ON ALMOST COMPLEX SYMPLECTIC MANIFOLDS AND AFFINE CONNECTIONS WITH RESTRICTED HOMOGENEOUS HOLONOMY GROUP $Sp(n, C)$

HIDEKIYO WAKAKUWA

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The purpose of this paper is at first to characterize a $4n$-dimensional affinely connected manifold (with or without torsion) whose restricted homogeneous holonomy group is the real representation of the complex symplectic group $Sp(n, C)$ or one of its subgroups. And conversely, we discuss to introduce in a $4n$-dimensional manifold an affine connection (with or without torsion) whose restricted homogeneous holonomy group is the real representation of $Sp(n, C)$ or one of its subgroups.

The *almost complex symplectic manifold* is equivalent to an almost quaternion manifold (§ 3), but the *natural affine connection* (§ 4) in an almost complex symplectic manifold is different from the *natural affine connection* ((φ, ψ)-connection by Obata's terminology, [5]) in an almost quaternion manifold[2]. They coincide if and only if the affine connection is a metric connection (with or without torsion) with respect to a *related Riemannian metric* (§ 3, Definition).

1. Preliminary remarks. Let C_{2n} be a complex $2n$-dimensional linear space. Complex symplectic group $Sp(n,C)$ in C_{2n} is the subgroup of $GL(2n,C)$ leaving invariant a bilinear form $z^s \wedge w^{s+n} = z^s w^{s+n} - z^{s+n} w^s$ [3] where (z^α) and (w^α) $(\alpha = 1, \ldots\ldots, 2n)$ are vectors in C_{2n}. Therefore if M_{2n} is a complex $(2n, 2n)$-matrix giving a transformation of $Sp(n,C)$, then $M_{2n} J_{2n} {}^t M_{2n} = J_{2n}$, where ${}^t M_{2n}$ denotes the transpose of M_{2n} and J_{2n} is a matrix such as $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$[4]. Conversely if M_{2n} satisfies the above relation, then it is a matrix giving a transformation of $Sp(n, C)$.

Next, we consider the real representation of $Sp(n, C)$ in a real $4n$-dimensional real linear space R^{4n}.

Put $\mathfrak{M} = \begin{pmatrix} M_{2n} & 0 \\ 0 & \overline{M_{2n}} \end{pmatrix}$, where $\overline{M_{2n}}$ denotes the complex conjugate of M_{2n},

1) We shall show that this manifold must be necessarily an "almost complex symplectic manifold" (§ 3).
2) Cf. Ehresmann [1]: Libermann [3], [4]; Obata [5].
3) S runs from 1 to n. In this paper we adopt the summation convention.
4) In this paper, E_N denotes a unit matrix of degree N.

then $\mathfrak{M}$ satisfies

$$\mathfrak{M}^{-1}\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}\mathfrak{M} = \begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}, \quad (i^2 = -1) \tag{1.1}$$

and

$$\mathfrak{M}\begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix}{}^t\mathfrak{M} = \begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix}. \tag{1.2}$$

If we perform a complex transformation to the matrix $\mathfrak{M}$ by a complex regular matrix of the form $\tau = \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}$, then we obtain $\mathfrak{M}' = \tau^{-1}\mathfrak{M}\tau = \begin{pmatrix} M'_{2n} & 0 \\ 0 & \overline{M'_{2n}} \end{pmatrix}$ and the matrix $\begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix}$ is transformed into an anti-symmetric regular complex matrix of the form $\begin{pmatrix} \sigma_{2n} & 0 \\ 0 & \overline{\sigma}_{2n} \end{pmatrix}$ (${}^t\sigma_{2n} = -\sigma_{2n}$). And $\mathfrak{M}'$ satisfies

$${}^t\mathfrak{M}'\begin{pmatrix} \sigma_{2n} & 0 \\ 0 & \overline{\sigma}_{2n} \end{pmatrix}\mathfrak{M}' = \begin{pmatrix} \sigma_{2n} & 0 \\ 0 & \overline{\sigma}_{2n} \end{pmatrix}. \tag{1.3}$$

Conversely, we can normalize this matrix $\mathfrak{M}'$ to a complex matrix $\mathfrak{M} = \begin{pmatrix} M_{2n} & 0 \\ 0 & \overline{M}_{2n} \end{pmatrix}$ satisfying (1.2) by a suitable complex transformation.

Therefore, with respect to complex bases, a transformation $\mathfrak{M}'$ belonging to the real representation of $GL(2n, C)$ gives a transformation of the real representation of $Sp(n, C)$ if and only if it satisfies (1.3) where σ_{2n} is an anti-symmetric regular complex matrix.

Suppose a complex matrix

$$I_{4n} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_{2n} & E_{2n} \\ -iE_{2n} & iE_{2n} \end{pmatrix}, \left(I_{4n}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_{2n} & iE_{2n} \\ E_{2n} & -iE_{2n} \end{pmatrix}\right),$$

then we have

$$M \equiv I_{4n}\,\mathfrak{M}\,I_{4n}^{-1} = \begin{pmatrix} H_{2n} & -K_{2n} \\ K_{2n} & H_{2n} \end{pmatrix},$$

where $M_{2n} = H_{2n} + iK_{2n}$, H_{2n} and K_{2n} being real matrices of degree $2n$ and M gives a transformation of the real representation of $Sp(n, C)$ with respect to real bases. We also have real matrices $\overset{(1)}{F}$ and $\overset{(2)}{F}$:

$$\overset{(1)}{F} \equiv I_{4n}\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}I_{4n}^{-1} = \begin{pmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{pmatrix}, \tag{1.4}$$

$$\overset{(2)}{F} \equiv I_{4n}\begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix}{}^t I_{4n} = \begin{pmatrix} J_{2n} & 0 \\ 0 & -J_{2n} \end{pmatrix}. \tag{1.5}$$

These $\overset{(1)}{F}$ and $\overset{(2)}{F}$ satisfy

$$\overset{(1)}{F}{}^2 = -E_{4n},\quad {}^t\overset{(2)}{F} = -\overset{(2)}{F},\quad {}^t(\overset{(1)}{F}\,\overset{(2)}{F}) = -(\overset{(1)}{F}\,\overset{(2)}{F}) \tag{1.6}$$

and on account of (1.1), (1.2), we see that

$$M^{-1}\overset{(1)}{F}M = \overset{(1)}{F},\quad {}^tM\,\overset{(2)}{F}\,M = \overset{(2)}{F}. \tag{1.7}$$

Conversely, if a transformation M in a real $4n$-dimensional linear space R^{4n} transforms $\overset{(1)}{F}$, $\overset{(2)}{F}$ of (1.4), (1.5) by (1.7), then we can introduce complex bases in R_{4n} in which M takes the form $\mathfrak{M}$ since M leaves invariant the $\overset{(1)}{F}$, and we can easily see that the transformation M belongs to the real representation of $Sp(n, C)$.

2. Characterizations. Let A_{4n} be an affinely connected manifold (with or without torsion) of class C^2 whose restricted homogeneous holonomy group h^0 is the real representation of $Sp(n, C)$ or one of its subgroups. At first, assume that A_{4n} be simply connected.

If we attach a suitable frame $[R_0]$ at a point O of A_{4n}, then the restricted homogeneous holonomy group $h^0(O)$ at O transforms the two matrices $\overset{(1)}{F}$, $\overset{(2)}{F}$ with components (1.4), (1.5) according to (1.7). And we attach to each point P of A_{4n} a frame obtained from $[R_0]$ by a parallel translation along an arbitrary but fixed curve joining O to P. Then we have frames of reference on A_{4n} and we see that there exist tensor fields $\overset{(1)}{F}$, $\overset{(2)}{F}$ whose components are given by (1.4), (1.5) respectively with respect to the frames of reference under consideration. We remark that $\overset{(1)}{F}$ is of type (1,1) and $\overset{(2)}{F}$ is of type (0,2), that is,

$$\overset{(1)}{F} = (\overset{(1)}{F}{}_i{}^h),\quad \overset{(2)}{F} = (\overset{(2)}{F}{}_{ih})^{5)}.$$

These two tensor fields are of maximal rank $4n$ and of null covariant derivative by virtue of (1.7).

With respect to general frames of reference, especially with respect to natural frames of reference, we see that there exist two tensor fields $\overset{(1)}{F} = (\overset{(1)}{F}{}_i{}^h)$, $\overset{(2)}{F} = (\overset{(2)}{F}{}_{ih})$ satisfying

$$\overset{(1)}{F}{}_i{}^a\,\overset{(1)}{F}{}_a{}^h = -\delta_i^h,\quad \overset{(2)}{F}{}_{ih} = -\overset{(2)}{F}{}_{hi},\quad \overset{(1)}{F}{}_i{}^a\overset{(2)}{F}{}_{ah} = -\overset{(1)}{F}{}_h{}^a\overset{(2)}{F}{}_{ai}, \tag{2.1}$$

$\overset{(2)}{F}{}_{ih}$ being of maximal rank $4n$ and

5) Throughout this paper, if otherwise stated, the latin indices $h, i, j, k, \cdots a, b, c, \cdots$ run from 1 to $4n$.

$$\nabla_j \overset{(1)}{F}{}_i{}^h = 0, \nabla_j \overset{(2)}{F}_{ih} = 0, \tag{2.2}$$

where ∇_j denotes the covariant differentiation with respect to the affine connection $\Gamma_{ji}{}^h$ of A_{4n}

If A_{4n} is not simply connected, consider the universal covering manifold $\tilde{A}_{4n}$ of A_{4n} in which there are introduced an affine connection naturally from that of A_{4n}. Then the conclusion for $\tilde{A}_{4n}$ induces the same conclusion for A_{4n}.

Assume conversely that there exist two tensor fields $\overset{(1)}{F} = (\overset{(1)}{F}{}_i{}^h)$, $\overset{(2)}{F}{}^* = (\overset{(2)}{F}_{ih})$ satisfying (2.1) and (2.2). Let $h^0(O)$ be the restricted homogeneous holonomy group at O. Then $h^0(O)$ leaves invariant two matrices $\overset{(1)}{F}_0 = (\overset{(1)}{F}{}_i{}^h)_0$, $\overset{(2)}{F}{}^*_0 = (\overset{(2)}{F}_{ih})_0$ satisfying

$$\overset{(2)}{F}{}^2_0 = - E_{4n}, \quad {}^t\overset{(2)}{F}{}^*_0 = - \overset{(2)}{F}{}^*_0, \quad \overset{(1)}{F}_0\overset{(2)}{F}{}^*_0 = - {}^t(\overset{(1)}{F}_0\overset{(2)}{F}{}^*_0), \tag{2.3}$$

where $\overset{(1)}{F}_0$, $\overset{(2)}{F}{}^*_0$ denote the values of $\overset{(1)}{F}$, $\overset{(2)}{F}{}^*$ at O. We can choose a frame $[R_0]$ at O such that the components of $\overset{(1)}{F}_0=(\overset{(1)}{F}{}_i{}^h)_0$ are given by the form $\begin{pmatrix} 0 & E_{4n} \\ -E_{2n} & 0 \end{pmatrix}$ and further, by a complex transformation of the frame, $\overset{(1)}{F}_0$ changes into

$$\overset{(1)}{\mathfrak{F}}_0 = I_{4n}^{-1} \overset{(1)}{F}_0 I_{4n} = \begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix},$$

I_{4n} being given in §1. With respect to this complex frames, let $\overset{(2)}{\mathfrak{F}}{}^*_0 = (\overset{(2)}{\mathfrak{F}}_{ih})_0$ be the matrix corresponding to $\overset{(2)}{F}{}^*_0$ and put

$$\overset{(2)}{\mathfrak{F}}{}^*_0 = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix},$$

whene f_1, f_2, f_3 and f_4 are complex matrices of degree $2n$. Since $\overset{(2)}{\mathfrak{F}}{}^*_0 = (\overset{(2)}{\mathfrak{F}}_{ih})_0$ is anti-symmetric in i and h, we have

$${}^tf_1 = - f_1, \quad {}^tf_4 = - f_4, \quad {}^tf_2 = - f_3$$

and further since $\overset{(1)}{\mathfrak{F}}_0 \overset{(2)}{\mathfrak{F}}{}^*_0 = \begin{pmatrix} -if_1 & -if_2 \\ if_3 & if_4 \end{pmatrix}$ is also anti-symmetric by virtue of (2.3) we have ${}^tf_2 = f_3$, and hence $f_2 = f_3 = 0$. That is, $\overset{(2)}{\mathfrak{F}}{}^*_0$ is of the form $\overset{(2)}{\mathfrak{F}}{}^*_0 = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix}$. Since

$$\overset{(2)}{F}{}^*_0 = I_{4n}\overset{(2)}{F}{}^*_0 {}^tI_{4n} = \frac{1}{2}\begin{pmatrix} f_1 + f_4 & i(f_1 - f_4) \\ -i(f_1 - f_4) & f_1 + f_4 \end{pmatrix}$$

must be real, we see that $f_4=\bar{f}_1$ and hence $\overset{(2)}{\mathfrak{F}}{}_0^*$ takes the form

$$\overset{(2)}{\mathfrak{F}}{}_0^* = \begin{pmatrix} f_1 & 0 \\ 0 & \bar{f}_1 \end{pmatrix}.$$

