# AN EXAMPLE OF GENERAL RECURSIVE WELL-ORDERING WHICH IS NOT PRIMITIVE RECURSIVE

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This note is to define a binary relation $x \prec y$ which is a general recursive well-ordering of the natural numbers but which is not primitive recursive. Peter [1] has constructed a general recursive function $\varphi(n, x, y)$ which enumerates all the primitive recursive functions of two variables so that given any primitive recursive function $F(x,y)$, there is a natural number $n_0$ such that $\varphi(n_0, x, y) = F(x,y)$. In terms of this enumerating function we define an ordering of the natural numbers in the following manner. We first arrange all the natural numbers in their natural order

$$0 < 1 < 2 < \ldots\ldots$$

Then for every natural number $i$ we change the places of $2i$ and $2i+1$ in the above series, if $\varphi(i, 2i, 2i+1) = 0$ and $\varphi(i, 2i+1, 2i) \neq 0$; otherwise, let $2i$ and $2i+1$ keep their natural order. For example, take $i = 0$. By the definition of $\varphi(i, x, y)$, $\varphi(0, x, y) = x$. Then $\varphi(0, 2.0, 2.0+1) = 0$ and $\varphi(0, 2.0+1, 2.0) = 1$, therefore $2.0$ and $2.0+1$ should change their places. According to this rule we arrange the natural numbers in a new order

$$1 \prec 0 \prec \ldots\ldots \prec 2i+1 \prec 2i \prec \ldots\ldots$$

where $\varphi(i, 2i, 2i+1) = 0$ and $\varphi(i, 2i+1, 2i) \neq 0$. Obviously $\prec$ is a well-ordering of type $\omega$.

To show that $\prec$ is not primitive recursive, let $W(x, y)$ be the representing function of $x \prec y$ so that $W(x,y) = 0$, if and only if $x \prec y$. Given any natural number $i$, we distinguish three cases.

Case 1. $\varphi(i, 2i, 2i+1) \neq 0$. In this case $2i \prec 2i+1$ and therefore $W(2i, 2i+1) = 0 \neq \varphi(i, 2i, 2i+1)$. Case 2. $\varphi(i, 2i, 2i+1) = 0$ and $\varphi(i, 2i+1, 2i) \neq 0$. In this case $2i+1 \prec 2i$, therefore $W(2i+1, 2i) = 0 \neq \varphi(i, 2i+1, 2i)$. Case 3. $\varphi(i, 2i, 2i+1) = 0$ and $\varphi(i, 2i+1, 2i) = 0$. In this case $2i \prec 2i+1$ but $\overline{2i+1 \prec 2i}$, therefore $W(2i+1, 2i) \neq 0 = \varphi(i, 2i+1, 2i)$. We have shown that in any case $W(x,y)$ is not identically equal to the primitive recursive function $\varphi(i, x, y)$ which is arbitrarily given. Hence $W(x, y)$ (and consequently $x \prec y$) is not primitive recursive.

The well-known primitive recursive function $[x, 2]$ means that whenever

$x$ is expressed as $x = 2i + j$ where $j < 2$ we have $i = [x, 2]$. Formally the function $W(x,y)$ can be defined as follows.

$$W(x,y) = \begin{cases} 0, & \text{if } [x,2] < [y,2] \\ 0, & \text{if } [x,2] = [y,2] \ \& \ x > y \ \& \\ & \quad \varphi([x,2],y,x) = 0 \ \& \ \varphi([x,2],x,y) \neq 0 \\ 0, & \text{if } [x,2] = [y,2] \ \& \ x < y \ \& \\ & \quad \varphi([x,2],x,y) = 0 \ \& \ \varphi([x,2],y,x) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

This definition shows that $W(x,y)$ is primitive recursive in $\varphi(i,x,y)$ and consequently that $W(x,y)$ is general recursive.

Thus $x \prec y$ is an example of general recursive well-ordering of the natural numbers which is not primitive recursive. Kleene [2] has shown that to any general recursive well-ordering of the natural numbers, there is a primitive recursive well-ordering, of the natural numbers, with the same order type. This implies that the types of primitive recursive well-orderings of the natural numbers have exhausted the types of general recursive well-orderings of the natural numbers. However, as the example of this note shows, primitive recursive well-orderings of the natural numbers do not exhaust general recursive well-orderings of the natural numbers.

## REFERENCES

[1] R. PETER, Konstruktion nichtrekursiver Funktionen, Math. Annalen 111(1935), 42-60.
[2] S. C. KLEENE, On the forms of the predicates in the theory of constructive ordinals, American Journal of Math., 77(1955), 405-428.

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