THE CAUCHY PROPERTY OF THE GENERALIZED APPROXIMATELY CONTINUOUS PERRON INTEGRAL

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1. Introduction. We say an integral has the Cauchy property, if it satisfies the following condition (C).

(C) If $f(x)$ is defined in $[a, b]$ and is integrable in each interval $[a+\varepsilon, b-\eta]$, where $a < a+\varepsilon < b-\eta < b$ and

$$\lim_{\epsilon, \eta \to 0} \int_{a+\epsilon}^{b-\eta} f(t)dt \qquad (*)$$

exists, then $f(x)$ is integrable in $[a, b]$ and the integral over $[a, b]$ is equal to the above limit.

Both the special and the general Denjoy integrals have this property. M. E. Grimshaw [1] proved that the approximately continuous Perron integral defined by J. C. Burkill [2] satisfies the condition (C) with the approximate limit instead of the ordinary limit in $(*)$.

By the use of a similar method we will show that the corresponding property is possessed by the generalized approximately continuous Perron integral dfiened by G. Sunouchi and M. Utagawa [3].

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2. Generalized approximately continuous Perron integral.

DEFINITION 2. 1. *$U(x)$ $[L(x)]$ is termed upper [lower] function of a measurable $f(x)$ in $[a, b]$, provided that*

(i) $U(a) = 0$ $[L(a) = 0]$,

(ii) $\underline{AD}\, U(x) > -\infty$ $[\overline{AD}\, L(x) < +\infty]$ *at each point x,*

(iii) $\underline{AD}\, U(x) \geqq f(x)$ $[\overline{AD}\, L(x) \leqq f(x)]$ *at each point x.*

DEFINITION 2. 2. *If $f(x)$ has upper and lower functions in $[a, b]$ and*

$$\text{l. u. b. } L(b) = \text{g. l. b. } U(b),$$

then $f(x)$ is termed integrable in AP-sense or AP-integrable. The common value of the two bounds is called the definite AP-integral of $f(x)$ and *denoted by* $(AP)\int_a^b f(t)dt$.

G. Sunouchi and M. Utagawa [3] have proved the following results.

THEOREM 2. 1. *The function* $U(x) - L(x)$ *is non-decreasing and non-negative.*

THEOREM 2. 2. *If* $f(x)$ *is AP-integrable in* $[a, b]$, *then* $f(x)$ *is so in every interval* $[a, x]$ *for* $a \leqq x \leqq b$.

THEOREM 2. 3. *The indefinite integral* $F(x) \equiv (AP)\int_a^x f(t)dt$ *is approximately continuous.*

THEOREM 2. 4. *The function* $F(x)$ *is approximately derivable almost everywhere and*

$$AD\ F(x) = f(x),\ a.e.$$

3. Cauchy property of *AP*-integral. We shall prove the Cauchy property of the *AP*-integral in the following form.

THEOREM 3. 1. *If* $f(x)$ *is AP-integrable in* $[a, \beta]$, *where* $a \leqq \beta < b$ *and has the integral* $F(x)$ *in the interval* $a \leqq x < b$, *and if*

$$\operatorname{ap}\lim_{x \to b} F(x) = 1,$$

then $f(x)$ *is AP-integrable in* $[a, b]$ *and*

$$(AP)\int_a^b f(t)dt = 1.$$

PROOF. We put $F(b) = 1$. Then $F(x)$ is approximately continuous at b since $\operatorname{ap}\lim_{x\to b} F(x) = 1$. Hence, there exists a certain set S which includes the point b and has unit density on the left at b, and on which

$$F(x) \to 1 \text{ as } x \to b.$$

Let $\{b_n\}$ $(n \geqq 1)$ be an increasing sequence of S converging to b, and put $b_0 = a$.

For any positive number ε, we can choose an upper function $U_n(x)$ for $f(x)$ on $[b_n, b_{n+1}]$ $(n \geqq 0)$, such that

$$0 \leqq U_n(x) - [F(x) - F(b_n)] < \frac{\varepsilon}{2^n}, \tag{1}$$

and

$$\underline{AD}\ U_n(x) > -\infty,\ \underline{AD}\ U_n(x) \geqq f(x). \tag{2}$$

We define the function $\bar{U}(x)$ for $a \leqq x < b$ as follows,

$$\begin{aligned}\overline{U}(x) &= U_0(x) && (a \leqq x < b_1)\\ &= \sum_{k=0}^{n-1} U_k(b_{k+1}) + U_n(x) && (b_n \leqq x < b_{n+1}).\end{aligned}$$

Then it follows from (1) and (2) that

$$0 \leqq \overline{U}(x) - F(x) < 2\,\varepsilon, \tag{3}$$

and

$$\underline{AD}\ \overline{U}(x) > -\infty,\ \underline{AD}\ \overline{U}(x) \geqq f(x) \tag{4}$$

for $a \leqq x < b$.