Consequently we can normalize this $\overset{(2)}{\mathfrak{F}}{}_0^*$ into the form $\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ by a suitable complex transformation given by a matrix of the form $\begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}$ under which the form of $\overset{(1)}{\mathfrak{F}}{}_0$ is unchanged. And hence $h^0(O)$ is the real representation of $Sp(n,C)$ or one of its subgroups taking account of the preliminaries of § 1. Thus we have

THEOREM 2. 1. *The necessary and sufficient condition that the restricted homogeneous holonomy group of a* 4 *n-dimensional affinely connected manifold* A_{4n} *(with or without torsion) be the real representation of* $Sp(n, C)$ *or one of its subgroups is that there exist two tensor fields* $\overset{(1)}{F}{}_i{}^h$, $\overset{(2)}{F}{}_{ih}$ *satisfying*

$$\text{(I)} \qquad \overset{(1)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = -\delta_i^h, \quad \overset{(2)}{F}{}_{ih} = -\overset{(2)}{F}{}_{hi}, \quad \overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_{ah} = -\overset{(1)}{F}{}_h{}^a \overset{(2)}{F}{}_{ai},$$

$\overset{(2)}{F}{}_{ih}$ *being of maximal rank* $4n$ *and*

$$\text{(II)} \qquad \nabla_j \overset{(1)}{F}{}_i{}^h = 0, \ \nabla_j \overset{(2)}{F}{}_{ih} = 0.$$

$\overset{(1)}{F}{}_i{}^h$ gives an almost complex structure and $\overset{(2)}{F}{}_{ih}$ gives an almost (real) symplectic structure[6]. If we put $\overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_{ah} \equiv \overset{(3)}{F}{}_{ih}$, then $\overset{(3)}{F}{}_{ih}$ is anti-symmetric and of maximal rank $4n$. It is also of null covariant derivative by virtue of (II). Hence we have

COROLLARY 2. 1. *Let the assumption for* A_{4n} *be the same as in the Theorem. Then there exist in* A_{4n} *three tensor fields satisfying*

$$\text{(I}'\text{)} \qquad \begin{cases} \overset{(1)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = -\delta_i^h, \ \overset{(2)}{F}{}_{ih} = -\overset{(2)}{F}{}_{hi}, \ \overset{(3)}{F}{}_{ih} = -\overset{(3)}{F}{}_{hi}, \\ \overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_{ah} = -\overset{(1)}{F}{}_h{}^a \overset{(2)}{F}{}_{ai} = \overset{(3)}{F}{}_{ih}, \ \overset{(1)}{F}{}_i{}^a \overset{(3)}{F}{}_{ah} = -\overset{(1)}{F}{}_h{}^a \overset{(3)}{F}{}_{ai} = -\overset{(2)}{F}{}_{ih}, \end{cases}$$

$\overset{(2)}{F}{}_i{}^h$, $\overset{(3)}{F}{}_{ih}$ *being of maximal rank and*

$$\text{(II}'\text{)} \qquad \nabla_j \overset{(1)}{F}{}_i{}^h = 0, \ \nabla_j \overset{(2)}{F}{}_{ih} = 0. \ \nabla_j \overset{(3)}{F}{}_{ih} = 0.$$

6) With respect to (real) symplectic structure, see Ehresmann [2] and Libermann [3], especially Chap. IV.

3. Almost complex sympectic structure. Let X_{4n} be a real $4n$-dimensional manifold of class C^2 admitting two tensor fields $\overset{(1)}{F}{}_i{}^h$, $\overset{(2)}{F}{}_{ih}$ satisfying (I) where $\overset{(2)}{F}{}_{ih}$ is of maximal rank, or necessarily admitting three tensor fields satisfying (I′) where $\overset{(2)}{F}{}_{ih}$, $\overset{(3)}{F}{}_{ih}$ are of maximal rank. We call such a manifold X_{4n} an *almost complex symplectic manifold* (or briefly *almost CS-manifold*) and further we call the set of two tensor fields $(\overset{(1)}{F}{}_i{}^h, \overset{(2)}{F}{}_{ih})$ (hence necessarily three tensor fields $\overset{(1)}{F}{}_i{}^h$, $\overset{(2)}{F}{}_{ih}$, $\overset{(3)}{F}{}_{ih}$) an *almost complex symplectic structure* (or briefly *almost CS-structure*).

As is known, an almost quaternion structure in a real $4n$-dimensional manifold X_{4n} is defined by a set of two tensor fields of (1,1)-type $(\overset{(1)}{F}{}_i{}^h, \overset{(2)}{F}{}_i{}^h)$ satisfying $\overset{(1)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = -\delta_i^h$, $\overset{(2)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = -\delta_i^h$, $\overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = -\overset{(2)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h$.[7] And the existence of such two tensor fields of (1, 1)-type implies necessarily the existence of the third tensor field $\overset{(3)}{F}{}_i{}^h$ of (1,1)-type which is an almost complex structure and in quaternic relations with $\overset{(1)}{F}{}_i{}^h$ and $\overset{(2)}{F}{}_i{}^h$:

$$\overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = -\overset{(2)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = \overset{(3)}{F}{}_i{}^h,\quad \overset{(2)}{F}{}_i{}^a \overset{(3)}{F}{}_a{}^h = -\overset{(3)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = \overset{(1)}{F}{}_i{}^h,\quad \overset{(3)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = -\overset{(1)}{F}{}_i{}^a \overset{(3)}{F}{}_a{}^h = \overset{(2)}{F}{}_i{}^h.$$

THEOREM 3.1. *In a differentiable $4n$-dimensional manifold X_{4n}, a given almost quaternion structure induces an almost complex symplectic structure and conversely from a given almost complex symplectic structure we can find an almost quaternion structure. That is, the two structures are equivalent.*

PROOF. Suppose at first that a differentiable X_{4n} admits an almost quaternion structure $(\overset{(1)}{F} = (\overset{(1)}{F}{}_i{}^h), \overset{(2)}{F} = (\overset{(2)}{F}{}_i{}^h))$ which satisfy

$$\overset{(1)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h = -\delta_i^h;\quad \overset{(2)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = -\delta_i^h,\quad \overset{(1)}{F}{}_i{}^a \overset{(2)}{F}{}_a{}^h = -\overset{(2)}{F}{}_i{}^a \overset{(1)}{F}{}_a{}^h.$$

Or, in matrix forms

$$\overset{(1)}{F}{}^2 = -E,\quad \overset{(2)}{F}{}^2 = -E,\quad \overset{(1)}{F}\overset{(2)}{F} = -\overset{(2)}{F}\overset{(1)}{F},$$

where we denote for brevity the unit matrix of degree $4n$ by E instead of E_{4n}.

If we put $\overset{(3)}{F} \equiv \overset{(1)}{F}\overset{(2)}{F} = -\overset{(2)}{F}\overset{(1)}{F}$, then we get the following relations:

$$\overset{(3)}{F}{}^2 = -E,\quad \overset{(3)}{F}\overset{(1)}{F} = -\overset{(1)}{F}\overset{(3)}{F} = \overset{(2)}{F},\quad \overset{(2)}{F}\overset{(3)}{F} = -\overset{(3)}{F}\overset{(2)}{F} = \overset{(1)}{F},$$

7) Ehresmann [1]; Libermann [3], [4]; Obata [5].

by virtue of the given conditions for $\overset{(1)}{F}$, $\overset{(2)}{F}$. Since in our X^{4n} there exists always a positive definite Riemannian metric $G=(g_{ji})$, we put

$$G^{*}=(g^{*}_{ij})=\frac{1}{4}(G+\overset{(1)}{F}G^{t}\overset{(1)}{F}+\overset{(2)}{F}G^{t}\overset{(2)}{F}+\overset{(3)}{F}G^{t}\overset{(3)}{F}).^{8)}$$

Then G^{*} is also positive definite and it is simultaneously hermitian with respect to $\overset{(1)}{F}, \overset{(2)}{F}, \overset{(3)}{F}$, i. e.

$$\overset{(1)}{F}G^{*t}\overset{(1)}{F}=G^{*},\qquad \overset{(2)}{F}G^{*t}\overset{(3)}{F}=G^{*},\qquad \overset{(3)}{F}G^{*t}\overset{(2)}{F}=G^{*},$$

or in tensor forms

$$g^{*}_{ab}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}=g^{*}_{ji},\qquad g^{*}_{ab}\overset{(2)}{F}_{j}{}^{a}\overset{(2)}{F}_{i}{}^{b}=g^{*}_{ji},\qquad g^{*}_{ab}\overset{(3)}{F}_{i}{}^{a}\overset{(2)}{F}_{i}{}^{b}=g^{*}_{ji}.$$

Hence, if we put

$$\overset{(2)}{F}G^{*}\equiv\overset{(2)}{F}{}^{*}\qquad(\overset{(2)}{F}_{i}{}^{a}g^{*}_{ah}=\overset{(2)}{F}_{ih})$$

then we can see that $\overset{(2)}{F}{}^{*}=(\overset{(2)}{F}_{ih})$ are anti-symmetric and of maximal rank. And we have

$$\overset{(1)(2)}{FF^{*}}=\overset{(1)(2)}{FFG^{*}}=-\overset{(2)(1)}{FFG^{*}}=-\overset{(2)}{F}G^{*t}\overset{(1)}{F}{}^{-1}=-{}^{t}(\overset{(1)(2)}{FF^{*}}),$$

or in tensor forms

$$\overset{(1)}{F}_{i}{}^{a}\overset{(2)}{F}_{ah}=-\overset{(1)}{F}_{h}{}^{a}\overset{(2)}{F}_{ai}.$$

That is, the tensor fields $\overset{(1)}{F}=(\overset{(1)}{F}_{i}{}^{h})$, $\overset{(2)}{F}{}^{*}=(\overset{(2)}{F}_{ih})$ gives an almost complex symplectic structure.

We will prove the converse. Let $\overset{(1)}{F}=(\overset{(1)}{F}_{i}{}^{h})$, $\overset{(2)}{F}{}^{*}=(\overset{(2)}{F}_{ih})$ be an almost complex symplectic structure:

$$\overset{(1)}{F}{}^{2}=-E,\quad {}^{t}\overset{(2)}{F}{}^{*}=-\overset{(2)}{F}{}^{*},\quad \overset{(1)(2)}{FF^{*}}=-{}^{t}(\overset{(1)(2)}{FF^{*}}),\tag{3.1}$$

$\overset{(2)}{F}{}^{*}=(\overset{(2)}{F}_{ih})$ being of maximal rank. We remark that the third condition of (3.1) can also be written as $\overset{(1)(2)}{FF^{*}}=\overset{(2)}{F}{}^{*t}\overset{(1)}{F}$.

Consider an arbitrary Riemannian metric $\overset{\circ}{G}=(\overset{\circ}{g}_{ji})$ in X_{4n}, then it is well known

$$G=(g_{ji})=\frac{1}{2}(\overset{\circ}{G}+\overset{(1)}{F}\overset{\circ}{G}\,{}^{t}\overset{(1)}{F})$$

8) Cf. Obata [5], Section 14 ; Wakakuwa [7], Lemma 1.3.

is a positive definite Riemannian metric hermitian with respect to $\overset{(1)}{F}$:

$$\overset{(1)}{F} G\, {}^t\overset{(1)}{F} = G. \tag{3.2}$$

Further, if we put

$$\tilde{G} = (\tilde{g}_{ji}) = -\overset{(2)}{F^*} G^{-1} \overset{(2)}{F^*} \qquad (\tilde{g}_{ji} = g^{ab} \overset{(2)}{F}_{aj} \overset{(2)}{F}_{bi}),$$

then $\tilde{G}$ is also positive definite, and consider the characteristic equation

$$|\tilde{G} - \rho G| = 0.$$[9]

Since $\tilde{G}$ are and G both positive definite, the ν different characteristic roots $\rho_u\,(u = 1,\ldots\ldots,\nu)$ are all positive and the elementary divisors are all simple because the matrix $(\tilde{G} - \rho G)$ is of $(0, 2)$-type. Let $R_u(u = 1,\ldots\ldots,\nu)$ be the characteristic root spaces corresponding to the different characteristic roots ρ_u.

Put $\overset{(2)}{F^*} G^{-1} = \overset{(2)}{F'} = (\overset{(2)}{F}{}_i'^{\,h})$ and let $x = (x^h)$ be an arbitrary vector in R_u, i. e.,

$$x\tilde{G} = \rho_u x G \quad \text{or} \quad -x\overset{(2)}{F^*} G^{-1} \overset{(2)}{F^*} = \rho_u x G,$$

then the vectors $x\overset{(1)}{F} = (\overset{(1)}{F}{}_a{}^h x^a)$, $x\overset{(2)}{F'} = (\overset{(2)}{F}{}_a'^{\,h} x^a)$ are also in R_u. For, using (3. 1) and (3. 2), we can see that

$$\begin{aligned}(x\overset{(1)}{F})\tilde{G} &= -x\overset{(1)}{F}\overset{(2)}{F^*} G^{-1} \overset{(2)}{F^*} = -x\overset{(2)}{F^*}{}^t\overset{(1)}{F}\, G^{-1} \overset{(2)}{F^*} \\ &= -x\overset{(2)}{F^*} G^{-1}\, {}^t\overset{(1)}{F}{}^{-1} \overset{(2)}{F^*} = -x\overset{(2)}{F^*} G^{-1} \overset{(2)}{F^*}{}^t\overset{(1)}{F}{}^{-1} \\ &= x\tilde{G}\, {}^t\overset{(1)}{F}{}^{-1} = \rho_u x\, G\, {}^t\overset{(1)}{F}{}^{-1} \\ &= \rho_u (x\overset{(1)}{F})\, G,\end{aligned}$$

and this shows that the vector $x\overset{(1)}{F} = (\overset{(1)}{F}{}_a{}^h x^a)$ is also in R_u. Similarly we can see that $x\overset{(2)}{F}$ lies in R_u, too.

