\ \overset{\rho}{\underset{\pi}{K}}{}^{y}_{\cdot x} = \delta^y_\pi\,\delta^\rho_x - (\delta^y_\sigma\,\delta^\sigma_x/d_2)\,\delta^\rho_\pi, \\ \overset{\sigma}{\underset{\sigma}{K}}{}^{y}_{\cdot x} = 0 \end{cases}$$

is a faithful representation of a linear Lie algebra. Then there are numbers $a^k_{\cdot\pi}$ such that

$$(3.\ 8)\qquad \overset{\rho}{\underset{\pi}{K}}{}^{k}_{\cdot\tau} = \delta^\rho_\pi\,a^k_{\cdot\tau} - d_2\,\delta^\rho_\tau\,a^k_{\cdot\pi}.$$

Moreover we have

$$(3.\ 9)\qquad \overset{\rho}{\underset{\pi}{K}}{}^{i}_{\cdot\varphi} = 0.$$

This lemma and the corollary can be obtained as a special case of Weyl's theorem. But the following proof will not be superfluous for our purpose.

PROOF. We immediately get $\mathfrak{L} \simeq \mathfrak{sl}(d_2)$ by virtue of (3. 7) and $\dim \mathfrak{L} = (d_2)^2 - 1$. Then we get $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$, which proves (3. 9). Now we obtain from (3. 7)

$$[\underset{\eta}{\overset{\xi}{K}},\ \underset{\pi}{\overset{\rho}{K}}]^{k}_{\cdot\tau}=\underset{\eta}{\overset{\xi}{K}}{}^{k}_{\cdot\sigma}\underset{\pi}{\overset{\rho}{K}}{}^{\sigma}_{\cdot\tau}-\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\sigma}\underset{\eta}{\overset{\xi}{K}}{}^{\sigma}_{\cdot\tau}$$

$$=\underset{\eta}{\overset{\xi}{K}}{}^{k}_{\cdot\pi}\,\delta^{\rho}_{\tau}-\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\eta}\,\delta^{\xi}_{\tau}-\frac{1}{d_2}(\delta^{\rho}_{\pi}\underset{\eta}{\overset{\xi}{K}}{}^{k}_{\cdot\tau}-\delta^{\xi}_{\eta}\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\tau}),$$

$$[\underset{\eta}{\overset{\xi}{K}},\ \underset{\pi}{\overset{\rho}{K}}]^{\zeta}_{\cdot\tau}=\underset{\eta}{\overset{\xi}{K}}{}^{\zeta}_{\cdot\sigma}\underset{\pi}{\overset{\rho}{K}}{}^{\sigma}_{\cdot\tau}-\underset{\pi}{\overset{\rho}{K}}{}^{\zeta}_{\cdot\sigma}\underset{\eta}{\overset{\xi}{K}}{}^{\sigma}_{\cdot\tau}$$

$$=\delta^{\xi}_{\pi}\,\delta^{\rho}_{\tau}\,\delta^{\zeta}_{\eta}-\delta^{\rho}_{\eta}\,\delta^{\xi}_{\tau}\,\delta^{\zeta}_{\pi}$$

$$=\delta^{\xi}_{\pi}\underset{\eta}{\overset{\rho}{K}}{}^{\zeta}_{\cdot\tau}-\delta^{\rho}_{\eta}\underset{\pi}{\overset{\xi}{K}}{}^{\zeta}_{\cdot\tau}.$$

The latter formulas show that

$$[\underset{\eta}{\overset{\xi}{K}},\ \underset{\pi}{\overset{\rho}{K}}]=\delta^{\xi}_{\pi}\underset{\eta}{\overset{\rho}{K}}-\delta^{\rho}_{\eta}\underset{\pi}{\overset{\xi}{K}},$$

from which we get

$$\delta^{\xi}_{\pi}\underset{\eta}{\overset{\rho}{K}}{}^{k}_{\cdot\tau}-\delta^{\rho}_{\eta}\underset{\pi}{\overset{\xi}{K}}{}^{k}_{\cdot\tau}=[\underset{\eta}{\overset{\xi}{K}},\ \underset{\pi}{\overset{\rho}{K}}]^{k}_{\cdot\tau}$$

$$=\underset{\eta}{\overset{\xi}{K}}{}^{k}_{\cdot\pi}\,\delta^{\rho}_{\tau}-\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\eta}\,\delta^{\xi}_{\tau}-\frac{1}{d_2}(\delta^{\rho}_{\pi}\underset{\eta}{\overset{\xi}{K}}{}^{k}_{\cdot\tau}-\delta^{\xi}_{\eta}\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\tau}).$$

Putting $\rho=\pi$ and summing we get

$$-\underset{\sigma}{\overset{\sigma}{K}}{}^{k}_{\cdot\eta}\,\delta^{\xi}_{\tau}+\frac{1}{d_2}\delta^{\xi}_{\eta}\underset{\sigma}{\overset{\sigma}{K}}{}^{k}_{\cdot\tau}=0,$$

hence

$$\underset{\sigma}{\overset{\sigma}{K}}{}^{k}_{\cdot\tau}=0.$$

Putting $\xi=\pi$ and summing and making use of the formula just obtained we get

$$d_2\underset{\eta}{\overset{\rho}{K}}{}^{k}_{\cdot\tau}=\underset{\eta}{\overset{\sigma}{K}}{}^{k}_{\cdot\sigma}\,\delta^{\rho}_{\tau}-\underset{\tau}{\overset{\rho}{K}}{}^{k}_{\cdot\eta},$$

which we write as follows

$$d_2\underset{\pi}{\overset{\rho}{K}}{}^{k}_{\cdot\tau}+\underset{\tau}{\overset{\rho}{K}}{}^{k}_{\cdot\pi}=\delta^{\rho}_{\tau}\underset{\pi}{\overset{\sigma}{K}}{}^{k}_{\cdot\sigma}.$$

Then putting

$$\underset{\tau}{\overset{\sigma}{K}}{}^{k}_{\cdot\sigma}=-[(d_2)^2-1]\,a^{k}_{\cdot\tau}$$

we get (3.8) immediately.