The function $\overline{U}(x) - F(x)$ is non-decreasing for $a \leqq x < b$ by Theorem 2. 1 and is bounded in any neighbourhood of b by (3), and so tends to a finite limit as x tends to b from below.

Since $F(x) \to 1$ as $x \to b$ on S, $U(x)$ converges to a finite limit as $x \to b$, x on S.

We define $\overline{U}(b) = \lim\limits_{x \to b} \overline{U}(x)\,(x \in S)$. Then, we obtain from (3)

$$0 \leqq \overline{U}(b) - F(b) \leqq 2\,\varepsilon. \tag{5}$$

Let $\chi(x)$ be a continuous, non-decreasing function in $[a, b]$ such that $\chi(a) = 0$, $\chi(b) = \varepsilon$, $\chi'(b) = +\infty$.

We select the integer p such that oscillation of $\overline{U}(x)$ on $S \cap [b_{p-1}, b]$ is less than ε. This is possible since $\overline{U}(x)$ tends to a finite limit as $x \in S$ tends to b.

Let ω_n be the oscillation of $U(x)$ on $S \cap [b_{n-1}, b]$ for $n \geqq p$. We define the function $\varphi(x)$ on $[b_n, b_{n+1}]$ for each $n \geqq p$ and at b as follows,

$$\begin{aligned}\varphi(b_n) &= \omega_n,\\ \varphi(b_{n+1}) &= \omega_{n+1},\\ \varphi(x) &= \text{linear} \qquad (b_n \leqq x \leqq b_{n+1}),\\ \varphi(b) &= 0.\end{aligned}$$

Finally, we set

$$\begin{aligned}U(x) &= \chi(x) + \overline{U}(x) && (a \leqq x < b_p)\\ &= \chi(x) + \overline{U}(x) + \varphi(b_p) - \varphi(x) && (b_p \leqq x \leqq b).\end{aligned}$$

Then, we obtain from (4)

$$\underline{AD}\ U(x) \geqq f(x),\ \underline{AD}\ U(x) > -\infty$$

for $a \leqq x < b$, since $\chi(x)$ and $-\varphi(x)$ are non-decreasing functions.

To verify $\underline{AD}\, U(b)$, we consider $\underline{AD}[\overline{U}(x) + \varphi(b_p) - \varphi(x)]$ at b. By the definition of φ, we obtain

$$\left\{x: \frac{\overline{U}(b) - \overline{U}(x) + \varphi(x)}{b - x} \geqq 0\right\} \supset S \cap [b_p, b]$$

and therefore the approximate lower derivate of $\{\overline{U}(x) + \varphi(b_p) - \varphi(x)\}$ at b is not negative. Since $\chi'(b) = \infty$, we have $\underline{AD}\, U(b) = \infty$. Thus, the function $U(x)$ is an upper function of $f(x)$ on $[a, b]$.

Finally, we have

$$U(b) = \overline{U}(b) + \chi(b) + \varphi(b_p) < \overline{U}(b) + 2\varepsilon$$

and hence by (5)

$$0 \leqq U(b) - F(b) < 4\varepsilon.$$

By constructing an upper function for $-f(x)$ in $[a, b]$, we obtain a lower function $L(x)$ such that

$$0 \geqq F(b) - L(b) > -4\varepsilon.$$

We have thus proved that $f(x)$ is AP-integrable on $[a, b]$ and that

$$(AP)\int_a^b f(t)dt = 1.$$

References

[1] M. E. GRIMSHAW, The Cauchy property of the generalized Perron integrals, Proc. Combridge Phil. Soc., 30(1933), 15-18.

[2] J. C. BURKILL, The approximately continuous Perron integral, Math. Zeit., 34(1931), 270-278.

[3] G. SUNOUCHI AND M. UTAGAWA, The generalized Perron integrals, Tôhoku Math. Jour., 1(1949), 95-99.

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