Hence if we choose the frames of reference $[e_i]$ such that $[e_{iu}]$ span the root space R_u, then $G = (g_{ji})$, $\overset{(1)}{F} = (\overset{(1)}{F}{}_i{}^h)$, $\overset{(2)}{F'} = (\overset{(2)}{F}{}_i'^{\,h})$ decomposes into ν blocks simultaneously, i. e , $(g_{ji}) = (g_{j_1 i_1}) \dotplus (g_{j_2 i_2}) \dotplus \cdots\cdots \dotplus (g_{j_\nu i_\nu})$, etc.,

Thus with respect to the frames of reference now introduced

$$G^* = (g^*_{ji}) = (\sqrt{\rho_1}\; g_{j_1 i_1}) \dotplus (\sqrt{\rho_2}\; g_{j_2 i_2}) \dotplus \cdots\cdots \dotplus (\sqrt{\rho_\nu}\; g_{j_\nu i_\nu})$$

defines a positive definite Riemannian metric such that

9) Cf. Iwamoto [8] ; Lichnerowicz [9].

$$\overset{(1)}{F}G^{*}\,{}^{t}\overset{(1)}{F} = G^{*}, \qquad -\overset{(2)}{F^{*}}G^{*-1}\overset{(2)}{F^{*}} = G^{*}.$$

Therefore, if we put

$$\overset{(2)}{F^{*}}G^{*-1} = \overset{(2)}{F},$$

then we can verify that

$$\overset{(1)}{F}{}^{2} = -E, \quad \overset{(2)}{F}{}^{2} = -E, \quad \overset{(1)}{F}\overset{(2)}{F} = -\overset{(2)}{F}\overset{(1)}{F}$$

by virtue of (3.1), or in tensor forms

$$\overset{(1)}{F}{}_{i}{}^{a}\overset{(1)}{F}{}_{a}{}^{h} = -\delta_{i}^{h}, \quad \overset{(2)}{F}{}_{i}{}^{a}\overset{(2)}{F}{}_{a}{}^{h} = -\delta_{i}^{h}, \quad \overset{(1)}{F}{}_{i}{}^{a}\overset{(2)}{F}{}_{a}{}^{h} = -\overset{(2)}{F}{}_{i}{}^{a}\overset{(1)}{F}{}_{a}{}^{h}.$$

Consequently, we can find an almost quaternion structure $\overset{(1)}{F} = (\overset{(1)}{F}{}_{i}{}^{h})$, $\overset{(2)}{F} = (\overset{(2)}{F}{}_{i}{}^{h})$ derived from the almost CS-structure. And hereby we have completed the proof of Theorem 3.1.

On account of the proof of the above Theorem, we see that there exists a positive definite Riemannian metric g^{*}_{ji} combining the almost quaternion structure $(\overset{(1)}{F}{}_{i}{}^{h}, \overset{(2)}{F}{}_{i}{}^{h})$ and the almost CS-structure $(\overset{(1)}{F}{}_{i}{}^{h}, \overset{(2)}{F}{}_{ih})$, such that

$$g^{*}_{ab}\,\overset{(1)}{F}{}_{j}{}^{a}\overset{(1)}{F}{}_{i}{}^{b} = g^{*}_{ij}, \quad \overset{(1)}{F}{}_{i}{}^{a}g^{*}_{ah} = \overset{(2)}{F}{}_{ih},$$

hence necessarily

$$\overset{(3)}{F}{}_{i}{}^{a}g^{*}_{ah} = \overset{(3)}{F}{}_{ih}.$$

DEFINITION. We call such an almost quaternion structure and an almost CS-structure to be *naturally related* and call g^{*}_{ji} he *related Riemannian metric.*

4. Natural affine connections in almost complex symplectic manifold.

Let X_{4n} be an almost CS-manifold with almost CS-structure $\overset{(1)}{F}{}_{i}{}^{h}$, $\overset{(2)}{F}{}_{ih}$:

$$(4.1) \qquad \overset{(1)}{F}{}_{i}{}^{a}\overset{(1)}{F}{}_{a}{}^{h} = -\delta_{i}^{h}, \quad \overset{(2)}{F}{}_{ih} = -\overset{(2)}{F}{}_{hi}, \quad \overset{(1)}{F}{}_{i}{}^{a}\overset{(2)}{F}{}_{ah} = -\overset{(1)}{F}{}_{h}{}^{a}\overset{(2)}{F}{}_{ai}.$$

It is noted that in X_{4n} there exists the third tensor field of $(0,2)$-type $\overset{(3)}{F}{}_{ih}$ satisfying

$$(4.2) \qquad \overset{(3)}{F}{}_{ih} = -\overset{(3)}{F}{}_{hi}, \; \overset{(1)}{F}{}_{i}{}^{a}\overset{(2)}{F}{}_{ah} = -\overset{(1)}{F}{}_{h}{}^{a}\overset{(2)}{F}{}_{ai} = \overset{(3)}{F}{}_{ih}, \; \overset{(1)}{F}{}_{i}{}^{a}\overset{(3)}{F}{}_{ah} = -\overset{(1)}{F}{}_{h}{}^{a}\overset{(3)}{F}{}_{ai} = -\overset{(3)}{F}{}_{ih}$$

$\overset{(3)}{F}{}_{ih}$ being of maxiaml rank.

There exists a related Riemannian metric g^*_{ji} (§ 3 Definition) such that

$$g^*_{ib}\overset{(1)}{F}_j{}^a\overset{(1)}{F}_i{}^b = g^*_{ji},\ \overset{(2)}{F}_{ia}g^{*ah} = \overset{(2)}{F}_i{}^h,\ \overset{(3)}{F}_{ia}g^{*ah} = \overset{(3)}{F}_i{}^h,$$

where $(\overset{(1)}{F}_i{}^h, \overset{(2)}{F}_i{}^h, \overset{(3)}{F}_i{}^h)$ gives an almost quaternion structure. If we put

$$\overset{(2)}{F}{}^{ih} = g^{*ia}\overset{(2)}{F}_a{}^h,\ \overset{(3)}{F}{}^{ih} = g^{*ia}\overset{(3)}{F}_a{}^h,$$

then $\overset{(2)}{F}{}^{ih}$, $\overset{(3)}{F}{}^{ih}$ are also anti-symmetric in i, h and we have

$$\text{(4.3)}\qquad \begin{cases} \overset{(2)}{F}_{ia}\overset{(2)}{F}{}^{ah} = -\delta^h_i,\ \overset{(3)}{F}_{ia}\overset{(3)}{F}{}^{ah} = -\delta^h_i, \\ \overset{(2)}{F}_{ia}\overset{(3)}{F}{}^{ah} = -\overset{(3)}{F}_{ia}\overset{(2)}{F}{}^{ah} = \overset{(1)}{F}_i{}^h. \end{cases}$$

We remark that although the related Riemannian metric g^*_{ji} is not unique, but the $\overset{(2)}{F}{}^{ih}$ and $\overset{(3)}{F}{}^{ih}$ are both unique for $\overset{(2)}{F}_{ih}, \overset{(3)}{F}_{ih}$ since $\overset{(2)}{F}_{ia}\overset{(2)}{F}{}^{ah} = -\delta^h_i$, and $\overset{(3)}{F}_{ia}\overset{(3)}{F}{}^{ah} = -\delta^h_i$. Hence with no use of g^*_{ij} we can define $\overset{(2)}{F}{}^{ih}$ such that $\overset{(2)}{F}_{ia}(-\overset{(2)}{F}{}^{ah}) = \delta^h_i$, since such a $\overset{(2)}{F}{}^{ih}$ is a tensor field. It is similar for $\overset{(3)}{F}{}^{hi}$.

If the covariant differentiation ∇_j with respect to an affine connection satisfies

$$\nabla_j\overset{(1)}{F}_i{}^h = 0,\ \nabla_j\overset{(2)}{F}_{ih} = 0 \text{ and hence necessarily } \nabla_j\overset{(3)}{F}_{ih} = 0,$$

then the restricted homogeneous holonomy group of the affine connection is the real representation of $Sp(n, C)$ or one of its subgroups. We call such an affine connection a *natral affine connection* or briefly *natural connection* of the almost complex symplectic manifold X_{4n}.

We can easily verify that *a natural affine connection in an almost CS-manifold coincides with a natural affine connection in an almost quaternion manifold* ((φ, ψ)-connection by Obata's terminology, Obata [5]) *if and only if the connection is a metric connection (with or without torsion) with respect to the related Riemannian metric.*

In the similar way as those of Schouten and Yano [10] and Obata [5], we introduce the following operations making use of $\overset{(1)}{F}_i{}^h$.

Let $P_{ji}{}^h$ be an arbitrary tensor field in X_{4n} and we define[10]

$$\overset{(1)}{\mathfrak{F}}P_{ji}{}^h = \frac{1}{2}(P_{ji}{}^h - \overset{(1)}{F}_i{}^b P_{jb}{}^a \overset{(1)}{F}_a{}^h)$$

10) These are the same as O^{ba}_{ia}, $*O^{dc}_{ac}$ of Schouten and Yano [10] or Φ_1, Φ_2, Φ_3, Φ_4 of Obata [5].

$$\overset{(1)}{\mathfrak{F}}\, P_{ji}{}^{h} = \frac{1}{2}(P_{ji}{}^{h} - \overset{(1)}{F}_{i}{}^{b} P_{jb}{}^{a} \overset{(1)}{F}_{a}{}^{h})$$

$$\overset{(1)}{\mathfrak{F}}{}^{*} P_{ji}{}^{h} = \frac{1}{2}(P_{ji}{}^{h} + \overset{(1)}{F}_{i}{}^{b} P_{jb}{}^{a} \overset{(1)}{F}_{a}{}^{h})$$

$$\overset{(1)}{\mathfrak{F}}{}' P_{ji}{}^{h} = \frac{1}{2}(P_{ji}{}^{h} - \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} P_{ab}{}^{h})$$

$$\overset{(1)}{\mathfrak{F}}{}'^{*} P_{ji}{}^{h} = \frac{1}{2}(P_{ji}{}^{h} + \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} P_{ab}{}^{h}).$$

And we also introduce anew by using $\overset{(2)}{F}_{i}{}^{h}$, $\overset{(2)}{F}_{ih}$ and $\overset{(3)}{F}_{ih}$, $\overset{(3)}{F}_{ih}$ the following operations

$$\begin{cases} \overset{(2)}{\mathfrak{F}} P_{ji}{}^{h} = \dfrac{1}{2}(P_{ji}{}^{h} - P_{jb}{}^{a} \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh}) \\[2ex] \overset{(2)}{\mathfrak{F}}{}' P_{ji}{}^{h} = \dfrac{1}{2}(P_{ji}{}^{h} + P_{jb}{}^{a} \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh}), \end{cases}$$

$$\begin{cases} \overset{(3)}{\mathfrak{F}} P_{ji}{}^{h} = \dfrac{1}{2}(P_{ji}{}^{h} - P_{jb}{}^{a} \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh}) \\[2ex] \overset{(3)}{\mathfrak{F}}{}' P_{ji}{}^{h} = \dfrac{1}{2}(P_{ji}{}^{h} + P_{jb}{}^{a} \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh}). \end{cases}$$

And we define operations $\overset{(1)}{\mathfrak{F}}$, $\overset{(2)}{\mathfrak{F}}$, $\overset{(3)}{\mathfrak{F}}$ for an arbitrary affine connection $\Gamma_{ji}{}^{h}$ as follows.

$$\overset{(1)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = \Gamma_{ji}{}^{h} - \frac{1}{2}(\nabla_{j} \overset{(1)}{F}_{i}{}^{a}) \overset{(1)}{F}_{a}{}^{h}$$

$$\overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = \Gamma_{ji}{}^{h} - \frac{1}{2}(\nabla_{j} \overset{(2)}{F}_{ia}) \overset{(2)}{F}{}^{ah}$$

$$\overset{(3)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = \Gamma_{ji}{}^{h} - \frac{1}{2}(\nabla_{j} \overset{(3)}{F}_{ia}) \overset{(3)}{F}{}^{ah}$$

Then, we see that *the operations* $\overset{(u)}{\mathfrak{F}}$ $(u = 1, 2, 3)$ *are linear for an affine connection* $\Gamma_{ji}{}^{h}$ *and a tensor* $P_{ji}{}^{h}$:

$$\overset{(u)}{\mathfrak{F}}(\Gamma_{ji}{}^{h} + P_{ji}{}^{h}) = \overset{(u)}{\mathfrak{F}}\Gamma_{ji}{}^{h} + \overset{(u)}{\mathfrak{F}} P_{ji}{}^{h} \qquad (u = 1, 2, 3).$$

These are shown by a direct calculation. For example, consider $\overset{(2)}{\mathfrak{F}}$. Denoting by $\overset{p}{\nabla}_{j}$ the covaraint differentiation with respect to $\Gamma_{ji}{}^{h} + P_{ji}{}^{h}$, we see that

$$\overset{(2)}{\mathfrak{F}}(\Gamma_{ji}{}^{h} + P_{ji}{}^{h}) = (\Gamma_{ji}{}^{h} + P_{ji}{}^{h}) - \frac{1}{2}(\overset{p}{\nabla}_{j} \overset{(2)}{F}_{ia}) \overset{(2)}{F}{}^{ah}$$

$$= \Gamma_{ji}{}^{h} + P_{ji}{}^{h} - \frac{1}{2}(\nabla_{i}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{an} + \frac{1}{2}(P_{ji}{}^{c}\overset{(2)}{F}_{ca} + P_{ja}{}^{c}\overset{(2)}{F}_{tc})\overset{(2)}{F}{}^{ah}$$

$$= \Gamma_{ji}{}^{h} - \frac{1}{2}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah} + \frac{1}{2}(P_{ji}{}^{h} - P_{jb}{}^{a}\overset{(2)}{F}_{at}\overset{(2)}{F}{}^{bh})$$

$$= \overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{h} + \overset{(2)}{\mathfrak{F}}P_{ji}{}^{h}.$$

The others are proved similarly.

