ON COHOMOTOPY GROUPS OF S. S. TRIAD

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1. Introduction. A definition of the homotopy groups of a semisimplicial complex, using only the simplicial structure, was given by D. M. Kan [8] and homotopy theory in the context of abstract semisimplicial theory was developed. On the other hand there exists an important group which was investigated by E. Spanier [14] for a compact pair (X, A) and was called Borsuk's cohomotopy group.

In this paper we shall introduce an n-dimensional cohomotopy group $\pi^n(K; L, M)$ of an s. s. triad $(K; L, M)$ which is the set of homotopy classes of maps from $(K; L, M)$ into the triad of singular complexes for some special spaces. It will be proved that $\pi^n(K; L, M)$ has a group structure and the cohomotopy sequence of an s. s. triad is exact. In order to prove these properties we define a one-to-one correspondence which is a generalization of Kan's one [9] and recall some results which were already proved in [1]. Moreover we shall introduce the other n-dimensional cohomotopy group $\bar{\pi}^n(K; L, M)$ of an s. s. triad $(K; L, M)$. It is not proved that the cohomotopy sequence for these groups is exact, however, it seems to me that this cohomotopy sequence is also exact. In the last section, we consider the other one-to-one correspondence and, using this correspondence, define the different definitions of $\pi^n(K; L, M)$ and $\pi_n(K; L, M)$. (Recently the author and others investigated the n-dimensional homotopy group $\pi_n(K; L, M)$ of an s. s. triad $(K; L, M)$ [2]). At last it will be proved that these definitions are equivalent respectively.

2. Preliminaries. Let C denote the space of sequences of real numbers $y = \{y_i\}$ $(i = 1, 2 \ldots\ldots)$ which are finitely non-zero. C is metrized by

$$\text{dist }(y, y') = \Big(\sum_i (y_i - y'_i)^2\Big)^{\frac{1}{2}}.$$

DEFINITION 2. 1. The sets below are defined by the corresponding condition on the right.

$$S^n = \Big\{y \in C \,|\, y_i = 0 \text{ for } i > n+1 \text{ and } \sum_{1 \leqq i \leqq n+1} y_i^2 = 1\Big\},$$

$$E^{n+1} = \left\{ y \in C \,|\, y_i = 0 \text{ for } i > n+1 \text{ and } \sum_{1 \leqq i \leqq n+1} y_i^2 \leqq 1 \right\},$$

$$E_+^n = \{y \in S^n \,|\, y_{n+1} \geqq 0\},$$

$$E_-^n = \{y \in S^n \,|\, y_{n+1} \leqq 0\}.$$

The following relations are obvious

$$\ldots \subset E_+^{n-1} \cup E_-^{n-1} = S^{n-1} = E_+^n \cap E_-^n \subset E_+^n \cup E_-^n = S^n = E_+^{n+1} \cap E_-^{n+1} \subset \ldots.$$

DEFINITION 2.2. A *topological pair* (X, A) is a topological space X and a closed subset A. A *map* f of a topological pair (X, A) into a topological pair (Y, B) is a continuous function from X into Y which maps A into B and base point to base point, and will be denoted by $f: (X, A) \to (Y, B)$.

A *topological triad* $(X; A, B)$ is a topological space X and closed subsets A and B which have a non-vacuous intersection $A \cap B \neq 0$. A *map* f of a topological triad $(X; A, B)$ into a topological triad $(Y; C, D)$ is a continuous function from X into Y which maps A, B into C, D respectively and base point into base point and will be denoted by $f: (X; A, B) \to (Y; C, D)$.

DEFINITION 2.3. A *CW-pair* (X, A) is a CW-complex X [17] and a subcomplex A and a *CW-triad* $(X; A, B)$ is a CW-complex X and subcomplexes A and B which have a non-vacuous intersection $A \cap B \neq 0$.

DEFINITION 2.4. An *s. s. pair* (K, L) is an s. s. complex K and a subcomplex L [7], [11]. An s.s. *map* f [8] of an s. s. pair (K, L) into an s. s. pair (P, Q) is a function from K into P which maps L into Q and base point to base point and commutes with all face and degeneracy operators. It will be denoted by $f: (K, L) \to (P, Q)$. An *s. s. triad* $(K; L, M)$ is an s. s. complex K and subcomplexes L and M which have a non-vacuous intersection $L \cap M \neq 0$. An s. s. *map* f of an s. s. triad $(K; L, M)$ into an s. s. triad $(P; Q, R)$ is a function from K into P which maps L, M into Q, R respectively and base point φ to base point x and commutes with all face and degeneracy operators and will be denoted by $f: (K; L, M) \to (P; Q, R)$.

DEFINITION 2.5. Let $(K; L, M)$ and $(P; Q, R)$ be two s. s. triads. The function complex $(P; Q, R)^{(K;L,M)}$ is the s. s. complex defined as follows [5], [11]: An n-simplex of $(P; Q, R)^{(K;L,M)}$ is any s. s. map $f: (K \times \Delta_n; L \times \Delta_n, M \times \Delta_n) \to (P; Q, R)$ such that $f(\varphi\eta^0 \ldots \eta^r, \sigma) = x\eta^0 \ldots \eta^r$ for all r simplex $\sigma \in \Delta_n$, where Δ_n is the standard n-simplex. Its faces $f\varepsilon^i$ and degeneracies $f\eta^i$ are the composite maps

$$(K \times \Delta_{n-1}; L \times \Delta_{n-1}, M \times \Delta_{n-1}) \xrightarrow{1 \times \lambda^i} (K \times \Delta_n; L \times \Delta_n, M \times \Delta_n)$$

$$\xrightarrow{f} (P;\ Q, R),$$

$$(K \times \Delta_{n+1};\ L \times \Delta_{n+1},\ M \times \Delta_{n+1}) \xrightarrow{1 \times \mu^i} (K \times \Delta_n;\ L \times \Delta_n,\ M \times \Delta_n)$$

$$\xrightarrow{f} (P;\ Q, R),$$

where $\lambda^i : \Delta_{n-1} \to \Delta_n$ and $\mu^i : \Delta_{n+1} \to \Delta_n$ are given by

$$\lambda^i(j) = j \ (j < i)$$
$$\lambda^i(j) = j + 1 \ (j \geqq i)$$
$$\mu^i(j) = j \ (j \leqq i)$$
$$\mu^i(j) = j - 1 \ (j > i) \qquad \text{for } i = 0, 1, 2, \ldots\ldots, n.$$

DEFINITION 2. 6. An s. s. triad $(P;\ Q, R)$ is said to *satisfy the extension condition* if P, Q, R and $Q \cap R$ satisfy the extension condition [7] and it will be called a Kan triad.

PROPOSITION 2. 7. *If X is a topological space, then the total singular complex SX [4] satisfies the extension condition.*

The proof of this proposition can be found in [12].

PROPOSITION 2. 8. *If $(P;\ Q, R)$ is a Kan triad, so is the function complex $(P;\ Q, R)^{(K;L,M)}$.*

This proposition is the generalization of Theorem 3 of [5] and its proof is completely similar to Gugenheim's one therefore it will be omitted.

PROPOSITION 2. 9. *Let $(X;\ A, B)$ be a topological triad. Then the function complex $(SX;\ SA, SB)^{(K;L,M)}$ is a Kan triad.*

PROOF. Proposition 2.6 asserts that SX, SA and SB satisfy the extension condition. On the other hand $SA \cap SB = S(A \cap B)$ and $S(A \cap B)$ satisfies extension condition. Hence $(SX;\ SA, SB)$ is a Kan triad. It follows from Proposition 2.8 that $(SX;\ SA, SB)^{(K;L,M)}$ is a Kan triad.