COROLLARY. *Under the assumption made in Lemma 2 consider a transformation* $\widetilde{S} = AS$ *such that*

$$A^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{i}\, b^{i}_{\cdot\sigma}\, \delta^{\tau}_{\mu} \tag{3. 10}$$

where

$$b^{i}_{\cdot\sigma} = - d_2\, a^{i}_{\cdot\sigma}. \tag{3. 11}$$

Then $\widetilde{\mathfrak{L}} = A^{-1}\mathfrak{L} A$ *is spanned by* $(d_2)^2$ *matrices such that*

$$(A^{-1})^{\lambda}_{\cdot\kappa} \underset{\pi}{\overset{\rho}{K}}{}^{\kappa}_{\cdot\nu}\, A^{\nu}_{\cdot\mu} = \delta^{\lambda}_{\pi}\,\delta^{\rho}_{\mu} - (\delta^{\lambda}_{\sigma}\,\delta^{\sigma}_{\mu}/d_2)\delta^{\rho}_{\pi}.$$

PROOF. The first member of this equality becomes

$$(\delta^{\lambda}_{\kappa} - \delta^{\lambda}_{i}\, b^{i}_{\cdot\sigma}\,\delta^{\sigma}_{\kappa}) \underset{\pi}{\overset{\rho}{K}}{}^{\kappa}_{\cdot\nu}\,(\delta^{\nu}_{\mu} + \delta^{\nu}_{j}\, b^{j}_{\cdot\tau}\,\delta^{\tau}_{\mu})$$

$$= \underset{\pi}{\overset{\rho}{K}}{}^{\lambda}_{\cdot\mu} - \delta^{\lambda}_{i}\, b^{i}_{\cdot\sigma} \underset{\pi}{\overset{\rho}{K}}{}^{\sigma}_{\cdot\mu}.$$

If we put $\mu = j$, this becomes zero. If we put $\lambda = y$, $\mu = x$, we get $\underset{\pi}{\overset{\rho}{K}}{}^{y}_{\cdot x}$. For $\lambda = k$, $\mu = \varphi$ and also for $\lambda = k$, $\mu = \tau$ this becomes zero since we have (3. 11).

3.3. Now $\mathfrak{K}_{[2]}$ satisfies the same condition as assumed to be satisfied by $\mathfrak{L}$ in Lemma 2. Hence according to the Corollary we can choose another base $\widetilde{S} = AS$ of R^n in such a way that we get

$$\widetilde{K}^{k}_{\cdot z} = 0 \quad \text{whenever } \widetilde{K} \in A^{-1}\mathfrak{K}_{[2]} A.$$

When we prove that such a base $\widetilde{S}$ is obtained among the bases of order 5[5], and when we use this base as the base S from the beginning, we get by virtue of (3. 4′)

$$K^{h_2}_{\cdot j_2} \underset{z}{\overset{k}{V}}{}^{j_2}_{\cdot h_2} = 0 \quad \text{whenever } K^{i_2}_{\cdot i_2} = 0.$$

Then we get the

LEMMA 3. *A base S of order* 5 *can be chosen in such a way that the 2-submatrices of all matrices of* $\mathfrak{V}$ *are scalar matrices.*

PROOF. Suppose that S is an arbitrary base of order 5. In order to obtain

5) Since d series is not uniquely determined by $\mathfrak{g}$, a base of order 5 may have no connection with another base of order 5. But we do not consider for the present a transformation between such bases. See the following proof of Lemma 3.

a base $\widetilde{S} = AS$ such that $\widetilde{K}^{k}_{\cdot z} \equiv [A^{-1}KA]^{k}_{\cdot z} = 0$ for all matrices K of $\mathfrak{K}_{[2]}$, we need only to put

$$A^{\lambda}_{\cdot \mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{i_1} b^{i_1}_{\cdot j_2} \delta^{j_2}_{\mu}.$$

As we are considering only the case of $d_1 = 1$, $1 \in C\,2$, this can be written as follows

$$A^{\lambda}_{\cdot \mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{1} b_{j_2} \delta^{j_2}_{\mu}.$$

Now what we must show is that $\widetilde{S}$ is a base of order 5.[6] We have $\widetilde{\mathfrak{V}} = \mathfrak{V}(\mathfrak{g}, \widetilde{S}) = A^{-1}\mathfrak{V}A$ and its subspaces $A^{-1}\mathfrak{V}_T A$, $A^{-1}\mathfrak{V}_{\hat{T}} A$ which we shall write as $\widetilde{\mathfrak{V}}_T$, $\widetilde{\mathfrak{V}}_{\hat{T}}$. But it is easy to see that $\widetilde{\mathfrak{V}}_T$, $\widetilde{\mathfrak{V}}_{\hat{T}}$ can play the same rôle in $\widetilde{\mathfrak{V}}$ as $\mathfrak{V}_T$, $\mathfrak{V}_{\hat{T}}$ respectively do in $\mathfrak{V}$. This is clear indeed with respect to $\widetilde{\mathfrak{V}}_{\hat{1}}$ and $\widetilde{\mathfrak{V}}_T$, $\widetilde{\mathfrak{V}}_{\hat{T}}$ such that $2 \leqq T \leqq P + 1$, for their second, ..., n th rows as well as the first columns remain intact. Consider $A^{-1}\mathfrak{V}_1 A$. The matrix $A^{-1}\underset{1}{V}A$ has quite similar form with $\underset{1}{V}$ and plays the same part in $\widetilde{\mathfrak{V}}_1$ as $\underset{1}{V}$ does in $\mathfrak{V}_1$. The matrix

$$A^{-1}\overset{1}{\underset{z}{V}}A + \delta^{j_2}_{z} b_{j_2} A^{-1}\underset{1}{V}A$$

has quite similar form with $\overset{1}{\underset{z}{V}}$ and plays the same part in $\widetilde{\mathfrak{V}}_1$ as $\overset{1}{\underset{z}{V}}$ does in $\mathfrak{V}_1$, though of course their 2-submatrices are not the same. Hence $\widetilde{S}$ is a base of order 5. This proves Lemma 3.

3.4. Consider a subspace of $\mathfrak{K}$ spanned by matrices K such that

$$(3.\ 12)\quad \begin{cases} K^{\lambda}_{\cdot \mu} = 0 \\ \text{for} \quad \lambda = n - n_1 + 1, \ldots\ldots, n\,; \quad \mu = 1, \ldots\ldots, n - n_1, \\ \qquad \ldots\ldots \quad \ldots\ldots \qquad\qquad \ldots\ldots \quad \ldots\ldots \\ \qquad \lambda = n - n_T + 1, \ldots\ldots, n\,; \quad \mu = n - n_{T-1} + 1, \ldots\ldots, n - n_T, \end{cases}$$

and call it $\mathfrak{K}_T$. A matrix K of $\mathfrak{K}_T$ is obtained by taking a matrix $K^{y_T}_{\cdot x_T}$ of degree n_T such that

$$(3.\ 13)\qquad K^{y_T}_{\cdot x_T} V^{x_T}_{\cdot y_T} = 0 \qquad (V \in \mathfrak{V}_{\hat{T}})$$