LEMMA 4.1. *For an affine connection or for a tensor,*

$$\overset{(u)}{\mathfrak{F}}{}^{2} = \overset{(u)}{\mathfrak{F}} \qquad (u = 1, 2, 3).$$

PROOF. For $\overset{(1)}{\mathfrak{F}}$, the property is alraedy known (for exp. [5]). We will prove for $\overset{(2)}{\mathfrak{F}}$. It is analoguous for $\overset{(3)}{\mathfrak{F}}$.

Put $\overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = \overset{(2)}{\Gamma}_{ji}{}^{h}$ and denoting by $\overset{(2)}{\nabla}_{j}$ the covaraint differentiation with respect to $\overset{(2)}{\Gamma}_{ji}{}^{h}$, then we see that

$$\overset{(2)}{\mathfrak{F}}(\overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{r}) = \overset{(2)}{\mathfrak{F}}\overset{(2)}{\Gamma}_{ji}{}^{h} = \overset{(2)}{\Gamma}_{ji}{}^{h} - \frac{1}{2}(\overset{(2)}{\nabla}_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah}$$

$$= (\Gamma_{j}{}^{ih} - \frac{1}{2}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah}) - \frac{1}{2}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah} - \frac{1}{4}$$

$$[((\nabla_{j}\overset{(2)}{F}_{tc})\overset{(2)}{F}{}^{cb})\overset{(2)}{F}_{ba} + ((\nabla_{j}\overset{(2)}{F}_{ac})\overset{(2)}{F}{}^{cb})\overset{(2)}{F}_{tb}]\ \overset{(2)}{F}{}^{ah}$$

$$= \Gamma_{ji}{}^{h} - (\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah} + \frac{1}{4}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah} + \frac{1}{4}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah}$$

$$= \Gamma_{ji}{}^{h} - \frac{1}{2}(\nabla_{j}\overset{(2)}{F}_{ta})\overset{(2)}{F}{}^{ah} = \overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{h}.$$

We can also verify for a tensor $P_{ji}{}^{h}$. Q.E.D.

The following Lemma is immediate from the definition of $\overset{(u)}{\mathfrak{F}}$ $(u = 1, 2, 3)$.

LEMMA 4.2. *Let* $\Gamma_{ji}{}^{h}$ *be an affine connection in* X_{4n} *and let* ∇_{j} *be the covariant differentiation with respect to* $\Gamma_{ji}{}^{h}$. *Then, in order that* $\nabla_{j}\overset{(1)}{F}_{i}{}^{h} = 0$, $\nabla_{j}\overset{(2)}{F}_{th} = 0$ *or* $\nabla_{j}\overset{(3)}{F}_{th} = 0$ *is that* $\overset{(1)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = 0$, $\overset{(2)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = 0$ *or* $\overset{(3)}{\mathfrak{F}}\Gamma_{ji}{}^{h} = 0$ *respectively.*

This Lemma is already known for $\overset{(1)}{F}_{i}{}^{h}$, $\overset{(1)}{\mathfrak{F}}$ (for exp. [5]).

LEMMA 4.3. *The operations* $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}, \overset{(3)}{\mathfrak{F}}$ *for an arbitrary affine connection* *satisfy*

$$\overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}\Gamma_{ji}^{\ h} = \Gamma_{ji}^{\ h} - \frac{1}{4}(\nabla_j \overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h - \frac{1}{4}(\nabla_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah} - \frac{1}{4}(\nabla_j \overset{(3)}{F}_{jb})\overset{(3)}{F}{}^{ah}$$

$$(u \neq v;\ \ u, v = 1, 2, 3).$$

And the operations $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}, \overset{(3)}{\mathfrak{F}}$ *for an arbitrary tensor field* $P_{ji}^{\ h}$ *satisfy*

$$\overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}P_{ji}^{\ h} = \frac{1}{4}(P_{ji}^{\ h} - \overset{(1)}{F}_i{}^b P_{jb}{}^a \overset{(1)}{F}_a{}^h - P_{jb}{}^a \overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - P_{jb}{}^a \overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh})$$

$$(u \neq v;\ \ u, v = 1, 2, 3).$$

PROOF. If we put

$$\overset{(2)}{\Gamma}_{ji}^{\ h} = \overset{(2)}{\mathfrak{F}}\Gamma_{ji}^{\ h} = \Gamma_{ji}^{\ h} - \frac{1}{2}(\nabla_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah},$$

and denote by $\overset{(2)}{\nabla}_j$ the covariant differentiation with respect to $\overset{(2)}{\Gamma}_{ji}^{\ h}$, then we see that

$$\overset{(1)}{\mathfrak{F}}(\overset{(2)}{\mathfrak{F}}\Gamma_{ji}^{\ h}) = \overset{(1)}{\mathfrak{F}}\overset{(2)}{\Gamma}_{ji}^{\ h} = \overset{(2)}{\Gamma}_{ji}^{\ h} - \frac{1}{2}(\overset{(2)}{\nabla}_j \overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h$$

$$= \overset{(2)}{\Gamma}_{ji}^{\ h} - \frac{1}{2}\Big[\nabla_j \overset{(1)}{F}_i{}^a - \frac{1}{2}((\nabla_j \overset{(2)}{F}_{bc})\overset{(2)}{F}{}^{ca})\overset{(1)}{F}_i{}^b + \frac{1}{2}((_j\nabla \overset{(2)}{F}_{ic})\overset{(2)}{F}{}^{cb})\overset{(1)}{F}_b{}^a\Big]\overset{(1)}{F}_a{}^h$$

$$= (\Gamma_{ji}^{\ h} - \frac{1}{2}(\nabla_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah}) - \frac{1}{2}(\nabla_j \overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h - \frac{1}{4}\times$$
$$(\nabla_j \overset{(2)}{F}_{bc})\overset{(3)}{F}{}^{ch}\overset{(1)}{F}_i{}^b + \frac{1}{4}\nabla(_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah}$$

$$= \Gamma_{ji}^{\ h} - \frac{1}{2}(\nabla_j \overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h - \frac{1}{4}(\nabla_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah} - \frac{1}{4}\times$$
$$(\nabla_j \overset{(1)}{F}_a{}^h)\overset{(1)}{F}_i{}^a + \frac{1}{4}(\nabla_j \overset{(3)}{F}{}^{ah})\overset{(3)}{F}_{ia}$$

$$= \Gamma_{ji}^{\ h} - \frac{1}{4}(\nabla_j \overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h - \frac{1}{4}(\nabla_j \overset{(2)}{F}_{ia})\overset{(2)}{F}{}^{ah} - \frac{1}{4}(\nabla_j \overset{(3)}{F}_{ia})\overset{(3)}{F}{}^{ah}.$$

We can verify that the other $\overset{(2)}{\mathfrak{F}}\overset{(1)}{\mathfrak{F}}\Gamma_{ji}^{\ h}$, $\overset{(2)}{\mathfrak{F}}\overset{(3)}{\mathfrak{F}}\Gamma_{ji}^{\ h}$, etc. are all equal to this quantity. The latter part of the Lemma is proved similarly.

From Lemma 4.1, 4.2, 4.3, we have the following Theorem

THEOREM 4.1. *Let* $\Gamma_{ji}^{\ h}$ *be an arbitrary affine connection*[11] *in an almost*

11) An affine connection always exists in our X_{4n}.

complex symplectic manifold X_{4n} *with almost complex symplectic structure* $(\overset{(1)}{F}_i{}^h, \overset{(2)}{F}_i{}^h, \overset{(3)}{F}_{ih})$ *and let* ∇_j *denote the covariant differentiation with respect to* Γ_{ji}^h. *Then the affine connection*

$$\Gamma'^{h}_{ji} = \Gamma_{ji}^h - \frac{1}{4}(\nabla_j\overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h - \frac{1}{4}(\nabla_j\overset{(2)}{F}_{ia})\overset{(2)}{F}^{ah} - \frac{1}{4}(\nabla_j\overset{(3)}{F}_{ia})\overset{(3)}{F}^{ah}$$

is a natural affine connection of X_{4n}, *that is, its restricted homogeneous holonomy group is the real representation of* $Sp(n,C)$ *or one of its subgroups.*

THEOREM 4.2. *The necessary and sufficient condition that an affine connection* Γ_{ji}^h *of* X_{4n} *be a natural affine connection is that* $\overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}\Gamma_{ji}^h = \Gamma_{ji}^h$ $(u \neq v;\ u, v = 1, 2, 3)$, *that is,*

$$(\nabla_j\overset{(1)}{F}_i{}^a)\overset{(1)}{F}_a{}^h + (\nabla_j\overset{(2)}{F}_{ia})\overset{(2)}{F}^{ah} + (\nabla_j\overset{(3)}{F}_{ia})\overset{(3)}{F}^{ah} = 0,$$

where ∇_j *denotes the covariant differentiation with respect to* Γ_{ji}^h.

The following Theorem is immediate from

$$\overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}(\Gamma_{ji}^h + P_{ji}^h) = \overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}\Gamma_{ji}^h + \overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}P_{ji}^h \quad (u \neq v;\ u, v = 1, 2, 3)$$

and from Lemma 4.3, Theorem 4.2.

THEOREM 4.3. *Let* Γ_{ji}^h *be a natural affine connection of an almost complex symplectic manifold* X_{4n} *and let* P_{ji}^h *be a tensor field over* X_{4n}. *Then the necessary and sufficient condition that the affine connection* $\Gamma_{ji}^h + P_{ji}^h$ *be again a natural affine connection is that* P_{ji}^h *satisfy* $\overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}}P_{ji}^h = P_{ji}^h$ $(u \neq v;\ u, v = 1, 2, 3)$, *that is,*

$$3P_{ji}^h + \overset{(1)}{F}_i{}^b P_{jb}{}^a\overset{(1)}{F}_a{}^h + P_{jb}{}^a\overset{(2)}{F}_{ai}\overset{(2)}{F}^{bh} + P_{jb}{}^a\overset{(3)}{F}_{ai}\overset{(3)}{F}^{bh} = 0.$$

This condition is equivalent to the following two conditions:

$$P_{ji}^h + \overset{(1)}{F}_i{}^b P_{jb}{}^a\overset{(1)}{F}_a{}^h = 0, \qquad P_{ji}^h + P_{jb}{}^a\overset{(2)}{F}_{ai}\overset{(2)}{F}^{bh} = 0,$$

which is verified by contracting $\overset{(1)}{F}_k\overset{(1)}{F}_h{}^i$ and $\overset{(2)}{F}_{hk}\overset{(2)}{F}^{il}$ to the equation indicated in the Theorem.

The following Theorem is also immediate from Lemma 4.1, 4.3 and Theorem 4.3.

THEOREM 4.4. *Let* Γ_{ji}^h *be a natural affine connection in an almost complex symplectic manifold* X_{4n} *and let* Q_{ji}^h *be an arbitrary tensor field over* X_{4n}. *Then*

$$\Gamma_{ji}{}^h + \overset{(u)}{\mathfrak{F}}\overset{(v)}{\mathfrak{F}} Q_{ji}{}^h = \Gamma_{ji}{}^h + \frac{1}{4}(Q_{ji}{}^h - \overset{(1)}{F}_i{}^b Q_{jb}{}^a \overset{(1)}{F}_a{}^h - Q_{jb}{}^a \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh} - Q_{jb}{}^a \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh})$$

$$(u \neq v;\ u, v = 1, 2, 3)$$

is also a natural affine connection.