DEFINITION 2. 10. [9] Two s. s. maps $f, g : (K;\ L, M) \to (SX;\ SA, SB)$ are called *homotopic* if there exists an s. s. map $f^I : (K \times I;\ L \times I,\ M \times I) \to (SX;\ SA, SB)$ such that

(i) $\quad f^I(\sigma,\ \varepsilon_1^0\eta^0 \ldots\ \eta^{n-1}) = f(\sigma),$

(ii) $\quad f^I(\sigma,\ \varepsilon_1^1\eta^0 \ldots \eta^{n-1}) = g(\sigma),$

(iii) $\quad f^I(\varphi\eta^0,\ \varepsilon_1) = x\eta^0$, where $I = \Delta_1$, and φ and x are base points of K and X respectively.

We then write $f^I: f \simeq g$ or $f \simeq g$.

PROPOSITION 2.11. *The homotopy relation $\simeq$ is an equivalence on the simplexes of* $(SX;\ SA, SB)^{(K,L,M)}$

PROOF. It follows from Proposition 2.8 that function complex $(SX;\ SA, SB)^{(K;L,M)}$ satisfies the extension condition. An s. s. map $f: (K;\ K, M) \to (SX;\ SA, SB)$ is a 0-simplex of the function complex $(SX; SA, SB)^{(K;L,M)}$. It is easy to see that for two s. s. maps $f, g: (K, L, M) \to (SX;\ SA, SB)$ such that $f^I: f \simeq g$, the homotopy f^I is a 1-simplex of the function complex such that $f^I\varepsilon^0 = f$, $f^I\varepsilon^1 = g$. Let $f: (K;\ L, M) \to (SX;\ SA, SB)$ be an s. s. map, then $f\eta^0\varepsilon^0 = f\eta^0\varepsilon^1 = f$. Hence $f\eta^0: f \simeq f$ i. e. the relation $\simeq$ is reflexive and it thus remains to show that $f \simeq g$ and $f \simeq h$ imply $g \simeq h$. Let $F_1, F_2 \in (SX, SA, SB)^{(K;L\ M)}$ be such that $F_1\varepsilon^0 = f$, $F_1\varepsilon^1 = g$, $F_2\varepsilon^0 = f$ and $F_2\varepsilon^1 = h$. Then 1-simplices F_1, F_2 "match" and application of the extension condition yields a 2-simplex F such that $F\varepsilon^0 = F_1$ and $F\varepsilon^1 = F_2$. A straightforward computation now yields $F\varepsilon^2\varepsilon^0 = g$, $F\varepsilon^2\varepsilon^1 = h$ and hence $F\varepsilon^2: g \simeq h$.

DEFINITION 2.12. Let $(K;\ L, M)$ be an s. s. triad and $(X;\ A, B)$ a topological triad. Then $\pi(K;\ L, M|SX;\ SA, SB)$ denotes the set of homotopy classes of maps $(K;\ L, M) \to (SX;\ SA, SB)$.

Let K be an s. s. complex and $|K|$ be its geometrical realization (Milnor, [10]). For an s. s. triad $(K;\ L, M)$, let $(|K|;\ |L|, |M|)$ be its geometrical realization. Let $(X; A, B)$ be a topological triad. Then (SX, SA, SB) is an s.s. triad.
Let $u: (|K|;\ |L|, |M|) \to (X;\ A, B)$ be a continuous map. We then define an s. s. map $\gamma(u): (K; L, M) \to (SX;\ SA, SB)$ as follows [9]. For an n-simplex $\sigma \in K$ let $\phi_\sigma: \Delta_n \to K$ be the s. s. map such that $\phi_\sigma\varepsilon_n = \sigma$ (where ε_n is the only non-degenerate n-simplex of Δ_n). Then the singular simplex $\gamma(u)\sigma \in SX$ is defined as the composition

$$|\Delta_n| \xrightarrow{|\phi_\sigma|} |K| \xrightarrow{u} X,$$

where $|\phi_\sigma|$ [10] is a continuous map induced by ϕ_σ.
If σ is an n-simplex of L (or M), then $\gamma(u)\sigma \in SA$ (or SB).

PROPOSITION 2.13. *The function γ establishes a one-to-one correspondence between continuous maps* $(|K|;|L|, |M|) \to (X;\ A, B)$ *and s. s. maps* $(K;\ L, M) \to (SX;\ SA, SB)$.

PROOF. It is clear that γ is a single valued function.
If $f: (K;\ L, M) \to (SX;\ SA, SB)$ is an s. s. map, we then define a continuous map $\bar{\gamma}(f): (|K|;\ |L|, |M|) \to (X;\ A, B)$ as follows. For an n-simplex $\sigma \in K$,

$f(\sigma)$ is a continuous map: $|\Delta_n| \to X$. We denote by $|\sigma, \mathcal{E}_n| \in |K|$ [10] the corresponding element to $\sigma \in K$. Let $\bar{\gamma}(f)$: $(|K|;\ |L|, |M|) \to (X;\ A, B)$ be the continuous map such that $\bar{\gamma}(f) | |\sigma, \mathcal{E}_n| = f(\sigma)\cdot|\psi_\sigma|$ where $\psi_\sigma : \sigma \to \Delta_n$ is the s. s. map such that $\psi_\circ\sigma = \mathcal{E}_n$. Therefore $\bar{\gamma}(f) : (|K|;\ |L|, |M|) \to (X;\ A, B)$ is a continuous map. It is easy to see that $\bar{\gamma}$ is a single valued function.

Now we prove that γ is an onto correspondence. Let $f: (|K|; |L|,\ |M|) \to (SX;\ SA, SB)$ be an s. s. map. For an n-simplex σ of $K, f(\sigma)$ is a continuous map: $|\Delta_n| \to X$ and $\bar{\gamma}(f)\,(|\sigma, \mathcal{E}_n|) = f(\sigma)\ (|\Delta_n|) \subset X$. If $\sigma \in L$ (or M), $\bar{\gamma}(f)$ $(|\sigma, \mathcal{E}_n|) \subset A$ (or B). Hence $\bar{\gamma}(f) : (|K|;\ |L|, |M|) \to (X;\ A, B)$ is a continuous map. Moreover for $\bar{\gamma}(f)$ we define $\gamma\,(\bar{\gamma}(f)) : (K; L, M) \to (SX; SA,\ SB)$. The singular simplex $\gamma(\bar{\gamma}(f))\,\sigma \ni SX$ is defined as the composition $|\Delta_n| \xrightarrow{|\phi^\tau|} |K| \xrightarrow{\bar{\gamma}(f)} X$ and if $\sigma \in L$ (or M), $\gamma(\bar{\gamma}(f))\,\sigma \in SA$ (or SB). By the definition of $\bar{\gamma}(f)$, $\gamma(\bar{\gamma}(f))\sigma : |\Delta_n| \to X$ (or A, or B) is equal to $f\ (\sigma)$. Then $\gamma\bar{\gamma} =$ identity and γ is onto. Therefore γ and $\bar{\gamma}$ are inverse to each other.

THEOREM 2. 14. *Let $(K; L, M)$ be an s. s. triad and $(X; A, B)$ be a topological triad. Then the function γ induces a one-to-one correspondence $\gamma^\#$ between the homotopy classes of continuous maps $(|K|; |L|, |M|) \to (X;\ A, B)$ and the homotopy classes of s. s. maps $(K; L, M) \to (SX, SA, SB)$.*