6) The set of matrices $\widetilde{K} = A^{-1}KA$ where $K \in \mathfrak{K}_{[2]}$ can play the same rôle as $\mathfrak{K}_{[2]}$ only when this is shown.

for all matrices V of $\mathfrak{V}_{\hat{T}}$ and by determining successively other elements with the use of formulas $(39)_T$ or $(40)_T$, $(39)_{T-1}$ or $(40)_{T-1}, \ldots\ldots$, $(39)_1$ or $(40)_1$ of [1]. The linear space spanned by the matrices $K^{y_T}_{\cdot x_T}$ satisfying (3. 13) was denoted by $\mathfrak{K}_{(T)}$. This is a linear Lie algebra homomorphic with $\mathfrak{K}_T$,

$$\mathfrak{K}_T \sim \mathfrak{K}_{(T)}. \tag{3. 14}$$

A homomorphic mapping f_h is obtained by taking from each matrix of $\mathfrak{K}_T$ its n_T-submatrix and is denoted by $f_h(n_T)$. Its inverse is obtained from formulas $(39)_T$, $(40)_T, \ldots\ldots$, $(39)_1$, $(40)_1$ of [1].

Let us assume that the 1-submatrix, ..., the T-submatrix are scalar matrices in each matrix V of $\mathfrak{V}$. Then it is clear that the kernel $\mathfrak{J}_T$ of (3. 14) contains $\mathfrak{K}_{[S]}$ for each S such that $1 \leqq S \leqq T$, where $\mathfrak{K}_{[S]}$ is the linear space spanned by the matrices K satisfying

$$K^{i_S}_{\cdot i_S} = 0,\ K^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{k_S} K^{k_S}_{\cdot j_S} \delta^{j_S}_{\mu}. \tag{3. 15$_S$}$$

This is correct because, if $(3. 15)_S$ is satisfied, we have

$$K^{k_S}_{\cdot j_S} V^{j_S}_{\cdot k_S} = 0 \quad \text{whenever } V \in \mathfrak{V}.$$

Moreover we can write

$$\mathfrak{J}_T = \sum_{1 \leqq S \leqq T} \mathfrak{K}_{[S]} + \mathfrak{J}'_T \tag{3. 16}$$

where $\mathfrak{J}'_T$ is a space such that the 1-submatrix, ..., the T-submatrix of each matrix of $\mathfrak{J}'_T$ are all scalar matrices and that at least one of the T submatrices mentioned above is not a zero matrix for each non zero matrix of $\mathfrak{J}'_T$.

The latter property of $\mathfrak{J}'_T$ is obtained since $\mathfrak{J}_T$ is the inverse image of the zero matrix of $\mathfrak{K}_{(T)}$ (See [1] (39) and (40).)

Consider a matrix $K^{\lambda}_{\cdot\mu}$ where the n_T-submatrix $K^{y_T}_{\cdot x_T}$ is a matrix of $\mathfrak{K}_{(T)}$ and other elements are determined successively from

$$K^{k_T}_{\cdot j_T} = 0,\ K^{k_T}_{\cdot z_T} = 0,\ K^{y_T}_{\cdot j_T} = 0 \qquad (\text{when } T \in C\,1) \tag{3. 17$_T$}$$

or

$$\left.\begin{aligned} K^{k_T}_{\cdot j_T} &= -\frac{1}{d_T} K^{y_T}_{\cdot x_T} V^{x_T}_{\cdot y_T} \delta^{k_T}_{\cdot j_T} \\ K^{k_T}_{\cdot z_T} &= - K^{y_T}_{\cdot x_T} V^{k_T\,x_T}_{z_T\,\cdot y_T}\,{}^{7)} \\ K^{y_T}_{\cdot j_T} &= 0 \end{aligned}\right\} \quad (\text{when } T \in C\,2) \tag{3. 17$'_T$}$$

7) This is equal to zero unless $d_T = 1$.

and $(3.17)_{T-1}$ or $(3.17')_{T-1}, \ldots\ldots, (3.17)_1$ or $(3.17')_1$. The set of all such matrices will be denoted by $\mathfrak{K}^E_T$. This is a linear space and $\mathfrak{K}_T$ is spanned by $\mathfrak{K}^E_T$ and $\mathfrak{J}_T$. Any matrix of $[\mathfrak{K}^E_T, \mathfrak{K}^E_T]$ is a matrix of $\mathfrak{K}_T$ and its S-submatrix is a zero matrix for every S, $1 \leqq S \leqq T$. Hence $[\mathfrak{K}^E_T, \mathfrak{K}^E_T] \subset \mathfrak{K}^E_T + \mathfrak{J}'_T$. We show that $[\mathfrak{K}^E_T, \mathfrak{K}^E_T] \subset \mathfrak{K}^E_T$.

Suppose that a matrix K has S-submatrices which are zero matrices for $1 \leqq S \leqq T$ and that $K = K' + K''$ where $K' \in \mathfrak{J}'_T$ and $K'' \in \mathfrak{K}^E_T$. Let S' be a number such that the S-submatrix of K' is a zero matrix when $S' < S \leqq T$ and the S'-submatrix of K' is not a zero matrix. Then the same is true for the matrix K''. From the property of $\mathfrak{J}'_T$ such a number S' must belong to C 1. But then $(3.17)_{S'}$ must be used and we find that the S'-submatrix of K'' is a zero matrix. Hence such a number S' can not exist and we get $K' = 0$. This proves $[\mathfrak{K}^E_T, \mathfrak{K}^E_T] \subset \mathfrak{K}^E_T$.

Thus, $\mathfrak{K}^E_T$ is a linear Lie algebra. Moreover, since $[\mathfrak{K}^E_T, \mathfrak{J}_T] \subset \mathfrak{J}_T$ and the S-submatrices of the left hand member are zero matrices for $1 \leqq S \leqq T$, we get

$$[\mathfrak{K}^E_T, \mathfrak{J}_T] = 0,$$

We also find that the decomposition $\mathfrak{K}_T = \mathfrak{K}^E_T + \mathfrak{J}_T$ has the following property

$$\mathfrak{K}^E_T \cong \mathfrak{K}_T/\mathfrak{J}_T \cong \mathfrak{K}_{(T)} \tag{3. 18}$$

where $\cong$ means isomorphism of Lie algebras.

An isomorphic mapping of $\mathfrak{K}^E_T$ onto $\mathfrak{K}_{(T)}$ is obtained by restricting $f_h(n_T)$ and is denoted by $f_i(n_T)$. Its inverse is obtained from (3. 17), (3. 17′).