5. Nijenhuis tensor of $\overset{(1)}{F}_i{}^h$ and tensors $\overset{(2)}{F}_{jih}$, $\overset{(3)}{F}_{jih}$. We introduce the Nijenhuis tensor $\overset{(1)}{N}_{ji}{}^h$ of the almost complex structure $\overset{(1)}{F}_i{}^h$:

$$\overset{(1)}{N}_{ji}{}^h \equiv \frac{1}{2}(\overset{(1)}{F}_{[j}{}^a \partial_{|a|} \overset{(1)}{F}_{i]}{}^h - \overset{(1)}{F}_{[j}{}^a \partial_{i]} \overset{(1)}{F}_a{}^h)$$

and if $\Gamma_{ji}{}^h$ is an arbitrary affine connection in X_{4n}, we can write

$$\overset{(1)}{N}_{ji}{}^h = \frac{1}{2}\Big[(\overset{(1)}{F}_{[j}{}^a \nabla_{|a|} \overset{(1)}{F}_{i]}{}^h - \overset{(1)}{F}_{[j}{}^a \nabla_{i]} \overset{(1)}{F}_a{}^h) + S_{ji}{}^h - 2\overset{(1)}{F}_{[j}{}^b S_{ib]}{}^a \overset{(1)}{F}_a{}^h - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^h\Big], \tag{5.1}$$

where ∇_a denotes the covariant differentiation with respect to $\Gamma_{ji}{}^h$ and $S_{ji}{}^h$ is the torsion tensor of $\Gamma_{ji}{}^h$.

As to $\overset{(2)}{F}_{ih}$ and $\overset{(3)}{F}_{ih}$, we put

$$\overset{(2)}{F}_{jih} \equiv \partial_{[j} \overset{(2)}{F}_{ih]} = \frac{1}{3}(\partial_j \overset{(2)}{F}_{ih} + \partial_i \overset{(2)}{F}_{hj} + \partial_h \overset{(2)}{F}_{ji}),$$

$$\overset{(3)}{F}_{jih} \equiv \partial_{[j} \overset{(3)}{F}_{ih]} = \frac{1}{3}(\partial_j \overset{(3)}{F}_{ih} + \partial_i \overset{(3)}{F}_{hj} + \partial_h \overset{(3)}{F}_{ji}).$$

Then, $\overset{(2)}{F}_{jih}$ and $\overset{(3)}{F}_{jih}$ are both tensor fields in X_{4n} and for an arbitrary affine connection $\Gamma_{ji}{}^h$ in X_{4n}, we can write

$$\overset{(2)}{F}_{jih} = \nabla_{[j} \overset{(2)}{F}_{ih]} + \frac{2}{3}(S_{ji}{}^a \overset{(2)}{F}_{ah} + S_{ih}{}^a \overset{(2)}{F}_{aj} + S_{hj}{}^a \overset{(2)}{F}_{ai}), \tag{5.2}$$

$$\overset{(3)}{F}_{jih} = \nabla_{[j} \overset{(3)}{F}_{ih]} + \frac{2}{3}(S_{ji}{}^a \overset{(3)}{F}_{ah} + S_{ih}{}^a \overset{(3)}{F}_{aj} + S_{hj}{}^a \overset{(3)}{F}_{ai}). \tag{5.3}$$

From these equations, we easily have

$$\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} = \nabla_{[j} \overset{(2)}{F}_{ia]} \overset{(2)}{F}{}^{ah} - \frac{2}{3}(S_{ji}{}^h + S_{jb}{}^a \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh} - S_{ib}{}^a \overset{(2)}{F}_{aj} \overset{(2)}{F}{}^{bh}),$$

$$\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah} = \nabla_{[j} \overset{(3)}{F}_{ia]} \overset{(3)}{F}{}^{ah} - \frac{2}{3}(S_{ji}{}^h + S_{jb}{}^a \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh} - S_{ib}{}^a \overset{(3)}{F}_{aj} \overset{(3)}{F}{}^{bh}).$$

If $S_{ji}{}^{h}$ is the torsion tensor of a natural connection, then we get

$$\overset{(1)}{N}_{ji}{}^{h} = \frac{1}{2}(S_{ji}{}^{h} - 2\overset{(1)}{F}_{[j}{}^{b}S_{i]}{}^{a}\overset{(1)}{F}_{a}{}^{h} - \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h}), \tag{5.4}$$

$$\overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah} = -\frac{2}{3}(S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - S_{ib}{}^{a}\overset{(2)}{F}_{aj}\overset{(2)}{F}{}^{bh}), \tag{5.5}$$

$$\overset{(2)}{F}_{jia}\overset{(3)}{F}{}^{ah} = -\frac{2}{3}(S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh} - S_{ib}{}^{a}\overset{(3)}{F}_{aj}\overset{(3)}{F}{}^{bh}). \tag{5.6}$$

For a tensor $P_{ji}{}^{h}$, we have

$$\overset{(1)}{\mathfrak{F}}{}^{*}\overset{(1)}{\mathfrak{F}}{}'P_{ji}{}^{h} = \frac{1}{4}(P_{ji}{}^{h} - \overset{(1)}{F}_{j}{}^{b}P_{ib}{}^{a}\overset{(1)}{F}_{a}{}^{h} + \overset{(1)}{F}_{i}{}^{b}P_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}P_{ab}{}^{h}),$$

and we obtain the following theorem.

THEOREM 5.1. *The Nijenhuis tensor $\overset{(1)}{N}_{ji}{}^{h}$ can be represented by means of the tensors $\overset{(2)}{F}_{jih}$, $\overset{(3)}{F}_{jih}$ as follows:*

$$\overset{(1)}{N}_{ji}{}^{h} = -\frac{3}{2}\overset{(1)}{\mathfrak{F}}{}^{*}\overset{(1)}{\mathfrak{F}}{}'(\overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah} + \overset{(3)}{F}_{jia}\overset{(3)}{F}{}^{ah})$$

$$= -\frac{3}{8}(P_{ji}{}^{h} - \overset{(1)}{F}_{j}{}^{b}P_{ib}{}^{a}\overset{(1)}{F}_{a}{}^{h} + \overset{(1)}{F}_{i}{}^{b}P_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}P_{ab}{}^{h}),$$

where $P_{ji}{}^{h} \equiv \overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah} + \overset{(3)}{F}_{jia}\overset{(3)}{F}{}^{ah}$.

PROOF. Let $\Gamma_{ji}{}^{h}$ be an arbitrary natural connection with torsion tensor $S_{ji}{}^{h}$ and at first we calculate $\overset{(1)}{\mathfrak{F}}{}^{*}\overset{(1)}{\mathfrak{F}}{}'(\overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah})$ taking account of (5.4), (5.5) and (5.6).

$$\overset{(1)}{\mathfrak{F}}{}^{*}\overset{(1)}{\mathfrak{F}}{}'(\overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah})$$

$$= -\frac{1}{6}[(S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - S_{ib}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh})$$

$$- \overset{(1)}{F}_{j}{}^{b}(S_{ib}{}^{a} + S_{id}{}^{c}\overset{(2)}{F}_{cb}\overset{(2)}{F}{}^{da} - S_{bd}{}^{c}\overset{(2)}{F}_{ci}\overset{(2)}{F}{}^{da})\overset{(1)}{F}_{a}{}^{h}$$

$$+ \overset{(1)}{F}_{i}{}^{b}(S_{jb}{}^{a} + S_{jd}{}^{c}\overset{(2)}{F}_{cb}\overset{(2)}{F}{}^{da} - S_{bd}{}^{c}\overset{(2)}{F}_{ci}\overset{(2)}{F}{}^{da})\overset{(1)}{F}_{a}{}^{h}$$

$$- \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}(S_{ab}{}^{h} + S_{ad}{}^{c}\overset{(2)}{F}_{cb}\overset{(2)}{F}{}^{dh} - S_{bd}{}^{c}\overset{(2)}{F}_{ca}\overset{(2)}{F}{}^{dh})]$$

$$= -\frac{1}{6}[(S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - S_{ib}{}^{a}\overset{(2)}{F}_{aj}\overset{(2)}{F}{}^{bh})$$

$$- (\overset{(1)}{F}_{j}{}^{b}S_{ib}{}^{a}\overset{(1)}{F}_{a}{}^{h} - S_{id}{}^{c}\overset{(3)}{F}_{cj}\overset{(3)}{F}{}^{dh} + \overset{(1)}{F}_{j}{}^{b}S_{bd}{}^{c}\overset{(2)}{F}_{ci}\overset{(3)}{F}{}^{dh})$$

$$+ (\overset{(1)}{F}_i{}^b S_{jb}{}^a \overset{(1)}{F}_a{}^h - S_{jd}{}^c \overset{(3)}{F}_{ci} \overset{(3)}{F}{}^{dh} + \overset{(1)}{F}_i{}^b S_{bd}{}^c \overset{(2)}{F}_{cj} \overset{(3)}{F}{}^{dh})$$

$$- (\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^h + \overset{(1)}{F}_j{}^a S_{ad}{}^c \overset{(3)}{F}_{ci} \overset{(2)}{F}{}^{dh} - \overset{(1)}{F}_i{}^b S_{bd}{}^c \overset{(3)}{F}_{cj} \overset{(2)}{F}{}^{dh})]$$

$$= -\frac{1}{6}[(S_{ji}{}^h - \overset{(1)}{F}_j{}^b S_{ib}{}^a \overset{(1)}{F}_a{}^h + \overset{(1)}{F}_i{}^b S_{jb}{}^a \overset{(1)}{F}_a{}^h - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^h)$$

$$+ (S_{ji}{}^h + S_{jb}{}^a \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh} - S_{ib}{}^a \overset{(2)}{F}_{aj} \overset{(2)}{F}{}^{bh})$$

$$- (S_{ji}{}^h + S_{jb}{}^a \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh} - S_{ib}{}^a \overset{(2)}{F}_{aj} \overset{(3)}{F}{}^{bh})$$

$$- \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (S_{ab}{}^h + S_{ad}{}^c \overset{(2)}{F}_{cb} \overset{(2)}{F}{}^{dh} - S_{bd}{}^c \overset{(2)}{F}_{ca} \overset{(2)}{F}{}^{dh})$$

$$+ \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (S_{ab}{}^h + S_{ad}{}^c \overset{(3)}{F}_{db} \overset{(3)}{F}{}^{dh} - S_{bd}{}^c \overset{(3)}{F}_{ca} \overset{(3)}{F}{}^{h})]$$

$$= -\frac{1}{6}\Big[2\overset{(1)}{N}_{ji}{}^h - \frac{3}{2}\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} + \frac{3}{2}\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah}$$

$$+ \frac{3}{2}\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(2)}{F}_{abc} \overset{(2)}{F}{}^{ch}) - \frac{3}{2}\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(3)}{F}_{abc} \overset{(3)}{F}{}^{ch})\Big]$$

$$= -\frac{1}{3}\overset{(1)}{N}_{ji}{}^h + \frac{1}{4}[\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(2)}{F}_{abc} \overset{(2)}{F}{}^{ch})]$$

$$- \frac{1}{4}[\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah} - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(3)}{F}_{abc} \overset{(3)}{F}{}^{ch})].$$

Analoguously we get

$$\overset{(1)}{\mathfrak{F}}{}^* \overset{(1)}{\mathfrak{F}}{}' (\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah}) = -\frac{1}{3}\overset{(1)}{N}_{ji}{}^h - \frac{1}{4}[\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(2)}{F}_{abc} \overset{(2)}{F}{}^{ch})]$$

$$+ \frac{1}{4}[\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah} - \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b (\overset{(3)}{F}_{abc} \overset{(3)}{F}{}^{ch})].$$

Consequently we have

$$\overset{(1)}{N}_{ji}{}^h = -\frac{3}{2}\overset{(1)}{\mathfrak{F}}{}^* \overset{(1)}{\mathfrak{F}}{}' (\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} + \overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah}). \qquad \text{Q. E. D.}$$

THEOREM 5.2. *The tensors* $\overset{(2)}{F}_{jih}$ *and* $\overset{(3)}{F}_{jih}$ *can be represented as follows*:

$$\overset{(2)}{F}_{jih} = 2P(\overset{(1)}{N}_{ji}{}^a \overset{(2)}{F}_{|a|h}) - \frac{3}{2}P(\overset{(1)}{F}_h{}^a \overset{(3)}{F}_{|a|ji}) - \frac{1}{2}\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b \overset{(1)}{F}_h{}^c \overset{(3)}{F}_{abc},$$

$$\overset{(3)}{F}_{jih} = 2P(\overset{(1)}{N}_{ji}{}^a \overset{(3)}{F}_{|a|h}) + \frac{3}{2}P(\overset{(1)}{F}_h{}^a \overset{(2)}{F}_{|a|ji}) + \frac{1}{2}\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b \overset{(1)}{F}_h{}^c \overset{(2)}{F}_{abc},$$

where

$$P(T_{jih}) = \frac{1}{3}(T_{jih} + T_{ihj} + T_{hji}),$$

for a tensor T_{jih}.