PROOF. Let $u, v : (|K|; |L|, |M|) \to (X;\ A, B)$ be continuous maps and $F : u \simeq v$. Now we consider non-degenerate $(n+1)$-simplex

$$\tau_i = (0,\dots, i-1, i, i, i+1,\dots, n) \times (0_1, 0_2\dots, 0_i, 1_{i+1},\dots,1_{n+1})$$

of $\Delta_n \times I$ for $i = 0, 1, 2, \dots, n$. Then a non-degenerate $(n+1)$-simplex of $K \times I$ may be given by $(\phi_\sigma \times 1)\ (\tau_i)$ for $\sigma \in K$. Now we denote by $[\sigma, \tau_i]$ this simplex and by $[\sigma, \tau_i\mathcal{E}^j]$ its face $[\sigma, \tau_i]\mathcal{E}^j$ for $j = 0, 1, 2,\dots, n+1$. On the other hand the n-simplex of $|K|$ is $|\sigma, \mathcal{E}_n|$ where $\sigma \in K_n$ and $\mathcal{E}_n \in \Delta_n$. Then non-degenerate $(n+1)$-simplex of $|K| \times |I|$ may be denoted by $[|\sigma, \mathcal{E}_n|,\ \tau_i]$. For an n-simplex $\sigma \in K$ let $\phi_{[\sigma,\tau_i]} : \Delta_{n+1} \to K \times I$ be the unique s. s. map such that $\phi_{[\sigma,\tau_i]}\ \mathcal{E}_{n+1} = [\sigma, \tau_i]$. Then the singular simplex $\gamma(F)\ [\sigma, \tau_i] \in SX$ is defined as the composition

$$|\Delta_{n+1}| \xrightarrow{|\phi_{[\sigma,\tau_i]}|} |K \times I| = |K| \times |I| \xrightarrow{F} X,$$

where $|K \times I|$ is canonically homeomorphic to $|K \times I|$ [10] and these may be identified each other. If σ is an n-simplex of L (or M), then $\gamma(F)\ (\sigma, \tau_i) \in$ SA (or SB). Therefore $\gamma(F) : (K \times I;\ L \times I, M \times I) \to (SX;\ SA, SB)$ is an

s. s. map. Since $\gamma(F)$ $(\sigma, \varepsilon_1^0\eta^0 \dots \eta^{n-1}) = \gamma(F)[\sigma, \tau_0\varepsilon^0] = \gamma(u)(\sigma)$, $\gamma(F)$ $(\sigma, \varepsilon_1^1\eta^0 \dots\dots \eta^{n-1}) = \gamma(F)[\sigma, \tau_n\varepsilon^{n+1}] = \gamma(v)(\sigma)$ and $\gamma(F)$ $(\varphi\eta^0, \varepsilon_1) = x\eta^0$, $\gamma(F): \gamma(u) \simeq \gamma(v)$. Hence γ induces a single valued correspondence $\gamma^{\#}$ between the homotopy classes of continuous maps $(|K|; |L|, |M|) \to (X; A, B)$ and the homotopy classes of s. s. maps $(K; L, M) \to (SX; SA, SB)$. Let $f, g: (K; L, M) \to (SX; SA, SB)$ be s. s. maps and $G: f \simeq g$. Now we define a continuous map $\overline{\gamma}(G)$: $(|K| \times |I|; |L| \times |I|, |M| \times |I|) \to (X; A, B)$ as follows. For an n-simplex $\sigma \in K$, $G[\sigma, \tau_i]$ is a continuous map: $\Delta_{n+1} \to X$. It is easy to see that if $\sigma \in L$(or M), $G[\sigma, \tau_i]$ is a continuous map: $\Delta_{n+1} \to A$(or B). Let $\overline{\gamma}(G)$: $(|K| \times |I|; |L| \times |I|, |M| \times |I|) \to (X; A, B)$ be the continuous map such that $\overline{\gamma}(G)$ $|[|\sigma, \varepsilon_n|, \tau_i] = G[\sigma, \tau_i] \cdot |\psi_{[\sigma,\tau_i]}|$. It is easy to verify $\overline{\gamma}(G)$ $[|\sigma, \varepsilon_n|, \tau_0\varepsilon^0] = \overline{\gamma}(f)| |\sigma, \varepsilon_n|$ and $\overline{\gamma}(G)$ $|[|\sigma, \varepsilon_n|, \tau_n\varepsilon^{n+1}] = \overline{\gamma}(g)| |\sigma, \varepsilon_n|$. Therefore $\overline{\gamma}(G): \overline{\gamma}(f) \simeq \overline{\gamma}(g)$. Hence $\overline{\gamma}$ induces a single valued correspondence $\overline{\gamma}_{\#}$ between the homotopy classes of s. s. maps $(K; L, M) \to (SX; SA, SB)$ and the homotopy classes of continuous maps $(|K|; |L|, |M|) \to (X; A, B)$. It is easy to see that $\gamma^{\#}$ and $\overline{\gamma}_{\#}$ are inverse to each other.

3. Group structure. Let $(X; A, B)$ be a CW-triad and $(Y; C, D)$ a topological triad. In [1] we denoted by $\pi(X; A, B|Y; C, D)$ the set of homotopy classes of maps of $(X; A, B) \to (Y; C, D)$ such that base point is mapped to base point. Let $f^{\#}: \pi(X'; A', B'|Y; C, D) \to \pi(X; A, B|Y; C, D)$ be induced map by a map $f: (X; A, B) \to (X'; A', B')$ and $\varphi_{\#}: \pi(X; A, B|Y; C, D) \to \pi(X; A, B|Y'; C', D')$ the one by a map $\varphi: (Y; C, D) \to (Y'; C', D')$ as usual. Let $\mathcal{S}_{\#}: \pi(X; A, B|Y; C, D) \to \pi(\mathcal{S}X; \mathcal{S}A, \mathcal{S}B|\mathcal{S}Y; \mathcal{S}C, \mathcal{S}D)$ be the function induced by suspension as in [15], [16].

Let $(Y; C, D)^{(X; A, B)}$ denote a function space of maps of $(X; A, B)$ into $(Y; C, D)$ with the compact-open topology. There is a function $\lambda: \pi_r((Y; C, D)^{(X; A, B)}) \to \pi(\mathcal{S}^r X; \mathcal{S}^r A, \mathcal{S}^r B|Y; C, D)$ [13] which is one-to-one and natural with respect to maps f and φ where $\mathcal{S}^r = \mathcal{S}(\mathcal{S}^{r-1})$.

The following theorem was proved in [1] using $\lambda^{-1} \circ \mathcal{S}_{\#}^2$.

THEOREM 3.1. *Let $(X; A, B)$ be a CW-triad and let Y, C and D be $(n-1)$, $(l-1)$ and $(m-1)$- connected topological spaces respectively, and assume that* $\dim X \leqq 2n-2$, $\dim A \leqq 2l-2$ *and* $\dim B \leqq 2m-2$. *Then we can introduce a group structure into $\pi(X; A, B|Y; C, D)$, which is Abelian and natural with respect to maps f and φ.*

PROPOSITION 3.2. *Let $|K|$ be a geometric realization of K, then $|K|$ is a CW-complex having one n-cell corresponding to each non-degenerate n-simplex of K.*

This Proposition was proved by J.Milnor in [10].

PROPOSITION 3. 3 *Let* $(K;\ L, M)$ *be an s. s. triad and* $(X;\ A, B)$ *be a topological triad where* X, A *and* B *be* $(n-1)$, $(l-1)$ *and* $(m-1)$*-connected respectively, and assume that* dim $K \leqq 2n-2$, dim $L \leqq 2l-2$ *and* dim $M \leqq 2m-2$. *Then* $\pi(K;\ L, M|SX, SA, SB)$ is an Abelian group.

PROOF. In Theorem 2.5 the function γ induced a one-to-one correspondence $\gamma_{\#}$ between the homotopy classes of continuous maps $(|K|;\ |L|, |M|) \to (X; A, B)$ and the homotopy classes of s.s. maps $(K; L, M) \to (SX; SA, SB)$. In its proof we defined $\bar{\gamma}$ which induces the inverse correspondence $\bar{\gamma}^{\#}$ of $\gamma^{\#}$. On the other hand $|K|$, $|L|$ and $|M|$ are CW-complexes and dim $|K| \leqq 2n-2$, dim $|L| \leqq 2l-2$ and dim $|M| \leqq 2m-2$ by Proposition 3.2. Then it follows from Theorem 3. 1 that $\pi(|K|; |L|, |M|\,|X;\ A, B)$ has a group structure. Therefore, using $\bar{\gamma}^{\#}$, we may define the multiplication in $\pi(K; L, M\,|SX;\ SA, SB)$. It is easy to see that $\pi(K;\ L, M|SX;\ SA,\ SB)$ is an Abelian group.