Considering the decomposition $\mathfrak{V}_{\hat{T}} = \mathfrak{V}_{T+1} + \mathfrak{V}_{(T\hat{+}1)}$ we can take as a matrix $K^{y_T}{}_{\cdot x_T}$ of $\mathfrak{K}_{(T)}$ an arbitrary matrix such that

$$\begin{aligned} &K^{i_{T+1}}{}_{\cdot t_{T+1}} = 0, \quad K^{k_{T+1}}{}_{\cdot x_{T+1}} = 0, \\ &K^{y_{T+1}}{}_{\cdot j_{T+1}} = 0, \quad K^{y_{T+1}}{}_{\cdot x_{T+1}} = 0. \end{aligned} \tag{3. 19}$$

A subspace of $\mathfrak{K}^E_T$ spanned by those matrices which are obtained from $K^{y_T}{}_{\cdot x_T}$ satisfying (3. 19) with the use of (3. 17), (3. 17′) will be denoted by $\mathfrak{K}_{[T+1]}$. Then we get

$$\mathfrak{K}_{[T+1]} \cong \mathfrak{sl}(d_{T+1})$$

because of (3. 18), hence

$$[\mathfrak{K}_{[T+1]}, \mathfrak{K}_{[T+1]}] = \mathfrak{K}_{[T+1]}.$$

This shows that all S-submatrices of $\mathfrak{K}_{[T+1]}$ are zero matrices for $1 \leqq S \leqq T$.

We can apply Lemma 2 and the Corollary to the n_{T-1}-submatrices of $\mathfrak{K}_{[T+1]}$

and find that, if the base S of R^n is changed to $\widetilde{S} = AS$ where

$$A^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{i_T} b^{i_T}_{\cdot j_{T+1}} \delta^{j_{T+1}}_{\mu}$$

and where the coefficients are taken suitably, then the only non zero elements of the n_{T-1}-submatrices of $A^{-1}\mathfrak{K}_{[T+1]}A$ are the elements $K^{k_{T+1}}_{\cdot\ j_{T+1}}$ which satisfy $K^{j_{T+1}}_{\cdot\ i_{T+1}} = 0$. Then we can apply Lemma 2 and the Corollary again to the n_{T-2}-submatrices of such $A^{-1}\mathfrak{K}_{[T+1]}A$ and so on. We thus find at last that there is a matrix

$$A^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{i_1} b^{i_1}_{\cdot j_{T+1}} \delta^{j_{T+1}}_{\mu} + \cdots\cdots + \delta^{\lambda}_{i_T} b^{i_T}_{\cdot j_{T+1}} \delta^{j_{T+1}}_{\mu} \tag{3. 20}$$

such that the only non zero elements of $A^{-1}\mathfrak{K}_{[T+1]}A$ are the elements

$$K^{k_{T+1}}_{\cdot\ j_{T+1}} \text{ which satisfy } K^{j_{T+1}}_{\cdot\ i_{T+1}} = 0.$$

The coefficients in (3. 20) must be chosen suitably, but, as the elements $K^{k_S}_{\cdot z_S}$ of $\mathfrak{K}_{[T+1]}$ are zero unless $d_S = 1$ and $S \in \mathrm{C}\,2$,[8] (3. 20) have the form

$$A^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\mu} + \Sigma^* \delta^{\lambda}_{i_S} b^{i_S}_{\cdot j_{T+1}} \delta^{j_{T+1}}_{\mu}$$

where Σ^* means summation over S such that $1 \leqq S \leqq T$, $d_S = 1$, $S \in \mathrm{C}\,2$ simultaneously. Hence (3. 20) is obtained by making a product of the matrices

$$\underset{S}{A}{}^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{\mu} + \delta^{\lambda}_{i_S} b^{i_S}_{\cdot j_{T+1}} \delta^{j_{T+1}}_{\mu} \tag{3. 21}$$

$$(1 \leqq S \leqq T,\ d_S = 1,\ S \in \mathrm{C}\,2).$$

We must only show that, if S is a base of order 5, then $\widetilde{S} = \underset{S}{A}S$ is also a base of order 5. This is easily done by examining the properties of $\widetilde{\mathfrak{B}}_1 = \underset{S}{A}{}^{-1}\mathfrak{B}_1\underset{S}{A}, \ldots\ldots, \widetilde{\mathfrak{B}}_T = \underset{S}{A}{}^{-1}\mathfrak{B}_T\underset{S}{A}, \ldots\ldots$ straightforwardly. Again, all matrices $\widetilde{V} = \underset{S}{A}{}^{-1}V\underset{S}{A}$ of $\widetilde{\mathfrak{B}}$ have 1-submatrices, ……, T-submatrices which are all scalar matrices. All these are true when $\underset{S}{A}$ is replaced by A.

The result obtained can be summed up as follows.

If we assume that S is a base of order 5 such that all matrices of $\mathfrak{B} = \mathfrak{B}(\mathfrak{g}, S)$ have 1-submatrices, ……, T-submatrices which are all scalar matrices, then we can change the choice of S in such a way that the matrices K of $\mathfrak{K}_{[T+1]}$ have the form

8) See (3. 17), (3. 17′).

$$K^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{k_{T+1}} K^{k_{T+1}}_{\cdot\ j_{T+1}} \delta^{j_{T+1}}_{\mu},$$

$$K^{i_{T+1}}_{\cdot\ i_{T+1}} = 0,$$

leaving other properties intact.

Then consider again $(3.17)_S$ and $(3.17')_S$ where $1 \leqq S \leqq T$. Since these must be satisfied by the matrices of $\mathfrak{K}_{[T+1]}$, we get, if $S \in C\,2$,

$$\underset{S}{V}^{k_{T+1}}_{\cdot\ j_{T+1}} \propto \delta^{k_{T+1}}_{j_{T+1}},$$

$$\overset{k_S}{\underset{z_S}{V}}{}^{h_{T+1}}_{\cdot\ j_{T+1}} \propto \delta^{h_{T+1}}_{j_{T+1}}.$$

We thus find that the $(T+1)$-submatrices of $\mathfrak{V}_1, \ldots\ldots, \mathfrak{V}_T$ are scalar matrices. As the $(T+1)$-submatrices of $\mathfrak{V}_{\hat{T}}$ are of course scalar matrices, this shows that the $(T+1)$-submatrices of $\mathfrak{V}$ are scalar matrices. This proves the following

THEOREM 2. *The base S of R^n of order* 5 *can be chosen in such a way that the* 1-*submatrices,* ......, $(P+1)$-*submatrices of all matrices V of $\mathfrak{V}$ are scalar matrices.*

Such a base S is called a *base of order* 6 and will be used constantly hereafter. Thus we obtain the next

THEOREM 3. *$\mathfrak{K}$ satisfies $\mathfrak{K}_{[S]} \subset \mathfrak{K}$, where $\mathfrak{K}_{[S]}$ is determined by* $(3.15)_S$ *for every S, $1 \leqq S \leqq P+1$.*

Of course $\mathfrak{K}_{[S]}$ is significant only when $d_S \geqq 2$. We get again the

COROLLARY. *As a linear space $\mathfrak{K}$ is decomposed into the direct sum*

$$\mathfrak{K} = \Sigma\, \mathfrak{K}_{[S]} + (\mathfrak{K} - \Sigma\, \mathfrak{K}_{[S]}),$$

where the S-submatrices of all matrices of $(\mathfrak{K} - \Sigma\, \mathfrak{K}_{[S]})$ are scalar matrices.