PROOF. Let $\Gamma_{ji}{}^h$ be an arbitrary natural connection with torsion tensor $S_{ji}{}^h$. Then by virtue of Theorem 4.4,

$$\Gamma'_{ij}{}^h = \Gamma_{ji}{}^h - \frac{2}{5}(S_{ji}{}^h - \overset{(1)}{F}_i{}^b S_{jb}{}^a \overset{(1)}{F}_a{}^h - S_{jb}{}^a \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh} - S_{ib}{}^a \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh})$$

is also a natural connection and taking account of (5.4), (5.5), (5.6), the torsion tensor $S'_{ji}{}^h$ of $\Gamma'_{ji}{}^h$ is calculated as follows:

$$\begin{aligned} S'_{ji}{}^h &= \frac{1}{5}[(S_{ji}{}^h - \overset{(1)}{F}_j{}^b S_{ib}{}^a \overset{(1)}{F}_a{}^h + \overset{(1)}{F}_i{}^b S_{jb}{}^a \overset{(1)}{F}_a{}^h) \\ &\qquad + (S_{ji}{}^h + S_{jb}{}^a \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh} - S_{ib}{}^a \overset{(2)}{F}_{aj} \overset{(2)}{F}{}^{bh}) \\ &\qquad + (S_{ji}{}^h + S_{jb}{}^a \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh} - S_{ib}{}^a \overset{(3)}{F}_{aj} \overset{(3)}{F}{}^{bh})] \\ &= \frac{1}{5}(2\overset{(1)}{N}_{ji}{}^h + \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^h - \frac{3}{2}\overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} - \frac{3}{2}\overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah}), \end{aligned}$$

from which we get

$$S'_{ji}{}^a \overset{(2)}{F}_{ah} = \frac{1}{5}(2\overset{(1)}{N}_{ji}{}^a \overset{(2)}{F}_{ah} + \overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^c \overset{(2)}{F}_{ch} + \frac{3}{2}\overset{(2)}{F}_{jih} - \frac{3}{2}\overset{(3)}{F}_{jia} \overset{(1)}{F}_h{}^a)$$

and hence

$$\begin{aligned} P(S'_{ji}{}^a \overset{(1)}{F}_{ah}) &= \frac{2}{5}P(\overset{(1)}{N}_{ji}{}^a \overset{(2)}{F}_{|a|h}) + \frac{1}{5}P(\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{|a||b|}{}^c \overset{(2)}{F}_{ch}) \\ &\qquad + \frac{3}{10}\overset{(2)}{F}_{jih} - \frac{3}{10}P(\overset{(3)}{F}_{ji|a|} \overset{(1)}{F}_h{}^a). \end{aligned}$$

Since $S'_{ji}{}^h$ is the torsion tensor of a natural connection, we have from (5.2)

$$P(S'_{ji}{}^a \overset{(2)}{F}_{ah}) = \frac{1}{3}(S'_{ji}{}^a \overset{(2)}{F}_{ah} + S'_{ih}{}^a \overset{(2)}{F}_{aj} + S'_{jh}{}^a \overset{(2)}{F}_{ai}) = \frac{1}{2}\overset{(2)}{F}_{jih},$$

and further from (5.3) we get

$$\begin{aligned} P(\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{|a||b|}{}^c \overset{(2)}{F}_{ch} =) &\frac{1}{3}(\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b S_{ab}{}^c \overset{(2)}{F}_{ch} + \overset{(1)}{F}_i{}^a \overset{(1)}{F}_h{}^b S_{ab}{}^c \overset{(2)}{F}_{cj} + \overset{(1)}{F}_h{}^a \overset{(1)}{F}_j{}^b S_{ab}{}^c \overset{(2)}{F}_{ci}) \\ &= -\frac{1}{2}\overset{(1)}{F}_j{}^a \overset{(1)}{F}_i{}^b \overset{(1)}{F}_h{}^c \overset{(3)}{F}_{abc}. \end{aligned}$$

Consequently, we obtain

$$\frac{1}{2}\overset{(2)}{F}_{jih} = \frac{2}{5}P(\overset{(1)}{N}_{ji}{}^{a}\overset{(2)}{F}_{|a|h}) - \frac{1}{10}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc}$$

$$+\frac{3}{10}\overset{(2)}{F}_{jih} - \frac{3}{10}P(\overset{(3)}{F}_{ji|a|}\overset{(1)}{F}_{h}{}^{a})$$

or

$$\overset{(2)}{F}_{jih} = 2P(\overset{(1)}{N}_{ji}{}^{a}\overset{(2)}{F}_{|a|h}) - \frac{3}{2}P(\overset{(1)}{F}_{h}{}^{a}\overset{(3)}{F}_{|a|ji}) - \frac{1}{2}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc}.$$

The representation of $\overset{(3)}{F}_{jih}$ is also obtained by a quite similar way.

Q. E. D.

COROLLARY. *If we put*

$$\overset{(2)}{E}_{jih} = \overset{(2)}{F}_{jih} - 3P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(2)}{F}_{|d||e|h}),\ \overset{(3)}{E}_{jih} = \overset{(3)}{F}_{jih} - 3P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(3)}{F}_{|d||e|h}),$$

then

$$\overset{(2)}{E}_{jih} = \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{E}_{abc},\qquad \overset{(3)}{E}_{jih} = -\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(2)}{E}_{jih}.$$

PROOF. From the second equation of Theorem 5.2, we have

$$\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc} = -2P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(1)}{N}_{|d||e|}{}^{a}\overset{(2)}{F}_{|a|h})$$

$$-\frac{3}{2}P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(2)}{F}_{|d||e|h}) - \frac{1}{2}\overset{(2)}{F}_{jih},$$

hence subtracting this equation from the first equation of the Theorem, we get

$$\overset{(2)}{F}_{jih} - \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc} = 2P((\overset{(1)}{N}_{ji}{}^{c} + \overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}N_{|d||e|}{}^{c})\overset{(2)}{F}_{|c|h})$$

$$-\frac{3}{2}P(\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{|c|ji}) - \frac{1}{2}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc}$$

$$+\frac{3}{2}P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(2)}{F}_{|d||e|h}) + \frac{1}{2}\overset{(2)}{F}_{jih}.$$

But since

$$\overset{(1)}{N}_{ji}{}^{c} + \overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(1)}{N}_{de}{}^{c} = 0$$

hlods true (Cf. [5], p. 55, Corollary 1), we obtain

$$\frac{1}{2}\overset{(2)}{F}_{jih} - \frac{3}{2}P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(2)}{F}_{|d||e|h}) = \frac{1}{2}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{abc} - \frac{3}{2}P(\overset{(1)}{F}_{h}{}^{c}\overset{(3)}{F}_{|c|ji})$$

or

$$\overset{(2)}{F}_{jih} - 3P(\overset{(1)}{F}_{j}{}^{d}\overset{(1)}{F}_{i}{}^{e}\overset{(2)}{F}_{|d||e|h}) = \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\overset{(1)}{F}_{h}{}^{c}[\overset{(3)}{F}_{abc} - 3P(\overset{(1)}{F}_{a}{}^{d}\overset{(1)}{F}_{b}{}^{e}\overset{(3)}{F}_{|d||e|c})].$$

Q.E.D.

From Theorem 5.1 and 5.2, we have

THEOREM 5.3. *If any two of* $\overset{(1)}{N}_{ji}{}^{h}$, $\overset{(2)}{F}_{jih}$, $\overset{(3)}{F}_{jih}$ *vanish, then the remaining one also vanishes.*

THEOREM 5.4. *There exists in* X_{4n} *a natural connection* $\Gamma'_{ji}{}^{h}$ *whose torsion tensor* $S'_{ji}{}^{h}$ *is given by*

$$S'_{ji}{}^{h} = \frac{1}{3}\overset{(1)}{N}_{ji}{}^{h} - \frac{5}{16}(\overset{(2)}{F}_{jia}\overset{(2)}{F}{}^{ah} + \overset{(3)}{F}_{jia}\overset{(3)}{F}{}^{ah}) - \frac{1}{16}\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}(\overset{(2)}{F}_{abc}\overset{(2)}{F}{}^{ch} + \overset{(3)}{F}_{abc}\overset{(3)}{F}{}^{ch}).$$

PROOF. Let $\Gamma_{ji}{}^{h}$ be an arbitrary natural affine connection in X_{4n}, then by virtue of Theorem 4.4,

$$\Gamma'_{ji}{}^{h} = \Gamma_{ji}{}^{h} + \frac{1}{4}(Q_{ji}{}^{h} - \overset{(1)}{F}_{i}{}^{b}Q_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - Q_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - Q_{jb}{}^{a}\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh})$$

is also a natural affine connection, where $Q_{ji}{}^{h}$ is an arbitrary tensor field. If we take

$$Q_{ji}{}^{h} = -\frac{1}{3}(5S_{ji}{}^{h} + \overset{(1)}{F}_{i}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h}),$$

where $S_{ji}{}^{h}$ is the torsion tensor of $\Gamma_{ji}{}^{h}$, then we can calculate

$$\frac{1}{4}(Q_{ji}{}^{h} - \overset{(1)}{F}_{i}{}^{b}Q_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - Q_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - Q_{jb}{}^{a}\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh})$$

$$= -\frac{1}{12}[(5S_{ji}{}^{h} + \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h}) - \overset{(1)}{F}_{i}{}^{b}(5S_{j}{}^{da} + \overset{(1)}{F}_{j}{}^{c}\overset{(1)}{F}_{b}{}^{d}S_{cd}{}^{a})\overset{(1)}{F}_{a}{}^{h}$$

$$- 5(S_{jb}{}^{a} + \overset{(1)}{F}_{j}{}^{c}\overset{(1)}{F}_{b}{}^{d}S_{cd}{}^{a})\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} - (5S_{jb}{}^{q} + \overset{(1)}{F}_{j}{}^{c}\overset{(1)}{F}_{bq}S_{cq}{}^{a})\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh}]$$

$$= -\frac{1}{12}[5S_{ji}{}^{h} + \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h} - 5\overset{(1)}{F}_{i}{}^{b}S_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} + \overset{(1)}{F}_{j}{}^{c}S_{ci}{}^{a}\overset{(1)}{F}_{a}{}^{h}$$

$$- 5S_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh} + \overset{(1)}{F}_{j}{}^{c}S_{cd}{}^{a}\overset{(2)}{F}_{ai}\overset{(3)}{F}{}^{dh} - 5S_{jb}{}^{a}\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh} - \overset{(1)}{F}_{j}{}^{c}S_{cd}{}^{a}\overset{(3)}{F}_{ai}\overset{(2)}{F}{}^{dh}]$$

$$= -\frac{1}{12}[12S_{ji}{}^{h} - (2S_{ji}{}^{h} - \overset{(1)}{F}_{j}{}^{b}S_{bi}{}^{a}\overset{(1)}{F}_{a}{}^{h} + 5\overset{(1)}{F}_{i}{}^{b}S_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - 2\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h})$$

$$- 5\left(\frac{1}{2}S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(2)}{F}_{ai}\overset{(2)}{F}{}^{bh}\right) - 5\left(\frac{1}{2}S_{ji}{}^{h} + S_{jb}{}^{a}\overset{(3)}{F}_{ai}\overset{(3)}{F}{}^{bh}\right)$$

$$- \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\left(\frac{1}{2}S_{ab}{}^{h} + S_{ad}{}^{c}\overset{(2)}{F}_{cb}\overset{(2)}{F}{}^{dh}\right) - \overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}\left(\frac{1}{2}S_{ab}{}^{h} + S_{ad}{}^{c}\overset{(3)}{F}_{cb}\overset{(3)}{F}{}^{dh}\right)\Big].$$

Hence we get

$$\Gamma_{ji}{}^{h} = \Gamma_{ji}{}^{h} - S_{ji}{}^{h} + \frac{1}{12}\Big[(2S_{ji}{}^{h} + \overset{(1)}{F}_{j}{}^{b}S_{ib}{}^{a}\overset{(1)}{F}_{a}{}^{h} + 5\overset{(1)}{F}_{i}{}^{b}S_{jb}{}^{a}\overset{(1)}{F}_{a}{}^{h} - 2\overset{(1)}{F}_{j}{}^{a}\overset{(1)}{F}_{i}{}^{b}S_{ab}{}^{h})$$

$$+ 5\left(\frac{1}{2} S_{ji}{}^{h} + S_{jb}{}^{a} \overset{(2)}{F}_{ai} \overset{(2)}{F}{}^{bh}\right) + 5\left(\frac{1}{2} S_{ji}{}^{h} + S_{jb}{}^{a} \overset{(3)}{F}_{ai} \overset{(3)}{F}{}^{bh}\right)$$

$$+ \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} \left(\frac{1}{2} S_{ab}{}^{h} + S_{ad}{}^{c} \overset{(2)}{F}_{cb} \overset{(2)}{F}{}^{dh}\right) + \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} \left(\frac{1}{2} S_{ab}{}^{h} + S_{ad}{}^{c} \overset{(3)}{F}_{cb} \overset{(3)}{F}{}^{dh}\right)\Bigg].$$

Let $S'_{ji}{}^{h}$ be the torsion tensor of $\Gamma'_{ji}{}^{h}$, then from (5.4), (5.5), (5.6) we see that

$$S'_{ji}{}^{h} = \frac{1}{12}\left[4 \overset{(1)}{N}_{ji}{}^{h} - \frac{15}{4} \overset{(2)}{F}_{jia} \overset{(2)}{F}{}^{ah} - \frac{15}{4} \overset{(3)}{F}_{jia} \overset{(3)}{F}{}^{ah}\right.$$

$$\left. - \frac{3}{4} \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} (\overset{(2)}{F}_{abc} \overset{(2)}{F}{}^{ch}) - \frac{3}{4} \overset{(1)}{F}_{j}{}^{a} \overset{(1)}{F}_{i}{}^{b} (\overset{(3)}{F}_{abc} \overset{(3)}{F}{}^{ch})\right].$$

Thus $\Gamma'_{ji}{}^{h}$ is a natural affine connection with torsion tensor $S'_{ji}{}^{h}$ of the required form. Q. E. D.