PROPOSITION 3. 4. *Let* $f\colon (K;\ L, M) \to (K';L',M')$ *be an s. s. map and* $\varphi : (X; A, B) \to (X', A'; B')$ *continuous map. Then induced maps* $f^{\#}$ *and* $(S\varphi)_{\#}$ *are homomorphisms.*

PROOF. Now we consider the following diagrams:

$$\begin{array}{ccc}
\pi(|K|; |L|, |M|\,|X; A, B) & \xrightarrow{\gamma^{\#}} & \pi(K;\ L, M|SX;\ SA, SB) \\
{\scriptstyle (|f|)^{\#}}\uparrow & & \uparrow{\scriptstyle f^{\#}} \\
\pi(|K'|; |L'|,\ |M'|\,|X; A, B) & \xleftarrow{\bar{\gamma}^{\#}} & \pi(K'; L', M'|SX, SA, SB),
\end{array}$$

and

$$\begin{array}{ccc}
\pi(|K|; |L|, |M|\,|X; A, B) & \xleftarrow{\bar{\gamma}^{\#}} & \pi(K; L, M|SX;\ SA, SB) \\
{\scriptstyle \varphi_{\#}}\downarrow & & \downarrow{\scriptstyle (S\varphi)_{\#}} \\
\pi(|K|; |L|, |M|\,|X'; A', B') & \xrightarrow{\gamma^{\#}} & \pi(K; L, M|SX'; SA', SB').
\end{array}$$

Clearly the commutative relations hold in these diagrams, and $(|f|)^{\#}$ and $\varphi_{\#}$ are homomorphisms [1]. Therefore $f^{\#}$ and $(S\varphi)_{\#}$ are homomorphisms.

PROROSITION 3. 5. *If* $f\colon (K;\ L, M) \to (K;\ L, M)$ *is the identity, then* $f_{\#}$ *is the identity.*

PROOF. Let α be an element of $\pi(K;\ L,\ M|SX; SA,\ SB)$ and h be its representative. Then $f^{\#}\alpha$ has a representative $h \circ f$. Since f is the identity,

$f^{\#}\alpha$ equals to α. Therefore $f^{\#}$ is the identity.

PROPOSITION 3.6. *$(g\circ f)^{\#} = f^{\#}\circ g^{\#}$ where f: $(K; L, M) \rightarrow (K'; L'M')$ and g: $(K'; L'M') \rightarrow (K''; L'', K'')$ are s. s. maps.*

PROOF. Let α be an element of $\pi(K''; L'', M''|SX; SA, SB)$ and h be its representative. Then $(g\circ f)^{\#}\alpha$ has a representative $h\circ g\circ f$. On the other hand $g^{\#}\alpha$ has a representative $h\circ g$ and $f^{\#}\circ g^{\#}$ has a representative $h\circ g\circ f$. Therefore $(g\circ f)^{\#} = f^{\#}\circ g^{\#}$.

PROPOSITION 3.7. *If f, g : $(K; L, M) \rightarrow (K'; L', M')$ are homotopic, then $f^{\#} = g^{\#}$.*

PROOF. Let α be an element of $\pi(K'; L', M'|SX; SA, SB)$ and h be its representative. Then $f^{\#}\alpha$ and $g^{\#}\alpha$ have representatives $h\circ f$ and $h\circ g$. Since $f \simeq g$, $h\circ f$ and $h\circ g$ are homotopic. Therefore $f^{\#} = g^{\#}$.

PROPOSITION 3.8. *If φ : $(X; A, B) \rightarrow (X; A, B)$ is the identity, then $(S\varphi)_{\#}$ is the identity.*

PROOF. Let β be any element of $\pi(K; L, M|SX; SA, SB)$ and u be its representative. Then $(S\varphi)_{\#}\beta$ has a representative $(S\varphi)\circ u$. Since $S\varphi =$ identity, $(S\varphi)_{\#}\beta$ is the homotopy class of u and hence $(S\varphi)_{\#}$ is the identity.

PROPOSITION 3.9. *$(S\psi\circ S\varphi)_{\#} = (S\psi)_{\#}\circ (S\varphi)_{\#}$, where φ: $(X; A, B) \rightarrow (X'; A', B')$ and ψ : $(X; A', B') \rightarrow (X''; A'', B'')$ are continuous maps.*

PROOF. Let β be any element of $\pi(K; L, M|SX; SA, SB)$ and u be its representative. Then $(S\psi\circ S\varphi)_{\#}\beta$ has a representative $S\psi\circ S\varphi\circ u$. On the other hand $(S\varphi)^{\#}\beta$ has a representative $S\varphi\circ u$ and $(S\psi)_{\#}\circ(S\varphi)_{\#}\beta$ has a representative $S\psi\circ S\varphi\circ u$. Therefore $(S\psi\circ S\varphi)_{\#} = (S\psi)_{\#}\circ(S\varphi)_{\#}$.

PROPOSITION 3.10. *If φ, ψ : $(X; A, B) \rightarrow (X'; A', B')$ are homotopic, then $S\varphi$ and $S\psi$ are homotophic.*

PROOF. Let $\Phi: (X \times |I|; A \times |I|, B \times |I|) \rightarrow (X'; A', B')$ be a homotopy between φ and ψ. For $[\sigma, \tau_i] \in SX \times I$ (for $\sigma \in SX$ and $\tau_i \in \Delta_n \times I$, non-degenrate $(n+1)$-simplex of $SX \times I$ may be denoted by $[\sigma, \tau_i]$) we define Θ: $SX \times I \rightarrow SX'$ by $\Theta[\sigma, \tau_i] = (S\Phi)[\sigma, \tau_i]$. Then $\Theta[\sigma, \tau_0\varepsilon^0] = (S\Phi)[\sigma, \tau_0\varepsilon^0] = \varphi\circ\sigma$ and $\Theta[\sigma, \tau_n\varepsilon^{n+1}] = (S\Phi)[\sigma, \tau_n\varepsilon^{n+1}] = \psi\circ\sigma$. If $[\sigma, \tau_i] \in SA \times I$ (or $SB \times I$), then $\Theta[\sigma, \tau_i] \in SA'$ (or SB'). Therefore Θ: $(SX \times I; SA \times I, SB \times I) \rightarrow (SX'; SA', SB')$ is a homotopy between $S\varphi$ and $S\psi$.

PROPOSITION 3.11. *If φ, ψ : $(X; A, B) \rightarrow (X'; A', B')$ are homotopic, then $(S\varphi)_{\#} = (S\psi)_{\#}$.*

PROOF. Since $\varphi \simeq \psi$, by Proposition 3. 10, $S\varphi$ and $S\psi$ are homotopic.

Let $\beta \in \pi(K; L, M|SX; SA, SB)$ be any element and u be its representative, then $(S\varphi)_{\#}\beta$ and $(S\psi)_{\#}\beta$ have representatives $S\varphi\circ u$ and $S\psi\circ u$ respectively and $S\varphi\circ u \simeq S\psi\circ u$. Therefore $(S\varphi)_{\#} = (S\psi)_{\#}$.

4. Exact sequences. Let $(K; L, M)$ be an s. s. triad and $(X; A, x)$ be a topological pair with base point x and $i: (M, L \cap M) \to (K, L)$ and $j: (K, L, k_0) \to (K; L, M)$ be inclusions where $k_0 \subset L \cap M$ is the minimal subcomplex of $L \cap M$ which contains base point k of $L \cap M$. Then i, j induces the inclusion $|i|, |j|$ respectively [10] and these maps induce the homomorphisms $|i|^{\#}: \pi(|K|; |L| |X, A) \to \pi(|M|, |L| \cap |M| |X, A)$ and $|j|^{\#}: \pi(|K|; |L|, |M| |X; A, x) \to \pi(|K|; |L|, |k| |X; A, x)$. On the other hand the homomorphism $\Delta: \pi(|M|, |L| \cap |M| | X, A) \to \pi(|K|; |L|, |M| |\mathcal{S}X; \mathcal{S}A, x)$ was defined in [1].