4. The operator $[T]$. Hereafter we shall consider only T's such that $d_T \geqq 2$. When T is fixed d_T will be denoted by d for simplicity.

Now we can take a matrix K of $\mathfrak{K}_{[T]}$ such that its T-submatrix has the form

$$(4.1)\qquad (\mathrm{a})\ \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 \\ & \cdots\cdots & & \cdots\cdots & \\ 0 & 0 & 0 & \cdots\cdots & 0 \end{pmatrix} \quad \text{or} \quad (\mathrm{b})\ \begin{pmatrix} 0 & 0 & 0 & \cdots\cdots & 0 \\ 1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 \\ & \cdots\cdots & & \cdots\cdots & \\ 0 & 0 & 0 & \cdots\cdots & 0 \end{pmatrix}.$$

Let V be a matrix of $\mathfrak{B}$ and K_1, K_2 be matrices of $\mathfrak{K}_{[T]}$ with T-submatrices (4. 1 a), (4. 1 b) respectively. Then $V' = [K_2, [K_1, V]]$ is a matrix of $\mathfrak{B}$ such that the second row in the rows of T and the first column in the columns of T respectively are the same in V and V', with the exception of the elements in the T-submatrix, which is a zero matrix in V' since the T-submatrix of V is a scalar matrix. Moreover, all other elements of V' are zero. This matrix will be denoted by $[T]^1_{\cdot 2}V$.

More generally, when M is a matrix of degree n, $[T]^1_{\cdot 2}M$ means a matrix such that the second row of T and the first column of T respectively are the same in M and $[T]^1_{\cdot 2}M$ with the exception of the elements of the T-submatrix which is a zero matrix in $[T]^1_{\cdot 2}M$, and moreover such that all other elements of $[T]^1_{\cdot 2}M$ are zero. Operators such as $[T]^2_{\cdot 3}$ are defined similarly. Then, if we put

$$[T] = [T]^1_{\cdot 2} + [T]^2_{\cdot 3} + \cdots\cdots + [T]^{d-1}_{\cdot\ d} + [T]^d_{\cdot 1}, \tag{4. 2}$$

we get

$$[T]\,V \in \mathfrak{B}, \qquad (\mathbf{1} - [T])\,V \in \mathfrak{B} \tag{4. 3}$$

for the matrices V of $\mathfrak{B}$. $[T]\,V$ is a matrix obtained from V by replacing all elements by zeroes except those elements which are in rows of T or in columns of T but not in the T-submatrix and consequently are left intact. $(\mathbf{1} - [T])\,V$ is a matrix obtained from V by replacing the elements in rows of T and columns of T but not in the T-submatrix by zeroes and leaving other elements intact.

5. Linear spaces $\mathfrak{L}'_T$, $\mathfrak{L}'^T$, $\mathfrak{M}'_T$. Let us study the matrices $[T]\,V$ more in detail.

We use indices in this section as follows

$$h, i, j, k, l, m \in I(T).$$

Then the matrix defined by

$$\overset{i}{\underset{j}{K}}{}^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{j}\,\delta^{i}_{\mu} \qquad (i \neq j) \tag{5. 1}$$

belongs to $\mathfrak{K}_{[T]}$. Let V be a matrix of $\mathfrak{B}$ and consider the matrix

$$[\overset{k}{\underset{h}{K}}, [\overset{i}{\underset{j}{K}}, V]] \in \mathfrak{B} \qquad (i \neq j,\ k \neq h), \tag{5. 2}$$

where the elements are

$$\delta^{\lambda}_{h}\,(\delta^{k}_{j}\,V^{i}_{\cdot\mu} - V^{k}_{\cdot j}\,\delta^{i}_{\mu}) - (\delta^{\lambda}_{j}\,V^{i}_{\cdot h} - V^{\lambda}_{\cdot j}\,\delta^{i}_{h})\,\delta^{k}_{\mu}. \tag{5. 3}$$

As $V^{i}_{\cdot h} \propto \delta^{i}_{h}$, the elements (5. 3) are equal to zero when either $\lambda, \mu \in I(T)$ or

$\lambda, \mu \notin I(T)$. On the other hand we have

$$[\underset{h}{\overset{k}{K}}, [\underset{j}{\overset{i}{K}}, V]]^{\lambda'}_{\cdot\, l} = V^{\lambda'}_{\cdot\, j}\, \delta^i_h\, \delta^k_l,$$

$$[\underset{h}{\overset{k}{K}}, [\underset{j}{\overset{i}{K}}, V]]^{m}_{\cdot \mu'} = \delta^n_h\, \delta^k_j\, V^i_{\cdot \mu'},$$

where λ', μ' are indices not belonging to $I(T)$. If $i = h$ and $j \neq k$, the matrix (5.2) has elements all of which are equal to zero except those in the k th column, which are $V^{\lambda}_{\cdot j} - v\, \delta^{\lambda}_j$ where $V^i_{\cdot j} = v\, \delta^i_j$. If $i \neq h$ and $j = k$, the matrix (5.2) has elements all of which are equal to zero except those in the h th row, which are $V^i_{\cdot \mu} - v\, \delta^i_\mu$. But, as we have assumed $i \neq j$ and $k \neq h$, such matrices can not exist unless $d_T \geqq 3$. When $d_T = 2$, $i = h$ induces $j = k$. Then the elements of the matrix (5.2) are equal to zero except those in the i th row and those in the j th column, the j th column being $V^{\lambda}_{\cdot j} - v\, \delta^{\lambda}_j$ and the i th row being $V^i_{\cdot \mu} - v\, \delta^i_\mu$.