COROLLARY. *In order that we can introduce in X_{4n} a natural affine connection without torsion is that the Nijenhuis tensor $\overset{(1)}{N}_{ji}{}^{h}$ of $\overset{(1)}{F}_{i}{}^{h}$ and the tenrors $\overset{(2)}{F}_{jih}$, $\overset{(3)}{F}_{jih}$ all vanish.*

PROOF. The necessity is evident from (5.4), (5.5), (5.6). We can also prove the sufficiency by virtue of the Theorem.

6. Complex frames and complex analytic cases with respect to $\overset{(1)}{F}_{i}{}^{h}$.

In general, let A_{2m} be a $2m$-dimensional almost complex manifold with natural affine connection[12], then the restricted homogeneous holonomy group is the real representration of $GL(m, C)$ or one of its subgroups.

If we choose complex frames of referfnce $[e^{\alpha}, e_{\bar{\alpha}}]$[13] in A_{2m}, the connection of A_{2m} can be given by

(6.1) $$dP = \pi^{\alpha} e_{\alpha} + \pi^{\bar{\alpha}} e_{\bar{\alpha}},\ de_{\beta} = \pi^{\alpha}_{\beta} e_{\alpha};\ \text{conj.}$$

where $e^{\alpha} = \overline{e_{\bar{\alpha}}}$, $\pi^{\alpha} = \overline{\pi^{\bar{\alpha}}}$. And if we put

$$\begin{cases} e_{\alpha} = \dfrac{1}{\sqrt{2}}(e'_{\alpha} - ie'_{\bar{\alpha}}),\ e_{\bar{\alpha}} = \dfrac{1}{\sqrt{2}}(e'_{\alpha} + ie'_{\bar{\alpha}}), \\ \pi^{\alpha} = \dfrac{1}{\sqrt{2}}(\omega^{\alpha} + i\omega^{\bar{\alpha}}) = \overline{\pi_{\bar{\alpha}}}, \\ \pi^{\alpha}_{\beta} = \omega^{\alpha}_{\beta} - i\omega^{\bar{\alpha}}_{\beta} = \omega^{\alpha}_{\beta} + i\omega^{\alpha}_{\bar{\beta}} = \overline{\pi^{\bar{\alpha}}_{\bar{\beta}}}, \end{cases}$$

12) The natural affine connection means the connection with respect to which the almost complex structure is of null covariant derivative.

13) The ranges of Greek indices are as follows.

$\alpha, \beta, \gamma, \cdots, \lambda, \mu, \nu, \cdots = 1, \cdots, m;\ \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots, \bar{\lambda}, \bar{\mu}, \bar{\nu}, \cdots = \alpha + m,\ \beta + m, \cdots,\ \lambda + m,\ \mu + m \cdots.$

then the real Pfaffians ω^{α}, $\omega^{\bar{\alpha}}$, $\omega^{\alpha}_{\beta}\,(=\omega^{\bar{\alpha}}_{\bar{\beta}})$, $\omega^{\bar{\alpha}}_{\beta}\,(=-\,\omega^{\alpha}_{\bar{\beta}})$ give the connection of A_{2m} with respect to real frames of reference $[e'_{\alpha}, e'_{\bar{\alpha}}]\left(e'_i = a^{a}_{i}(x)\dfrac{\partial}{\partial x^{a}}\,;\ i,\ a = 1,\ldots,\ 2m\right)$.

If $m = 2n$ and if the restricted homogeneous holonomy group h^0 of $A_{2m} = A_{4n}$ is the real representation of $Sp(n, C)$, then with respect to the connection (6.1), an anti-symmetric tensor field[14] of the form

$$
(6.2)\qquad \begin{pmatrix} f_{\mu\lambda} & 0 \\ 0 & f_{\bar{\mu}\bar{\lambda}} \end{pmatrix},\qquad (f_{\mu\lambda} = \overline{f_{\bar{\mu}\bar{\lambda}}}\,;\ \det|f_{\mu\lambda}| \neq 0)^{15)}
$$

is of null covariant derivative (Cf. § 1). And according to § 1, we see that: *Let A_{4n} be an almost complex manifold with natural affine connection and consider complex frames of reference such as* (6.1). *Then the necessary and sufficient condition that the restricted homogeneous holonomy group h^0 of A_{4n} be contained in the real representation of $Sp(n, C)$ is that there exists an anti-symmetric tensor field*[15] *with null covariant derivative whose components are given by* (6.2), *with respect to the complex frames of reference under consideration.*

We can normalize the tensor (6.2) by a suitable complex change of frames of reference.

Now, let X_{4n} be an almost CS-manifold with almost CS-structure $(\overset{(1)}{F}{}_i{}^h, \overset{(2)}{F}{}_{ih})$ and consider the case where X_{4n} is complex analytic, $\overset{(1)}{F}{}_i{}^h$ giving the complex analytic structure of X_{4n}. The Nijenhuis tensor $\overset{(1)}{N}{}_{ji}{}^h$ of $\overset{(1)}{F}{}_i{}^h$ necessarily vanishes.

We call such an X_{4n} a *complex almost symplectic manifold* and in this case we call the structure $(\overset{(1)}{F}{}_i{}^h, \overset{(2)}{F}{}_{ih})$ a *complex almost symplectic structure.*

And further, if $\overset{(2)}{F}{}_{jih} = 0$ in complex almost symplectic X_{4n}, then we call such an X_{4n} a *complex symplectic manifold* with *complex symplectic structure* $(\overset{(1)}{F}{}_i{}^h, \overset{(2)}{F}{}_{ih})$. In this case, we have necessarily $\overset{(3)}{F}{}_{jih} = 0$ by virtue of Theorem 5.3.

In a complex almost symplectic manifold X_{4n}, if we introduce a complex analytic coordinate system $(z^{\alpha}, z^{\bar{\alpha}})$, $(z^{\bar{\alpha}} = \overline{z^{\alpha}})$ then the tensor field $\overset{(1)}{F}{}_i{}^h$ takes

14) The components are complex, the real and imaginary parts being functions of the initial real coordinate system.

15) In case of $m = 2n$, the Greek indices run as follows:

$$\alpha, \beta, \gamma, \cdots, \lambda, \mu, \nu, \cdots = 1, \cdots 2n;\ \bar{\alpha}, \bar{\beta}, \cdots, \bar{\lambda}, \bar{\mu}, \cdots = \alpha+2n,\ \beta+2n, \cdots, \lambda+2n,\ \mu+2n, \cdots.$$

the numerical components of the form $\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}$ and we denote this tensor field anew by $I = (I_i^{\ h})$. With respect to the complex coordinate system under consideration, put $\overset{(2)}{F} = (\overset{(2)}{F}_{ih}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A,B,C,D are complex matrices of degree $2n$. Then since $(\overset{(2)}{F}_{ih})$ is anti-symmetric, we have

$$^{t}A = -A,\ ^{t}D = -D,\ ^{t}B = -C,$$

and further since $I\overset{(2)}{F} = \begin{pmatrix} -iA & -iB \\ iC & iD \end{pmatrix}$ is also anti-symmetric from the definition of the almost CS-structure, we have

$$^{t}B = C$$

and hence $B = C = 0$.

Therefore, if we denote the tensor $(\overset{(2)}{F}_{ih})$ with respect to the complex coordinate system by $f = (f_{ih})$, we see that

$$f = (f_{ih}) = \begin{pmatrix} f_{\mu\lambda} & 0 \\ 0 & f_{\bar{\mu}\bar{\lambda}} \end{pmatrix},$$

where $f_{\mu\lambda} = f_{\mu\lambda}(z, \bar{z})$ and $f_{\bar{\mu}\bar{\lambda}} = f_{\bar{\mu}\bar{\lambda}}(z, \bar{z})$ are anti-symmetric in λ, μ and $\bar{\lambda}$, $\bar{\mu}$ respectively. Since (f_{ih}) must have real representations it is self-adjoint:

$$(6.3) \qquad f = (f_{ih}) = \begin{pmatrix} f_{\mu\lambda} & 0 \\ 0 & f_{\bar{\mu}\bar{\lambda}} \end{pmatrix}, \ (f_{\mu\lambda} = \overline{f_{\bar{\mu}\bar{\lambda}}}\,;\ f_{\mu\lambda} = -f_{\lambda\mu}).$$

Hence in a complex almost symplectic case, we denote the complex almost symplectic structure with respect to a complex coordinate system by $(I_i^{\ h}, f_{ih})$, (f_{ih}) being of the form (6. 3).

Hereafter we confine ourselves to such complex analytic coordinate systems if otherwise stated.

If we put

$$f_{jih} = \partial_{[j} f_{ih]},$$

then of course this f_{jih} is no other than the $\overset{(2)}{F}_{jih}$ in general real coordinate system. f_{jih} is also self-adjoint, and

$$f_{\bar{\nu}\mu\lambda} = \frac{1}{3}\partial_{\bar{\nu}} f_{\mu\lambda};\ \text{conj.}$$

taking account of (6. 3).

A tensor field whose mixed components vanish is called *pure*. And we can easily see that: *The necessary and sufficient condition that* $f_{\mu\lambda}(f_{\bar{\mu}\bar{\lambda}})$ *do not contain* $z^{\alpha}(\bar{z}^{\alpha})$ *is that the tensor* f_{jih} *be pure. Hence if the manifold is complex symplectic, i.e., if* $f_{jih}=0$, *then* $f_{\mu\lambda}=f_{\mu\lambda}(z)$, $f_{\bar{\mu}\bar{\lambda}}=f_{\bar{\mu}\bar{\lambda}}(\bar{z})$.

If we put

$$f'=(f'_{ih})=(I_i^{a}f_{ah})=\begin{pmatrix}-if_{\mu\lambda} & 0\\ 0 & if_{\bar{\mu}\bar{\lambda}}\end{pmatrix}=(-if_{ih}),$$

then f'_{ih} corresponds to the $\overset{(3)}{F}_{ih}$ in the real case and

$$f'_{jih}=\partial_{[j}f_{ih]}$$

corresponds to the $\overset{(3)}{F}_{jih}$. We see that

$$f'_{\nu\mu\lambda}=-if_{\nu\mu\lambda};\ \text{conj.}$$

$$f'_{\bar{\nu}\mu\lambda}=\frac{1}{3}\partial_{\bar{\nu}}f_{\mu\lambda}=-\frac{i}{3}\partial_{\bar{\nu}}f_{\mu\lambda}=-if_{\bar{\nu}\mu\lambda};\ \text{conj.}$$

Hence we have

PROPOSITION 6.1. *In an* X_{4n} *with complex almost symplectic structure* $(I_i^{\ h}, f_{ih}, f'_{ih})$, *we have*

$$f'_{\nu\mu\lambda}=-if_{\nu\mu\lambda},\ f'_{\bar{\nu}\mu\lambda}=-if_{\bar{\nu}\mu\lambda};\ \textit{conj.}$$

This corresponds to Theorem 5.2 or to its Corollary.

In general, in a complex analytic manifold with complex coordinate system $(z^{\alpha}, z^{\bar{\alpha}})$, the natural affine connection is given by $(\Gamma_{j\mu}^{\ \ \lambda}, \Gamma_{j\bar{\mu}}^{\ \ \bar{\lambda}})$, the other Γ's being all zero. And we remark that $(\Gamma_{\nu\mu}^{\ \ \lambda}, \Gamma_{\bar{\nu}\bar{\mu}}^{\ \ \bar{\lambda}})$ give also components of a natural affine connection and $(\Gamma_{\bar{\nu}\mu}^{\ \ \lambda}, \Gamma_{\nu\bar{\mu}}^{\ \ \bar{\lambda}})$ are components of a mixed tensor.

PROPOSITION 6.2. *Let* A_{4n} *be a complex analytic manifold with complex coordinate system* $(z^{\alpha}, z^{\bar{\alpha}})$ *and with natural affine connection* $(\Gamma_{j\mu}^{\ \ \lambda}, \Gamma_{j\bar{\mu}}^{\ \ \bar{\lambda}})$. *Then the necessary and sufficient condition that the restricted homogeneous holonomy group* h^0 *is contained in the real representation of* $Sp(n, C)$ *is that there exist an anti-symmetric self-adjoint tensor field* $(f_{\mu\lambda}, f_{\bar{\mu}\bar{\lambda}})$ $(f_{\mu\bar{\lambda}}=f_{\bar{\mu}\lambda}=0)$ *with null covariant derivatives.*

That is, the A_{4n} is necessarily a complex almost symplectic manifold and the connection is a natural affine connecton with respect to the complex almost symplectic structure $(I_i^{\ h}, f_{ih})$.