THEOREM 4. 1. *Let $(K; L, M)$ be an s. s. triad with base point k and $(X; A, x)$ be a topological pair with base point x and assume that X and A are $(n-1)$ and $(m-1)$-connected respectively and that* $\dim K \leqq 2n-2$, $\dim L \leqq 2m-2$. *Then the following sequence is exact and natural with respect to maps $|f|: (|K|; |L|, |M|) \to (|K'|; |L'|, |M'|)$ and $\varphi: (X, A) \to (X', A')$:*

$$\pi(|K|; |L|, |M| | X; A, x) \xrightarrow{|j|^{\#}} \pi(|K|; |L|, |k_0| | X; A, x)$$
$$\xrightarrow{|i|^{\#}} \pi(|M|; |L| \cap |M|, |k_0| | X; A, x)$$
$$\xrightarrow{\Delta} \pi(|K|; |L|, |M| | SX; SA, x) \to \ldots\ldots$$

PROOF. It follows from Proposition 3.2 that $\dim |K| \leqq 2n-2$ and $\dim |L| \leqq 2m-2$. Therefore this theorem is an immediate consequence of Theorem 2. 5 in [1].

THEOREM 4. 2. *Let $(K; L, M)$ be an s.s.triad with base point k and (X, A, x) be a topological pair with base point x and assume that X and A are $(n-1)$ and $(m-1)$-connected respectively and that* $\dim L \leqq 2m-2$. *Then then following sequence is exact and natural with respect to maps $f: (K; L, M) \to (K'; L', M')$ and $\varphi: (X, A) \to (X', A')$:*

$$\pi(K; L, M|SX; SA, Sx) \xrightarrow{j_1^{\#}} \pi(K; L, k_0|SX; SA, Sx)$$
$$\xrightarrow{i_1^{\#}} \pi(M; L \cap M, k_0|SX; SA, Sx)$$
$$\xrightarrow{\Delta_1} \pi(K; L, M|S\mathcal{S}X; S\mathcal{S}A, Sx) \to$$

where $i_1^{\#}$ and $j_1^{\#}$ are induced by inclusion $i_1: (M; L \cap M, k_0) \to (K; L, k_0)$ and j_1: $(K; L, k_0) \to (K; L, M)$ *respectively and Δ_1 is defined so that $\Delta_1 = \gamma^{\#} \Delta \gamma^{\#}$*

PROOF. Now we consider the following diagram:

$$
\begin{tikzcd}
\pi(|K|;\ |L|,\ |M|\ |X;\ A, x) \arrow[d, "|j|^{\#}"] & \pi(K;\ L, M|SX;\ SA, Sx) \arrow[l, "\bar{\gamma}^{\#}"'] \arrow[d, "j_1^{\#}"] \\
\pi(|K|;\ |L|,\ |k_0|\ |X;\ A, x) \arrow[d, "|i|^{\#}"] \arrow[r, shift right, "\gamma^{\#}"'] & \pi(K;\ L, k_0|SX;\ SA, Sx) \arrow[l, shift right, "\bar{\gamma}^{\#}"'] \arrow[d, "i_1^{\#}"] \\
\pi(|M|;\ |L| \cap M,\ |k_0|\ |X; A, x) \arrow[d, "\Delta"] \arrow[r, shift right, "\gamma^{\#}"'] & \pi(M;\ L \cap M, k_0\ |SX;\ SA, Sx) \arrow[l, shift right, "\bar{\gamma}^{\#}"'] \arrow[d, "\Delta_1"] \\
\pi(|K|;\ |L|,\ |M|\ \ |\mathfrak{s}X;\ \mathfrak{s}A, x) \arrow[d, "|j|^{\#}"] \arrow[r, "\gamma^{\#}"] & \pi(K;\ L, M|\ \ S\mathfrak{s}X;\ S\mathfrak{s}A, Sx) \arrow[d, "j_1^{\#}"] \\
\vdots & \vdots
\end{tikzcd}
$$

Since commutative relations hold in this diagram, this theorem is a direct consequence of Theorem 4. 1.

5. $\pi^n(K;\ L, M)$ and $\bar{\pi}^n(K;\ L, M)$.

DEFINITION 5.1. Let $(X; A, B)$ be a CW-triad and $(K; L, M)$ be an s. s. triad. Then we define $\pi^n(X;\ A, B)$ and $\pi^n(K;\ L, M)$ by

$$\pi^n(X; A, B) = \pi(X; A, B\,|\,E_+^n;\ S^{n-1}, p),$$
$$\pi^n(K; L, M) = \pi(K; L, M\,|\,SE_+^n;\ SS^{n-1}, Sp).$$

PROPOSITION 5. 2. *Let $(X;\ A, B)$ be a CW-triad such that* dim $A \leqq 2n-4$. *Then $\pi^n(X; A, B)$ is an Abelian group.*

PROOF. Since E_+^n is contractible and S^{n-1} is $(n-2)$-connected and dim $A \leqq 2n-4$, this proposition is a direct consequence of Theorem 3. 1.

Similarly to Proposition 5.2, by Proposition 3. 3. we have

PROPOSITION 5. 3. *If $(K; L, M)$ is an s.s. triad such that* dim $L \leqq 2n-4$, *then $\pi^n(K;\ L, M)$ is an Abelian group.*

DEFINITION 5. 4. Let $(X; A, B)$ be a CW-triad and $(K;\ L, M)$ be an s. s. triad. Then we define $\bar{\pi}^n(X; A, B)$ and $\bar{\pi}^n(K;\ L, M)$ by

$$\bar{\pi}^n(X; A, B) = \pi(X;\ A, B\,|\,E_+^n;\ E_+^{n-1},\ E_-^{n-1}),$$

$$\bar{\pi}^n(K; L, M) = \pi(K; L, M\,|\,SE_+^n;\ \ SE_+^{n-1},\ SE_-^{n-1}).$$

PROPOSITION 5. 5. *Let $(X; A, B)$ be a CW-triad. Then $\bar{\pi}^n(X; A, B)$ is an*

Abelian group.

PROOF. Since E^n_+, E^{n-1}_+, E^{n-1}_- are all contractible, this proposition is a direct consequence of Theorem 3. 1.

Similarly to Proposition 5. 5, by Proposition 3. 3, we have

PROPOSITION 5. 6. *Let* $(K; L, M)$ *be an s. s. triad. Then* $\overline{\pi}^n(K; L, M)$ *is an Abelian group.*

DEFINITION 5. 7. (F.P.Peterson)[13] [14] Let (X, A) be a CW-pair. Then we define $\pi^n(X, A)$ by

$$\pi^n(X, A) = \pi(X, A \mid S^n, p).$$

DEFINITION 5. 8. Let (K, L) be an s. s. pair. Then we define $\pi^n(K, L)$ by

$$\pi^n(K, L) = \pi(K, L \mid SS^n, Sp).$$

PROPOSITION 5. 9. *Let* (X, A) *be a* CW*-pair such that* $\dim X \leqq 2n-2$. *Then* $\pi^n(X, A)$ *is an Abelian group.*

This Proposition is a direct consequence of Theorem 3. 1.

Similarly, Proposition 3. 3 implies

PROPOSITION 5.10. *Let* (K, L) *be an s. s. pair such that* $\dim K \leqq 2n - 2$. *Then* $\pi^n(K, L)$ *is an Abelian group.*

DEFINITION 5.11. Let (X, A) be a CW-pair and (K, L) be an s. s. pair. Then we define $\overline{\pi}^n(X, A)$ and $\overline{\pi}^n(K, L)$ by

$$\overline{\pi}^n(X, A) = \pi(X, A \mid E^n_+, S^{n-1})$$

$$\overline{\pi}^n(K, L) = \pi(K, L \mid SE^n_+, SS^{n-1}).$$

It is easy to verify that following Propositions hold.