Thus, if $d_T \geqq 3$, we obtain a set of matrices of $\mathfrak{V}$ such that in each matrix V' of the set either all rows are composed of 0's except one or all columns are so except one. But, if $d_T = 2$, we obtain a set of matrices of $\mathfrak{V}$ such that in each matrix V' the elements are zeroes except those which are in one row and those which are in one column. In both cases such exceptional rows and columns are only rows and columns of T, and besides, the T-submatrix is a zero matrix in each matrix V'. Moreover it is important to notice that, if V is a matrix of $\mathfrak{V}$ and $d_T \geqq 3$, we obtain a matrix V' by replacing all rows by zero rows except some one among the rows of T and leaving intact those elements of this row which are not in the T-submatrix. This holds good if we replace "row" by "column". If $d_T = 2$, such proposition must be naturally changed as it is clear from the above explanation. We get only $[T]^1_{\cdot 2} V$ and $[T]^2_{\cdot 1} V$.

The matrix $[\underset{l}{\overset{m}{K}}, V']$ where $m \neq l$ is a matrix such that its l th row is the same with the m th row of V' and such that its m th column is the same with the l th column of V' but bears opposite sign.

Hence, *if $d_T \geqq 3$, a matrix obtained by shifting the non zero row or column of V' within the rows or columns of T is also a matrix of $\mathfrak{V}$. If $d_T = 2$, a matrix obtained from V' by permuting the rows of T and permuting the columns of T with change of sign simultaneously is also a matrix of $\mathfrak{V}$.*

Thus we obtain following lemmas.

LEMMA 4_T. *Suppose $d_T \geqq 3$ for the value of T under consideration, and take the first column of T in every $V \in \mathfrak{V}$, replacing all elements in the*

T-submatrix by zeroes. The linear space spanned by the column vectors thus obtained is denoted by $\mathfrak{L}'_T$. *Then, if* $v_T|^\lambda$ *is an arbitrary vector of* $\mathfrak{L}'_T$, *the matrix* V *where*

$$V^{\lambda}_{\cdot\mu} = v_T|^\lambda \delta^i_\mu \qquad (i \in I(T)) \tag{5. 4}$$

is a matrix of $\mathfrak{V}$. *Similarly we obtain a linear space* $\mathfrak{L}'^T$ *such that, if* $u^T|_\mu$ *is an arbitrary vector of* $\mathfrak{L}'^T$, *then the matrix* V *where*

$$V^{\lambda}_{\cdot\mu} = \delta^\lambda_j u^T|_\mu \qquad (j \in I(T)) \tag{5. 5}$$

is a matrix of $\mathfrak{V}$. *Moreover, for any matrix* V *of* $\mathfrak{V}$, $[T]V$ *is a linear combination of matrices such as are given by* (5. 4), (5. 5).

LEMMA 5_T. *Suppose* $d_T = 2$ *for the value of* T *under consideration, and take the first column of* T *and the second row of* T *in every* $V \in \mathfrak{V}$, *replacing all elements in the T-submatrix by zeroes. The column vector and the row vector thus obtained are arranged to form a vector* $(v_T|^\lambda, u^T|_\mu)$ *with formally* $2n$ *components. The linear space spanned by such vectors is denoted by* $\mathfrak{M}'_T$. *Then, if* $(v_T|^\lambda, u^T|_\mu) \in \mathfrak{M}'_T$, *the matrices*

$$v_T|^\lambda \delta^{1'}_\mu + \delta^\lambda_{2'} u^T|_\mu \tag{5. 6}$$

$$-v_T|^\lambda \delta^{2'}_\mu + \delta^\lambda_{1'} u^T|_\mu \tag{5. 7}$$

$$(1' = d_1 + \dots\dots + d_{T-1} + 1,\ 2' = d_1 + \dots\dots + d_{T-1} + 2)$$

belong to $\mathfrak{V}$. *Moreover, for any matrix* V *of* $\mathfrak{V}$, $[T]V$ *is a linear combination of matrices such as* (5. 6), (5. 7).

REMARK TO LEMMA 4. We can take any one column (row) of T instead of the first column (row).

REMARK TO LEMMA 5. We can take the second column and the first row instead of the first column and the second row. Only the sign of either the column vector or the row vector must be changed then to obtain a vector of $\mathfrak{M}'_T$.

6. $\begin{bmatrix} S \\ T \end{bmatrix}$**-submatrices of** $\mathfrak{V}$. Hereafter indices such as $a, b, \dots\dots$ take only numbers belonging to no one of $I(T)$'s such that $d_T \geqq 2$. This fact will be denoted by $a, b, \dots\dots \in I(R)$. On the other hand *indices such as* S, T, U *will take only numbers such that* $d_S, d_T, d_U \geqq 2$ unless otherwise specified. $(S \cup T)$-submatrix is the matrix of degree $d_S + d_T$ that is the crossing of the rows of S and T with the columns of S and T, while $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrix is the crossing of the rows of S with the columns of T. Hence for example the non zero elements of $[S][T]V$ are only elements of the $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrix and elements of the $\begin{bmatrix} T \\ S \end{bmatrix}$-submatrix. A row (column) of R is a row (column) which is not a row (column) of T for any T. Submatrices such as R-submatrix, $\begin{bmatrix} R \\ T \end{bmatrix}$-submatrix will be understood similarly. When all numbers $T\,(d_T \geqq 2)$ are arranged as follows $T_1 < \dots\dots < T_Q$, the set of all rows of R will be divided into $Q+1$ components $R_1, \dots\dots, R_{Q+1}$ some of which may be vacuous.[9] Submatrices such as $\begin{bmatrix} R_i \\ R_j \end{bmatrix}$-submatrix will also be understood.

Now suppose $d_T \geqq 3$ for fixed T, and consider a matrix of $\mathfrak{V}$ as given by (5.4), that is,

$$V^{\lambda}_{\cdot\mu} = v_T|^{\lambda}\,\delta^{i}_{\mu} \qquad (v_T|^{\lambda} \in \mathfrak{L}'_T,\ i \in I(T)). \tag{6.1}$$

Let $h \in I(S)$. Then, as the hth row vector is $v_T|^{h}\,\delta^{i}_{\mu}$ where μ indicates individual components, we get $\delta^{i}_{\mu} \in \mathfrak{L}'^{s}$ as long as $v_T|^{h} \neq 0$. This shows the following fact. If $\mathfrak{L}'_T$ contains a vector $v_T|^{\lambda}$ such that for some value of h belonging to $I(S)$ the component $v_T|^{h}$ is not equal to zero, then $\delta^{i}_{\mu} \in \mathfrak{L}'^{S}$. We thus obtain

$$\delta^{\lambda}_{h}\,\delta^{i}_{\mu} \in \mathfrak{V} \qquad (h \in I(S),\ i \in I(T)). \tag{6.2}$$

Furthermore, once we get (6.2) for some value of h in $I(S)$ and for some value of i in $I(T)$, this holds good for all $h \in I(S)$ and all $i \in I(T)$. This is obtained from the italicized sentense just before Lemma 4. Moreover this is true even if $d_S = 2$ as the columns of the matrix (6.1) are composed of 0's except the ith column.