The condition $\nabla_j f_{ih}=0$ are written out fully as follows:

$$(6.4)\qquad \begin{cases}\Delta_{\nu}f_{\mu\bar{\lambda}}=\nabla_{\nu}f_{\bar{\mu}\lambda}=0;\ \text{conj.} & \text{(identically satisfied)}\\ \nabla_{\nu}f_{\mu\lambda}=\partial_{\nu}f_{\mu\lambda}-\Gamma_{\nu\mu}^{\ \ \omega}f_{\omega\lambda}-\Gamma_{\nu\lambda}^{\ \ \omega}f_{\mu\omega}=0;\ \text{conj.} & \\ \nabla_{\bar{\nu}}f_{\mu\lambda}=\partial_{\bar{\nu}}f_{\mu\lambda}-\Gamma_{\bar{\nu}\mu}^{\ \ \omega}f_{\omega\lambda}-\Gamma_{\bar{\nu}\lambda}^{\ \ \omega}f_{\mu\omega}=0;\ \text{conj.} & \end{cases}$$

Since $\partial_{\bar\nu} f_{\mu\lambda} = 0$ if and only if f_{jih} is pure, we can easily obtain from (6.4).

PROPOSITION 6.3. *Let X_{4n} be a complex almost symplectic manifold with complex almost symplectic structure $(I_i^{\ h}, f_{ih})$. Then in order that there exist a natural connection of the type $(\Gamma_{\nu\mu}{}^{\lambda}, \Gamma_{\bar\nu\bar\mu}{}^{\bar\lambda})$ with respect to $(I_i^{\ h}, f_{ih})$, it is necessary and sufficient that the tensor f_{jih} be pure.*

Hence, in a complex symplectic X_{4n} $(f_{jih} = 0)$, there exists always a natural connection of the type $(\Gamma_{\nu\mu}{}^{\lambda}, \Gamma_{\bar\nu\bar\mu}{}^{\bar\lambda})$ with respect to the structure $(I_i^{\ h}, f_{ih})$.

We can define a tensor f^{ih} such that $f_{ia}(-f^{ah}) = \delta_i^h$ since such an f^{ih} has a tensor character, and we see that f^{ih} is also self-adjoint and anti-symmetric in i, h.

PROPOSITION 6.4. *In an X_{4n} with complex almost symplectic structure $(I_i^{\ h}, f_{ih})$, there exists a natural connection with respect to $(I_i^{\ h}, f_{ih})$ whose torsion tensor $S'_{ij}{}^{h}$ is given by*

$$S'_{\nu\mu}{}^{\lambda} = -\frac{1}{2} f_{\nu\mu\alpha} f^{\alpha\lambda}, \quad S'_{\bar\nu\mu}{}^{\lambda} = -\frac{3}{4} f_{\bar\nu\mu\alpha} f^{\alpha\lambda}; \text{ conj.}$$

PROOF. Let $\Gamma_{ji}{}^{h} = (\Gamma_{j\mu}{}^{\lambda}, \Gamma_{j\bar\mu}{}^{\bar\lambda})$ be an arbitrary natural connection with respect to $(I_i^{\ h}, f_{ih})$, then it satisfies (6. 4). Since $(\Gamma_{\nu\mu}{}^{\lambda}, \Gamma_{\bar\nu\bar\mu}{}^{\bar\lambda})$ is an affine connection leaving invariant the $I_i^{\ h}$ and $(\Gamma_{\bar\nu\mu}{}^{\lambda}, \Gamma_{\nu\bar\mu}{}^{\bar\lambda})$ is a tensor, an affine connection $(\Gamma'_{j\mu\lambda}$ $\Gamma'_{j\bar\mu}{}^{\bar\lambda})$ such that

$$\begin{cases} \Gamma'_{\nu\mu}{}^{\lambda} = \Gamma_{\nu\mu}{}^{\lambda} - \dfrac{2}{3}(S_{\nu\mu}{}^{\lambda} - S_{\nu\beta}{}^{\alpha} f_{\alpha\mu} f^{\beta\lambda}); \quad \text{conj.} \\ \Gamma'_{\bar\nu\mu}{}^{\lambda} = \Gamma_{\bar\nu\mu}{}^{\lambda} - (S_{\bar\nu\mu}{}^{\lambda} - S_{\bar\nu\beta}{}^{\alpha} f_{\alpha\mu} f^{\beta\lambda}) = \dfrac{1}{2}\Gamma_{\bar\nu\mu}{}^{\lambda} + \dfrac{1}{2}\Gamma_{\bar\nu\mu}{}^{\alpha} f_{\alpha\mu} f^{\beta\lambda}; \quad \text{conj.} \end{cases}$$

is also an affine connection leaving invariant the $I_i^{\ h}$. And further we can see that this affine connection is indeed a natural connection with respect to $(I_i^{\ h}, f_{ih})$, by a simple calculation making use of (6.4). Taking account of (5.5) and (6.3), the components of the tensor $S'_{ji}{}^{h}$ are given by

$$S'_{\nu\mu}{}^{\lambda} = \frac{1}{3}(S_{\nu\mu}{}^{\lambda} + S_{\nu\beta}{}^{\alpha} f_{\alpha\mu} f^{\beta\lambda} - S_{\mu\beta}{}^{\alpha} f_{\alpha\nu} f^{\beta\lambda}) = -\frac{1}{2} f_{\nu\mu\alpha} f^{\alpha\lambda}; \quad \text{conj.}$$

$$S'_{\bar\nu\mu}{}^{\lambda} = \frac{1}{2}(S_{\bar\nu\mu}{}^{\lambda} + S_{\bar\nu\beta}{}^{\alpha} f_{\alpha\mu} f^{\beta\lambda}) = -\frac{3}{4} f_{\bar\nu\mu\alpha} f^{\alpha\lambda}; \quad \text{conj.} \qquad \text{Q. E. D.}$$

REMARK. This Proposjtion corresponds to the Corollary of Theorem 5.4. The natural connection $(\Gamma'_{j\mu}{}^{\lambda}, \Gamma'_{j\bar\mu}{}^{\bar\lambda})$ possesses a freedom of tensors such as $P_{ji}{}^{h}$ symmetric in j and i and satisfying $P_{\nu\mu}{}^{\omega} f_{\omega\lambda} + P_{\nu\lambda}{}^{\omega} f_{\mu\omega} = 0$; conj.

APENDIX

In connection with the almost complex case, we state several propositions on affinely connected manifolds with restricted homogeneous holonomy $Sp(m, R)$, the real symplectic group in $2m$-dimensional real linear space or one of its subgroups. Hereafter the class of the manifolds in consideration are C^2. The following Proposition is easily obtained.

PROPOSITION 1. *The necessary and sufficient condition that the restricted homogeneous holonomy group of a $2m$-dimensional affinely connected manifold A_{2m}(with or without torsion) be $Sp(m, R)$ or one of its subgroups is that there exists over A_{2m} an anti-symmetric tensor field F_{ih} of maximal rank $2m$ satisfying*

$$(1) \qquad \nabla_j F_{ih} = 0,$$

where ∇_j denotes the covariant differentiation with respect to the affine connection of A_{2m}.

A tensor field F_{ih} anti-symmetric in i and h of maximal rank in a $2m$-dimensional manifold is called a *null-system*. A differentiable manifold admitting a null-system F_{ih} $(= - F_{hi})$ or an exterior 2-form $F_{ih}\, dx^i \wedge dx^h$ of maximal rank is called an *almost symplectic manifold* (variété presque symplectique) ([2]; [3], especially Chap. IV), and the null-system or the 2-form is called *almost symplectic structure.*

In an almost symplectic manifold, we can always introduce a positive definite Riemannian metric such that $F_{ia}\, F_{hb} g^{ab} = g_{ih}$([9], Section 14) and $F_i^{\ h} = F_{ia} g^{ah}$ gives an almost complex structure for which the metric g_{ih} is hermitian. If we put $g^{ai} F_a^{\ h} = F^{ih}$, then $F_{ia} F^{ah} = - \delta_i^h$. We call an affine connection satisfying (1) for an almost symplectic structure F_{ih} a *natural affine connection of F_{ih}*. The restricted homogeneous holonomy group of a natural affine connection is $Sp(m, R)$ or one of its subgroups.

We also remark that the above Riemannian metric g_{ih} is not unique, but F^{ih} is uniquely determined from the given F_{ih} since $F_{ia} F^{ah} = - \delta_i^h$ (Cf. § 5 and § 6).

We can prove the following two propositions and a corollary by direct calculations.

PROPOSITION 2. *Let $\Gamma_{ji}^{\ \ h}$ be an affine connection in an almost symplectic A_{2m} admitting an almost symplectic structure F_{ih}. Then the affine connection $\Gamma'^{\ h}_{ji}$ such that*

$$\Gamma'^{\ h}_{ji} = \Gamma_{ji}^{\ \ h} - \frac{1}{2} (\nabla_j F_{ia}) F^{ah}$$

is a natural affine connection of F_{ih}, that it, a connection whose restricted homogeneous holonomy group is $Sp(m, R)$ or one of its subgroups.

PROPOSITION 3. *Let $\Gamma_{ji}{}^h$ be a natural affine connection of F_{ih} and let $P_{ji}{}^h$ be an arbitrary tensor field. Then,*

$$\Gamma'_{ji}{}^h = \Gamma_{ji}{}^h + \frac{1}{2}(P_{ji}{}^h - P_{jb}{}^a F_{ai} F^{bh})$$

is also a natural affine connection of F_{ih}.

COROLLARY. *For an arbitrary natural affine connection $\Gamma_{ji}{}^h$, in order that*

$$\Gamma'_{ji}{}^h = \Gamma_{ji}{}^h + Q_{ji}{}^h \quad (Q_{ji}{}^h : \text{a tensor})$$

be also a natural affine connection, it is necessary and sufficient that the tensor $Q_{ji}{}^h$ satisfy

$$Q_{ji}{}^a F_{ah} + Q_{jh}{}^a F_{ia} = 0 \quad \text{or} \quad Q_{ji}{}^h + Q_{jb}{}^a F_{ai} F^{bh} = 0.$$

Now, put

$$F_{jih} = \partial_{[j} F_{ih]} = \frac{1}{3}(\partial_j F_{ih} + \partial_i F_{hj} + \partial_h F_{ji}),$$

then F_{jih} is a tensor field. If $\Gamma_{ji}{}^h$ is an arbitrary affine connection in A_{2m} with torsion tensor $S_{ji}{}^h$, then we have

$$F_{jih} = \nabla_{[j} F_{ih]} + \frac{2}{3}(S_{ji}{}^a F_{ah} + S_{ih}{}^a F_{aj} + S_{hj}{}^a F_{ai}).$$

If $\Gamma_{ji}{}^h$ is a natural affine connection of F_{ih}, we get

$$\frac{3}{2} F_{jih} = S_{ji}{}^a F_{ah} + S_{ih}{}^a F_{al} + S_{hj}{}^a F_{ai},$$

from which

$$\frac{3}{2} F_{jia} F^{ah} = -S_{ji}{}^h + S_{ib}{}^a F_{aj} F^{bh} - S_{jb}{}^a F_{ai} F^{bh}.$$

On the other hand, if we put

$$\Gamma'_{ji}{}^h = \Gamma_{ji}{}^h - \frac{2}{3}(S_{ji}{}^h - S_{bj}{}^a F_{ai} F^{bh}),$$

then $\Gamma'_{ji}{}^h$ is also a natural connection by virtue of Proposition 3 and its torsion tensor $S'_{ji}{}^h$ is given by

$$S'_{ji}{}^h = \Gamma'_{[ji]}{}^h = \frac{1}{3}(S_{ji}{}^h + S_{jb}{}^a F_{ai} F^{bh} - S_{ib}{}^a F_{aj} F^{bh})$$

$$= -\frac{1}{2} F_{jia} F^{ah}.$$

Thus we have

PROPOSITION 4. *The necessary and sufficient condition that it be possible to introduce a natural affine connection of* F_{ih} *without torsion in* A_{2m} *is that the tensor* $F_{jih} = \partial_{[j} F_{ih]}$ *vanish identically.*

COROLLARY. *In our* A_{2m}, *there exists a natural affine connection of* F_{ih} *with torsion tensor*

$$S'^{\,h}_{ji} = -\frac{1}{2} F_{jia} F^{ah}.$$

If $F_{jih} = \partial_{[j} F_{ih]} = 0$, then the 2-form $F_{ih} dx^i \wedge dx^h$ is closed and in this case F_{ih} is called a symplectic structure and the manifold is called a symplectic manifold.

EXAMPLE. Consider an almost Kaehlerian manifold with metric tensor g_{ji} hermitian with respect to its almost complex structure $\phi_i^{\ h}$. Then the almost symplectic structure $\phi_{ji} = \phi_j^{\ a} g_{ai}$ satisfies $\partial_{[j} \phi_{ih]} = 0$, hence by Proposition 4 there exists a symmetric natural affine connection of ϕ_{ji}, but this natural connection *does not leave invariant the individual* g_{ji} *and* $\phi_i^{\ h}$ unless $\phi_i^{\ h}$ is integrable, i.e., unless the manifold is pseudo-Kaehlerian.

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