PROPOSITION 5.12. *Let* (X, A) *be a* CW*-pair such that* $\dim A \leqq 2n-4$. *Then* $\overline{\pi}^n(X, A)$ *is an Abelian group.*

PROPOSITION 5.13. *Let* (K, L) *be an s. s.-pair such that* $\dim L \leqq 2n - 4$. *Then* $\pi^n(K, L)$ *is an Abelian group.*

Let $\Phi : (E^n_+ \times I, S^{n-1} \times I) \to (S^n, E^n_-)$ be defined so that if $\Phi_t = \Phi \mid (E^n_+ \times t)$, then

$\Phi_0 =$ identity map of (E^+_n, S^{n-1}) into (S^n, E^n_-),

$\Phi_1 : (E^n_+, S^{n-1}) \to (S^n, p)$ is a homeomorphism of $E^n_+ - S^{n-1}$ onto $S^n - p$.

Let (X, A) be a CW-pair and $f, g : (X, A) \to (E^n_+, S^{n-1})$ be continuous maps.

PROPOSITION 5.14. *If f and g are homotopic, then $\phi_1 \circ f \simeq \phi_1 \circ g$.*

It is easily seen that this proposition holds.

DEFINITION 5.15. The function ϕ_1 induces a set transformation from $\overline{\pi}^n(X, A)$ to $\pi^n(X, A)$. Hereafter this transformation will also be denoted by ϕ_1.

PROPOSITION 5.16. *$\phi_1 : \overline{\pi}^n(X, A) \to \pi^n(X, A)$ is a homomorphism.*

PROOF. We considered the following diagram in [1]:

$$\pi(K; L, M \mid X; Y, Z) \xrightarrow{\mathscr{S}^2_\#} \pi(\mathscr{S}^2K; \mathscr{S}^2L, \mathscr{S}^2M \mid \mathscr{S}^2X, \mathscr{S}^2Y, \mathscr{S}^2Z)$$
$$\xleftarrow{\lambda} \pi_2((\mathscr{S}^2X;\ \mathscr{S}^2Y, \mathscr{S}^2Z)^{(K;L,M)}).$$

Now we consider the special cases as follows:

$$\begin{array}{ccccc}
\overline{\pi}^n(X, A) & \xrightarrow{\mathscr{S}^2_\#} & \pi(\mathscr{S}^2X, \mathscr{S}^2A \mid \mathscr{S}^2E^n_+,\ \mathscr{S}^2S^{n-1}) & \xleftarrow{\lambda} & \pi_2((\mathscr{S}^2E^+_n, \mathscr{S}^2S^{n-1})^{(X,A)}) \\
\phi_1 \downarrow & & (\mathscr{S}^2\phi_1)_\# \downarrow & & (\mathscr{S}^2\phi_1)_* \downarrow \\
\pi^n(X, A) & \xrightarrow{\mathscr{S}^2_\#} & \pi(\mathscr{S}^2X, \mathscr{S}^2A \mid \mathscr{S}^2S^n, p) & \xleftarrow{\lambda} & \pi_2((\mathscr{S}^2S^n, p)^{(X,A)}).
\end{array}$$

In virtue of Theorem 1.1 and Theorem 1.2 in [1], $\mathscr{S}_\#$ and λ are natural with respect to map ϕ_1 and the map $(\mathscr{S}^2\phi_1)_* : \pi_2((\mathscr{S}^2E^n_+, \mathscr{S}^2S^{n-1})^{(X,A)}) \to \pi_2((\mathscr{S}^2S^n, p)^{(X,A)})$ is a homomorphism.

We defined group structures in $\overline{\pi}^n(X, A)$ and $\pi^n(X, A)$ using $\lambda^{-1} \circ \mathscr{S}^2_\#$ in [1]. Therefore $\phi_1 : \overline{\pi}^n(X,A) \to \pi^n(X, A)$ is a natural homomorphism induced by

$$(\mathscr{S}^2\phi_1)_* : \pi_2((\mathscr{S}^2E^n_+,\ \mathscr{S}^2S^{n-1})^{(X,A)}) \to \pi_2((\mathscr{S}^2S^n,\ p)^{(X,A)}).$$

Let $\overline{\Psi} : (E^n_+ \times I;\ E^{n-1}_+ \times I,\ E^{n-1}_- \times I) \to (E^n_+,\ S^{n-1}, E^{n-1}_-)$ be defined so that if $\psi_t = \overline{\Psi} \mid (E^+_n \times t)$, then

$\psi_0 \simeq$ identity map of $(E^n_+;\ E^{n-1}_+,\ E^{n-1}_-)$ into $(E^n_+;\ S^{n-1},\ E^{n-1}_-)$

$\psi_1 : (E^n_+; E^{n-2}_+,\ E^{n-1}_-) \to (E^n_+; S^{n-1},\ p)$ is a homeomorphism of $E^n_+ - E^{n-1}_-$ onto $E^n_+ - p$.

Let $(X; A, B)$ be a CW-triad and $f, g : (X;\ A, B) \to (E^n_+;\ E^{n-1}_+,\ E^{n-1}_-)$ be continuous maps.

PROPOSITION 5.17. *If f and g are homotopic, then $\psi_1 \circ f \simeq \psi_1 \circ g$.*

It is easily seen that this proposition holds.

DEFINITION 5.18. The function ψ_1 induces a set transformation from $\bar{\pi}^n(X; A, B)$ to $\pi^n(X; A, B)$ by Proposition 5.17. Hereafter this transformation will also be denoted by ψ_1.

Similarly to Proposition 5.16, we have

PROPOSITION 5.19. *$\psi_1 : \bar{\pi}^n(X; A, B) \to \pi^n(X; A, B)$ is a homomorphism.*

DEFINITION 5.20. Let (K, L) be an s. s. pair such that dim $K \leqq 2n-2$ dim $L \leqq 2n-4$.

Now we consider the following diagram:

$$\begin{array}{ccc} \bar{\pi}^n(|K|, |L|) & \xleftarrow{\bar{\gamma}^{\#}} & \bar{\pi}^n(K, L) \\ \phi_1 \downarrow & & (\phi_1)_{\#} \downarrow \\ \pi^n(|K|, |L|) & \xrightarrow{\gamma^{\#}} & \pi^n(K, L) \end{array}$$

We define $(\phi_1)_{\#} : \bar{\pi}^n(K,L) \to \pi^n(K, L)$ such that $(\phi_1)_{\#} = \gamma^{\#}\phi_1\bar{\gamma}^{\#}$. Then $(\phi_1)^{\#}$ is a homomorphism.

Similarly we define a homomorphism $(\psi_1)_{\#} : \bar{\pi}^n(K; L, M) \to \pi^n(K; L, M)$, where $(K; L, M)$ is an s. s. triad.

THEOREM 5.21. *Let $(K; L, M)$ be an s. s. triad and assume that* dim $L \leqq 2n-4$. *Then the following sequence is exact and natural with respect to a map $f:(K; L, M) \to (K'; L', M')$:*

$$\pi^n(K; L, M) \xrightarrow{j_1^{\#}} \bar{\pi}^n(K, L) \xrightarrow{i_1^{\#}} \bar{\pi}^n(M, L \cap M) \xrightarrow{\Delta_1} \bar{\pi}^{n+1}(K; L, M) \xrightarrow{j_1^{\#}} \cdots.$$

PROOF. In virtue of Proposition 5.3, and Proposition 5.10, $\pi^2(K; L, M)$ and $\pi^2(K, L)$ are abelian groups for $q \geqq n$ and since dim $(L \cap M) \leqq$ dim $L \leqq 2n-4$, $\pi^2(M, L \cap M)$ is an abelian group for $q \geqq n$. Then this theorem is a direct consequence of Theorem 4.2.

DEFINITION 5.22. An *s. s. triple* (K, L, M) is an s. s. complex K and a subcomplex L and a subcomplelx $M(\subset L)$.

THEOREM 5.23. *Let (K, L, M) be an s. s. triple and assume that* dim $K \leqq 2n-2$. *Then the following sequence is exact:*

$$\pi^n(K, L) \xrightarrow{j_1^{\#}} \pi^n(K, M) \xrightarrow{i_1^{\#}} \pi^n(L, M) \xrightarrow{\Delta_1} \pi^{n+1}(K, L) \to \cdots,$$

where $i_1^{\#}$ and $j_1^{\#}$ are inclusion maps and Δ_1 is defined similarly to Δ_1 of Theorem 4.2.