We cannot obtain (6.2) only when $v_T|^{h} = 0$ for all $v_T|^{\lambda} \in \mathfrak{L}'_T$ and for all $h \in I(S)$. Hence from the definition of $\mathfrak{L}'_T$ we find that

If $d_T \geqq 3$ and $d_S \geqq 2$, then (6.2) holds good, or else we have

$$V^{h}_{\cdot i} = 0 \quad \text{for all} \quad V \in \mathfrak{V},\ i \in I(T),\ h \in I(S). \tag{6.3}$$

This fact will be also expressed as follows, the $\begin{bmatrix} S \\ T \end{bmatrix}$ region of $\mathfrak{V}$ is *all* or *none*.

Similarly we find from (5.5) that the $\begin{bmatrix} T \\ S \end{bmatrix}$ region of $\mathfrak{V}$ is also all or none if $d_T \geqq 3$ and $d_S \geqq 2$.

On the other hand, when $S < T$, the matrix V where the elements are shown by (6.2) is, if it exists, a matrix of $\mathfrak{V}_{(P\hat{+}1)}$, so that it must satisfy $(1.1)_S$, which contradicts (6.2). Thus we find that, if $S < T$, we must get

9) For example, if $T_i \cap T_j$, then $R_{i+1}, \dots\dots, R_j$ are vacuous.

(6.3). But, if $S > T$, (6.3) fails to hold on account of some matrices of $\mathfrak{B}_T$. Hence we must get (6.2) then. Thus we find that, if $d_S \geqq 2$ and $d_T \geqq 3$, the $\begin{bmatrix} S \\ T \end{bmatrix}$ region of $\mathfrak{B}$ is none or all according as $S < T$ or $S > T$ respectively. We get a similar result when we start from (5.5).

Gathering the results we can say as follows.

In case $d_S \geqq 2$ and $d_T \geqq 3$ and also in case $d_S \geqq 3$ and $d_T \geqq 2$, the $\begin{bmatrix} S \\ T \end{bmatrix}$ region of $\mathfrak{B}$ is all if $S > T$ and none if $S < T$.

Next let us assume $d_S = d_T = 2$ for the given S and T and study the behavior of the $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrix and the $\begin{bmatrix} T \\ S \end{bmatrix}$-submatrix then. For the present we use indices as follows.

$$p, p' \in I(S), \quad p \neq p',$$
$$q, q' \in I(T), \quad q \neq q',$$
$$p'' = p' - d_1 - \dots\dots - d_{S-1}, \; q'' = q' - d_1 - \dots\dots - d_{T-1}.$$

Then, if $(v_T|^\lambda, u^T|_\mu)$ is a vector of M'_T, the matrix

$$V^\lambda_{\cdot\mu} = v_T|^\lambda \delta^q_\mu + (-1)^{q''} \delta^\lambda_{q'} u^T|_\mu \tag{6.4}$$

belongs to $\mathfrak{B}$ by virtue of Lemma 5. Hence, considering the p th column and the p' th row of this matrix, we get

$$((-1)^{q''} \delta^\lambda_{q'} u^T|_p, (-1)^{p''} v_T|^{p'} \delta^q_\mu) \in \mathfrak{M}'_S. \tag{6.5}$$

When p is given, we can take a vector $(v_T|^\lambda, u^T|_\mu)$ of $\mathfrak{M}'_T$ in such a way that at least one of $v_T|^{p'}$ and $u^T|_p$ is not zero, unless the $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrix and the $\begin{bmatrix} T \\ S \end{bmatrix}$-submatrix are both zero matrices for every matrix of $\mathfrak{B}$. Assume for example $S > T$. Then some matrices of $\mathfrak{B}_T$ have $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrices that are not zero matrices. Hence it is immediately ascertained that for any given $p \in I(S)$ there is a vector $(v_T|^\lambda, u^T|_\mu)$ of $\mathfrak{M}'_T$ such that $v_T|^{p'} = 1$. On the other hand we can apply Lemma 5_S with respect to the vector (6.5) and obain

$$((-1)^{q''} \delta^\lambda_{q'} u^T|_p) \delta^p_\mu + (-1)^{p''} \delta^\lambda_{p'} ((-1)^{p''} v_T|^{p'} \delta^q_\mu) \in \mathfrak{B}.$$

Hence, if we put $v_T|^{p'} = 1$ and $u^T|_p = t_p$, we get

$$(-1)^{q''} \delta^\lambda_{q'} t_p \delta^p_\mu + \delta^\lambda_{p'} \delta^q_\mu \in \mathfrak{B}.$$

Again, if we interchange p and p' we get

$$(-1)^{q''} \delta^\lambda_{q'} \delta^{p'}_\mu t_{p'} + \delta^\lambda_p \delta^q_\mu \in \mathfrak{B}.$$

Then, by virtue of the italicized sentense just before Lemma 4, we find that we can put $t_{p'} = (-1)^{p''} t$ and obtain

$$\delta_p^\lambda \delta_\mu^l + (-1)^{p''+q''} t\, \delta_{q'}^\lambda \delta_\mu^{p'} \in \mathfrak{B}. \tag{6. 6}$$

But the matrix (6. 6) belongs to $\mathfrak{B}_T$ because of the assumption $S > T$. Since the base S of R^n in use is a base of order 5, the n_{T-1}-submatrix of any matrix of $\mathfrak{B}_T$ must have special form as stated in § 2. Accordingly we must have $t = 0$ or $t = 1$. $t = 0$ shows that $S \overline{\cap} T$, while $t = 1$ that $S \cap T$. We immediately find that, if $S \cap T$, then $(-1)^{p''+q''} = (-1)^{p+q}$.

We then obtain the

THEOREM 4. *If $d_S \geqq 2$, $d_T \geqq 2$, $S > T$, and moreover $S \overline{\cap} T$, then the $\begin{bmatrix} S \\ T \end{bmatrix}$ region of $\mathfrak{B}$ is all, while the $\begin{bmatrix} T \\ S \end{bmatrix}$ region of $\mathfrak{B}$ is none. If $d_S = d_T = 2$, $S > T$ and moreover $S \cap T$, then*

$$\delta_p^\lambda \delta_\mu^q + (-1)^{p+q} \delta_{q'}^\lambda \delta_\mu^{p'} \in \mathfrak{B} \tag{6. 7}$$

$$(p, p' \in I(S)\,;\ q, q' \in I(T)\,;\ p \neq p',\ q \neq q'),$$

and besides,

$$V^{q'}_{\cdot p'} = (-1)^{p+q} V^p_{\cdot q} \tag{6. 8}$$

for every $V \in \mathfrak{B}$.