PROOF. By virtue of Theorem 14.3 in [14], the following sequence is exact:

$$\pi^n(|K|,|M|) \xrightarrow{j^{\#}} \pi^n(|K|,|M|) \xrightarrow{i^{\#}} \pi^n(|L|,|M|) \xrightarrow{\Delta} \pi^{n+1}(|K|,|L|) \to \dots.$$

In virtue of the isomorphism $\gamma^{\#}$, it is easy to see that this theorem holds.

6. $\pi_n(K; L, M)$.

DEFINITION 6.1. A *CW-tetrad* $(X; A, B, C)$ is a CW-complex X and subcomplexes A, B and C which have a non-vacuous intersection $A \cap B \cap C \neq 0$. A *map* f of a CW-tetrad $(X; A, B, C)$ into a CW-tetrad $(Y; D, E, F)$ is a continuous function from X to Y which maps A, B, C into D, F, E respectively and base point to base point and will be denoted by f: $(X; A, B, C) \to (Y; D, E, F)$.

DEFINITION 6.2. An *s. s. tetrad* $(K; L, M, N)$ is an s. s. complex K and subcomplexes L, M and N which have a non-vacuous intersection $L \cap M \cap N \neq 0$. An *s. s. map* f of an s. s. tetrad $(K; L, M, N)$ into an s. s. tetrad $(S|P|; S|Q|, S|R|, S|T|)$ is a function from K to $S|P|$ which maps L, M, N into $S|Q|, S|R|, S|T|$ respectively and base point to base point and commutes with all face and degeneracy operators and will be denoted by f: $(K; L, M, N) \to (S|P|; S|Q|, S|R|, S|T|)$.

Similarly to a case of triads, the following theorem holds.

THEOREM 6.3. *Let $(K; L, M, N)$ and $(P; Q, R, T)$ be s. s. tetrads. Then the function γ induces a one-to-one correspondence $\gamma^{\#}$ between the homotopy classes of continuous maps $(|K|; |L|, |M|, |N|) \to (|P|; |Q|, |R|, |T|)$ and the homotopy classes of s.s. maps $(K; L, M, N) \to (S|P|; S|Q|, S|R|, S|T|)$.*

DEFINITION 6.4. Let $(P; Q, R, p)$ be an s. s. triad with base point p. Then we define for $n \geqq 2$, $\pi_n(|P|; |Q|, |R|)$ and $\pi_n(P; Q, R)$ by

$$\pi_n(|P|: |Q|, |R|) = \pi(|\Delta_n|; |\partial_0\Delta_n|, |\partial_1\Delta_n|, |\sum_{i\neq 0,1} \partial_i\Delta_n| \, \Big| \, |P|; |Q|, |R|, |p_0|),$$

$$\pi_n(P; Q, R) = \pi(\Delta_n; \partial_0\Delta_n, \partial_1\Delta_n, \sum_{i\neq 0,1} \partial_i\Delta_n \, \Big| \, S|P|; S|Q|, S|R|, S|p_0|).$$

This set $\pi_n(|P|; |Q|, |R|)$ will be canonically identified to n-dimensional homotopy group in the sense of A. L. Blakers-W.S.Massey [3].

THEOREM 6.5. *Let $(P; Q, R)$ be an s. s. triad. Then the following sequence is exact:*

$$\pi_n(P; Q, R) \xrightarrow{\Delta_2} \pi_{n-1}(Q, Q \cap R) \xrightarrow{i_2} \pi_{n-1}(P, Q) \xrightarrow{j_2} \pi_{n-1}(P; Q, R) \to \cdots,$$

where i_2 and j_2 are inclusions and Δ_2 is defined so that $\Delta_2 = \gamma^{\#}\beta_+ \gamma^{\#}$ [3].

PROOF. Consider the following diagram:

$$\begin{array}{ccccccccc}
\pi^n(P; Q, R) & \xrightarrow{\Delta_2} & \pi_{n-1}(Q, Q \cap R) & \xrightarrow{i_2} & \pi_{n-1}(P, Q) & \xrightarrow{j_2} & \pi_{n-1}(P; Q, R) & \to \cdots \\
\downarrow{\scriptstyle \overline{\gamma}^{\#}} & & \downarrow{\scriptstyle \overline{\gamma}^{\#}} & & \downarrow{\scriptstyle \overline{\gamma}^{\#}} & & \downarrow{\scriptstyle \overline{\gamma}^{\#}} & \\
\pi^n(|P|; |Q|, |R|) & \xrightarrow{\beta_+} & \pi_{n-1}(|Q|, |Q| \cap |R|) & \xrightarrow{i_{2*}} & \pi_{n-1}(|P|, |Q|) & \xrightarrow{j_{2*}} & \pi_{n-1}(|P|; |Q|, |R|) & \to \cdots .
\end{array}$$

The lower sequence is exact [3] and $\overline{\gamma}^{\#}$ is a one-to-one correspondence. Then every terms of the upper sequence have a group structure in virtue of $\gamma^{\#}$ and the exactness of the upper sequence is an immediate consequence of that of the lower sequence.

Let Sd K be the subdivision [7] of K and $\mathrm{Sd}^r K$ be the r-fold subdivison of K and let u: $(\mathrm{Sd}^r K;\ \mathrm{Sd}^r L,\ \mathrm{Sd}^r M,\ \mathrm{Sd}^r N) \to (P;\ Q;\ R;\ T)$ be an s. s. map. Then we shall define

$$\omega^{\#}\colon \pi(\mathrm{Sd}^r K;\ \mathrm{Sd}^r L,\ \mathrm{Sd}^r M,\ \mathrm{Sd}^r N \mid P;\ Q, R, T) \to$$
$$\pi(|K|:|L|, |M|, |N| \mid |P|;\ |Q|, |R|, |T|).$$

In order to define this map we prepare the following proposition.

PROPOSITION 6.6. *Let $(K; L, M, N)$ be an s. s. tetrad. Then the continuous map $|dK|$: $(|\mathrm{Sd}K|;\ |\mathrm{Sd}L|,\ |\mathrm{Sd}M|,\ |\mathrm{Sd}N|) \to (|K|;\ |L|, |M|, |N|)$ is a homotopy equivalence.*

This proposition is the generalization of Lemma 7.5 in [7] and its proof is similar to Kan's one, therefore it will be omitted.

COROLLARY 6.7. *Let $(K;\ L, M, N)$ be an s. s. tetrad. Then the continuous map $|d^r K|$: $(|\mathrm{Sd}^r K|; |\mathrm{Sd}^r L|, |\mathrm{Sd}^r M|, |\mathrm{Sd}^r N|) \to (|K|; |L|, |M|, |N|)$ is a homotopy equivalence.*

This corollary is an immediate consequence of Proposition 6.6. For u we define $\omega^{\#}$ by $\omega^{\#}([u]) = [|u| \circ \eta]$, where η is an inverse map of $[d^r K]$ in Corollary 6. 7. It follows from the following proposition which asserts that $\omega^{\#}$ is well defined.

PROPOSITION 6.8. *If u, v: $(\mathrm{Sd}^r K;\ \mathrm{Sd}^r L, \mathrm{Sd}^r M, \mathrm{Sd}^r N) \to (P; Q, R, T)$ are homotopic s. s. maps, then $|u| \circ \eta \simeq |v| \circ \eta$.*

PROOF. Let F be a homotopy between u and v. Similarly to the result of [10], F induces an ordinary homotopy

$$(|\mathrm{Sd}^rK| \times |I|; |\mathrm{Sd}^rL| \times |I|, |\mathrm{Sd}^rM| \times |I|, |\mathrm{Sd}^rN| \times |I|) \to (|P|; |Q|, |R|, |T|).$$

Hence $|u| \simeq |v|$. Therefore $|u| \circ \eta \simeq |v| \circ \eta$.

Let f: $(|K|; |L|, |M|, |N|) \to (|P|; |Q|, |R|, |T|)$ be a continuous map Then we shall define a correspondence $\bar{\omega}^{\#}$: $\pi(|K|; |L|, |M|, |N| \,|\, |P|; |Q|, |R| \,|T|) \to \pi(\mathrm{Sd}^rK; \mathrm{Sd}^rL, \mathrm{Sd}^rM, \mathrm{Sd}^rN | P; Q, R, T)$. In order to define this correspondence we prepare the following theorem.

THEOREM 6.9. *Let $(K; L, M, N)$ and $(P; Q. R, T)$ be s. s. tetrads and K finite. Then for every continuous map $f: (|K|; |L|, |M|, |N|) \to (|P|; |Q|, |R|, |T|)$, there exist an integer $r > 0$ and an s.s. map $h: (\mathrm{Sd}^rK; \mathrm{Sd}^rL, \mathrm{Sd}^rM, \mathrm{Sd}^rN) \to (P; Q, R, T)$ such that the diagram*

$$\begin{tikzcd}
(|K|; |L|, |M|, |N|) \arrow[r, "f"] & (|P|; |Q|, |R|, |T|) \\
(|\mathrm{Sd}^rK|; |\mathrm{Sd}^rL|, |\mathrm{Sd}^rM|, |\mathrm{Sd}^rN|) \arrow[u, "|d^rK|"] \arrow[ur, "|h|"'] &
\end{tikzcd}$$

is commutative up to homotopy, i. e., $|h| \simeq f \circ |d^rK|$.

This theorem is the generalization of Theorem 8.5 in [7] and its proof is completely similar to Kan's one, therefore it will be omitted.

Let f and g be homotopic continuous maps $(|K|; |L|, |M|, |N|) \to (|P|; |Q|, |R|, |T|)$ and F be its homotopy. Then F is a continuous map

$$(|K| \times |I|; |L| \times |I|, |M| \times |I|, |N| \times |I|) \to (|P|; |Q|, |R|, |T|)$$

such that $F[|\sigma, \mathcal{E}_n|, \tau_0\mathcal{E}^0] = f| |\sigma, \mathcal{E}_n|$ and $F|[|\sigma, \mathcal{E}^n|, \tau_n\mathcal{E}^{n+1}] = g| |\sigma, \mathcal{E}_n|$. Since $|K| \times |I|$ is canonically homeomorphic to $|K \times I|$, these spaces may be identified. By Theorem 6.9, for a continuous map F, there exist an integer $r > 0$ and an s. s. map $H: (\mathrm{Sd}^r(K \times I); \mathrm{Sd}^r(L \times I), \mathrm{Sd}^r(M \times I), \mathrm{Sd}^r(N \times I)) \to (P; Q, R, T)$ such that $|H| \simeq F \circ |d^r(K \times I)|$. Let h_0 and h_1 be s. s. maps $H|\mathrm{Sd}^rK \times (0)$ and $H|\mathrm{Sd}^rK \times (1)$. Then for $[f]$ we define $\bar{\omega}^{\#}([f]) = [h_0]$. Since $F: f \simeq g$ implies $H: h_0 \simeq h_1$, $\bar{\omega}^{\#}$ is well defined. It is easy to see that $|h_0| \simeq f \circ |d^rK|$.

THEOREM 6.10. *Let $(K; L, M, N)$ and $(P; Q, R, T)$ be s. s. tetrads and K finite. Then correspondences $\omega^{\#}$ and $\bar{\omega}^{\#}$ are one-to-one and $\bar{\omega}^{\#}$ is an inverse of $\omega^{\#}$.*

PROOF. It was proved already that $\omega^{\#}$ and $\overline{\omega}^{\#}$ are single valued. Let f: $(|K|; |L|, |M|, |N|) \to (|P|; |Q|, |R|, |T|)$ be a continuous map. By those difinitions $\overline{\omega}^{\#}([f]) = [h_0]$ and $\omega^{\#}(\overline{\omega}^{\#}[f]) = [h_0 \circ \eta] = [f \circ |d^r K| \circ \eta]$. By corollary 6.7, $d^r K \circ \eta \simeq 1$. Therefore $\omega^{\#}(\overline{\omega}^{\#}[f]) = [f]$ and $\omega^{\#}$ is onto.

Let I^n be the set $\{y \in C \mid y^i = 0$ for $i > n$ and $-1 \leqq y^i \leqq 1$ $(\leqq i \leqq n)\}$ and $\dot{I}^n$ be its boundary and I^n_1, $\dot{I}^n_1$ be simplicial subdivisions of I^n, $\dot{I}^n$ respectively. It follows that I^n_1 and $\dot{I}^n_1$ may be considered as s. s. complexes [12].

PROPOSITION 6.11 *Let* $(K; L, M)$ *be an s. s. triad and* K *finite. Then* $\pi^n(K; L, M)$, *given in Definition* 5.1, *may also be defined by*

$$\pi^n(K; L, M) = \pi(\mathrm{Sd}^r K; \mathrm{Sd}^r L, \mathrm{Sd}^r M \mid I^n_1; \dot{I}^n_1, q_0)$$

for sufficiently large integer, $r > 0$ *where* q_0 *is a base point of* I^n_1.

PROOF. By its definition $\pi^n(K; L, M) = \pi(K; L, M \mid SE^n; SS^{n-1}, Sp)$ and there exists a one-to-one correspondence
$\gamma^{\#}: \pi(K; L, M \mid SE^n; SS^{n-1}, Sp) \to \pi(|K|; |L|, |M| \mid E^n; S^{n-1}, p)$. On the other hand there exists a one-to-one correspondence

$$\omega^{\#}: \pi(\mathrm{Sd}^r K; \mathrm{Sd}^r L, \mathrm{Sd}^r M \mid I^n_1, \dot{I}^n_1, q_0) \to \pi(|K|; |L|, |M| \mid |I^n_1|; |\dot{I}^n_1|, |q_0|).$$

The image of $\overline{\gamma}^{\#}$ is canonically identified with the image of $\omega^{\#}$. Therefore this proposition holds.

PROPOSITION 6.12. *Let* $(P; Q, R, p)$ *be an s. s. triad with base point* p. *Then* $\pi_n(P; Q, R)$, *given in Definition* 6.4, *may also be defined by*

$$\pi_n(P; Q, R) = \pi\Big(\mathrm{Sd}^r \Delta_n; \mathrm{Sd}^r(\partial_0 \Delta_n), \mathrm{Sd}^r(\partial_1 \Delta_n), \mathrm{Sd}^r\Big(\sum_{i \neq 0,1} \partial_i \Delta_n\Big) \Big| P; Q, R, p_0\Big).$$

PROOF. By its definition $\pi(P; Q, R) = \pi\Big(\Delta_n; \partial_0 \Delta_n, \partial_1 \Delta_n, \sum_{i \neq 0,1} \partial_i \Delta_n \Big| S|P|; S|Q|,$ $S|R|, Sp_0|\Big)$ and there exists a one-to-one correspondence $\overline{\gamma}^{\#}: \pi_n(P; Q, R) \to$ $\pi_n(|P|; |Q|, |R|)$. On the other hand there exists a one-to-one correspondence

$$\omega^{\#}: \pi\Big(\mathrm{Sd}^r \Delta_n; \mathrm{Sd}^r(\partial_0 \Delta_n), \mathrm{Sd}^r(\partial_1 \Delta_n), \mathrm{Sd}^r\Big(\sum_{i \neq 0,1} \partial_i \Delta_n\Big) \Big| P; Q, R, p_0\Big) \to$$

$$\pi\Big(|\Delta_n|; |\partial_0 \Delta_n|, |\partial_1 \Delta_n|, \Big|\sum_{i \neq 0,1} \partial_i \Delta_n\Big| \Big| P; Q, R, p_0\Big)$$

The image of $\overline{\gamma}^{\#}$ is canonically identified with the image of $\omega^{\#}$. Therefore this proposition holds.

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