Proof is needed only for the last enunciation.

According to Lemma 5_T, the matrix[10]

$$V'^\lambda_{\cdot\mu} = (V^\lambda_{\cdot q} - \delta_q^\lambda V^q_{\cdot q}) \delta_\mu^q + \delta_{q'}^\lambda (V^{q'}_{\cdot\mu} - V^{q'}_{\cdot q'} \delta_\mu^{q'})$$

belongs to $\mathfrak{B}$ if $V^\lambda_{\cdot\mu} \in \mathfrak{B}$. Then applying Lemma 5_S to this matrix V' we find that the matrix

$$(V'^\lambda_{\cdot p'} - \delta_{p'}^\lambda V'^{p'}_{\cdot p'}) \delta_\mu^{p'} + \delta_p^\lambda (V'^p_{\cdot\mu} - V'^p_{\cdot p} \delta_\mu^p)$$

also belongs to $\mathfrak{B}$. Hence we get

$$\delta_{q'}^\lambda V^{q'}_{\cdot p'} \delta_\mu^{p'} + \delta_p^\lambda V^p_{\cdot q} \delta_\mu^q \in \mathfrak{B}$$

and then

$$(V^{q'}_{\cdot p'} - (-1)^{p+q} V^p_{\cdot q}) \delta_{q'}^\lambda \delta_\mu^{p'} \in \mathfrak{B}$$

by virtue of (6. 7). If (6. 8) does not hold good for some V in $\mathfrak{B}$, then we get $\delta_{q'}^\lambda \delta_\mu^{p'} \in \mathfrak{B}$ and consequently $\delta_p^\lambda \delta_\mu^q \in \mathfrak{B}$ which contradicts $S \cap T$.

We also obtain the following

10) The summation convension is not used with respect to p, p', q, q'.

COROLLARY. *Let S and T be fixed. Then the linear space $\mathfrak{B}$ is spanned by the following two subspaces. The matrices of one subspace are such that their $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrices and $\begin{bmatrix} T \\ S \end{bmatrix}$-submatrices are zero matrices. The matrices of the other subspace are such that their elements are zeroes except those of $\begin{bmatrix} S \\ T \end{bmatrix}$-submatrices and those of $\begin{bmatrix} T \\ S \end{bmatrix}$-submatrices. These matrices are linear combinations of* (6.7) *if $S \cap T$ or of* (6.2) *if $S \bar{\cap} T$ and $S > T$.*

The space spanned by matrices

$$V^{\lambda}_{\cdot\mu} = \delta^{\lambda}_{j_S}\, \delta^{i_T}_{\mu} \qquad (S \bar{\cap} T)$$

for given S and T will be denoted by $\mathfrak{B}\begin{bmatrix} S \\ T \end{bmatrix}$ while the space spanned by the matrices (6.7) will be denoted by $\mathfrak{B}_{[S,T]}$.

7. Linear spaces $\mathfrak{L}_T$, $\mathfrak{L}^T$, $\mathfrak{M}_T$. Let T be a fixed number and let V be a matrix (5.4) or (5.5) if $d_T \geqq 3$, or a matrix (5.6) or (5.7) if $d_T = 2$. Applying all operators $\mathbf{1} - [S]$ such that $S \neq T$ upon V we get

$$\left\{ \prod_{S \neq T} (\mathbf{1} - [S]) \right\} V \in \mathfrak{B}. \tag{7. 1}$$

This calls forth the following notion.

When $d_T \geqq 3$ we take an arbitrary column vector of $\begin{bmatrix} R \\ T \end{bmatrix}$-submatrix in any matrix V of $\mathfrak{B}$. The set of all such vectors is a linear space, which will be denoted by $\mathfrak{L}_T$. Similarly we can define $\mathfrak{L}_T$. When $d_T = 2$ we take the first column vector of $\begin{bmatrix} R \\ T \end{bmatrix}$-submatrix and the second row vector of $\begin{bmatrix} T \\ R \end{bmatrix}$-submatrix in any matrix V of $\mathfrak{B}$ and arrange them in this order. The set of all compound vectors obtained in such a way is a linear space, which will be denoted by $\mathfrak{M}_T$.

Then, if $d_T \geqq 3$, a matrix such that the column vectors of its $\begin{bmatrix} R \\ T \end{bmatrix}$-submatrix are vectors of $\mathfrak{L}_T$ while all other elements are zeroes is a matrix of $\mathfrak{B}$. A matrix such that the row vectors of its $\begin{bmatrix} T \\ R \end{bmatrix}$-submatrix are vectors of $\mathfrak{L}^T$ while all other elements are zeroes is also a matrix of $\mathfrak{B}$. On the other hand, if $d_T = 2$, then

$$\delta^{\lambda}_{a}\, v_T|^{a}\, \delta^{p}_{\mu} + (-1)^{p''}\, \delta^{\lambda}_{p'}\, u^{T}|_{b}\, \delta^{b}_{\mu} \in \mathfrak{B}$$

as long as $(v_T|^{a},\ u^{T}|_{b}) \in \mathfrak{M}_T$.

Let us denote the spaces spanned by such matrices by

$$\mathfrak{B}\begin{bmatrix} R \\ T \end{bmatrix}, \qquad \mathfrak{B}\begin{bmatrix} T \\ R \end{bmatrix}, \qquad \mathfrak{B}_{[T,R]}$$

respectively.

The linear space spanned by those matrices of $\mathfrak{B}$ in which non zero elements are found only in R-submatrix and T-submatrices will be denoted by $\mathfrak{B}_{[R,D]}$. Then we get the

THEOREM 5. *The linear space $\mathfrak{B}$ is decomposed into the direct sum as follows*

$$\mathfrak{B} = \mathfrak{B}_{[R,D]} + \sum_{d_T \geqq 3} \left(\mathfrak{B}\begin{bmatrix} R \\ T \end{bmatrix} + \mathfrak{B}\begin{bmatrix} T \\ R \end{bmatrix} \right) + \sum_{d_T = 2} \mathfrak{B}_{[T,R]} + \sum{}^{*} \mathfrak{B}\begin{bmatrix} S \\ T \end{bmatrix} + \sum{}^{\dagger} \mathfrak{B}_{[S,T]}, \tag{7. 2}$$

where Σ^* is the sum with respect to S, T such that $S \,\overline{\cap}\, T$ and $S > T$. and where $\Sigma^\dagger$ is the sum with respect to S, T such that $S \cap T$ and $S > T$.

To be continued.

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