ON DIFFERENTIABLE MANIFOLDS WITH CERTAIN STRUCTURES WHICH ARE CLOSELY RELATED TO ALMOST CONTACT STRUCTURE. II

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1. Introduction. In the previous paper I [2], one of the authors defined the notions of manifolds with (ϕ, ξ, η)-structure and of manifolds with (ϕ, ξ, η, g)-structure and studied some algebraic properties of these manifolds. By definition, a differentiable manifold M^{2n+1} with (ϕ, ξ, η)-structure is a manifold with three tensor fields ϕ^i_j, ξ^i and η_j defined over M^{2n+1} which satisfy the relations

$$\text{rank } |\phi^i_j| = 2n, \tag{1.1}$$

$$\xi^i \eta_i = 1, \tag{1.2}$$

$$\phi^i_j \xi^j = 0, \tag{1.3}$$

$$\phi^i_j \eta_i = 0, \tag{1.4}$$

$$\phi^i_j \phi^j_k = - \delta^i_k + \xi^i \eta_k. \tag{1.5}$$

Every differentiable manifold with (ϕ, ξ, η)-structure has a positive definite Riemannian metric g such that

$$g_{ij} \xi^j = \eta_i, \tag{1.6}$$

$$g_{ij} \phi^i_h \phi^j_k = g_{hk} - \eta_h \eta_k, \tag{1.7}$$

$$g_{ih} \phi^h_j = - g_{jh} \phi^h_i \ (\equiv \phi_{ij}). \tag{1.8}$$

We call such metric g an associated Riemannian metric of the (ϕ, ξ, η)-structure. Any manifold with (ϕ, ξ, η)-structure and its associated Riemannian metric is called a manifold with (ϕ, ξ, η, g)-structure.

In this paper, we shall study mainly about some tensor fields defined by (ϕ, ξ, η)-structures and connections which leave ϕ^i_j, ξ^i and η_i covariant constant. Notations are same as in I.

2. Some tensors on manifolds with (ϕ, ξ, η)-structure. Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η)-structure and R be a real line, and consider the product manifold $M^{2n+1} \times R$. We take a sufficiently fine open covering $\mathfrak{U}$ of M^{2n+1} by coordinate neighborhoods. If we denote coordinates of

U in $\mathfrak{U}$ by x^i $(i, j, k = 1, 2, \ldots\ldots, 2n+1)$ and a cartesian coordinate of R by x^∞, then (x^i, x^∞) can be considered as a set of coordinates of $U \times R$ and $\{U \times R \mid U \in \mathfrak{U}\}$ constitutes an open covering of $M^{2n+1} \times R$ by coordinate neighborhoods.

Now, suppose that U, U' $(U \cap U' \neq \phi)$ belong to $\mathfrak{U}$ and $x^i, x^{i'}$ are their coordinates and let

$$x^{i'} = x^{i'}(x^1, \ldots\ldots, x^{2n+1}). \tag{2.1}$$

be the coordinate transformation in $U \cap U'$. We define the coordinate transformation between $U \times R$ and $U' \times R$ by

$$\begin{cases} x^{i'} = x^{i'}(x^1, \ldots\ldots\ldots, x^{2n+1}), \\ x^{\infty'} = x^\infty. \end{cases} \tag{2.2}$$

Making use of the product manifold $M^{2n+1} \times R$ and the pseudo-group of the type (2.2), we shall define four tensors N^i_{jk}, N^i_j, N_{jk} and N_j over M^{2n+1}. We begin with the following

LEMMA 1. *If we put*

$$F^i_j = \phi^i_j,\ F^i_\infty = \xi^i,\ F^\infty_j = -\eta_j,\ F^\infty_\infty = 0 \tag{2.3}$$

in coordinate neighborhoods $\{U \times R \mid U \in \mathfrak{U}\}$, *then* $F^A{}_B$ $(A, B, C = 1, 2, \ldots, 2n+1, \infty)$ *defines a field of mixed tensor over* $M^{2n+1} \times R$ *with respect to the pseudo-group of transformations of the type* (2.2), *and* $F^A{}_B$ *gives an almost complex structure on* $M^{2n+1} \times R$.

PROOF. As the Jacobian matrix of the coordinate transformation (2.2) is given by

$$\begin{pmatrix} \dfrac{\partial x^{i'}}{\partial x^j} & 0 \\ 0 & 1 \end{pmatrix}$$

we get

$$\begin{aligned} F^{i'}_{j'} &= \phi^{i'}_{j'} = \phi^i_j \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}} = F^i_j \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}} \\ &= F^A_B \frac{\partial x^{i'}}{\partial x^A}\frac{\partial x^B}{\partial x^{j'}}, \\ F^{i'}_{\infty'} &= \xi^{i'} = \xi^i \frac{\partial x^{i'}}{\partial x^i} = F^A_B \frac{\partial x^{i'}}{\partial x^A}\frac{\partial x^B}{\partial x^{\infty'}}, \\ F^{\infty'}_{j'} &= -\eta_{j'} = -\eta_j \frac{\partial x^j}{\partial x^{j'}} = F^A_B \frac{\partial x^{\infty'}}{\partial x^A}\frac{\partial x^B}{\partial x^{j'}}, \end{aligned}$$

$$F^{\infty'}_{\infty'} = F^A_B \frac{\partial x^{\infty'}}{\partial x^A} \frac{\partial x^B}{\partial x^{\infty'}},$$

which shows that F^A_B defines a tensor field on the product manifold $M^{2n+1} \times R$.

Making use of the properties (1. 1)~(1. 5), we can easily see that the tensor F^A_B satisfies

$$F^A_B F^B_C = - \delta^A_C. \tag{2.4}$$

Now the Nijenhuis tensor of this almost complex structure is given by

$$N^A_{BC} = F^E_C(\partial_E F^A_B - \partial_B F^A_E) - F^E_B(\partial_E F^A_C - \partial_C F^A_E). \tag{2.5}$$

If we calculate the components of this tensor by grouping their indices, in two groups $(1, 2, \ldots\ldots, 2n + 1)$ and ∞, we get

$$\left\{\begin{aligned} N^i_{jk} &= \phi^h_k(\partial_h \phi^i_j - \partial_j \phi^i_h) - \phi^h_j(\partial_h \phi^i_k - \partial_k \phi^i_h) \\ &\qquad - \eta_j \partial_k \xi^i + \eta_k \partial_j \xi^i, \\ N_{jk} &\equiv N^\infty_{jk} = \phi^h_k(\partial_j \eta_h - \partial_h \eta_j) - \phi^h_j(\partial_k \eta_h - \partial_h \eta_k), \\ N^i_j &\equiv N^i_{j\infty} = \xi^h(\partial_h \phi^i_j - \partial_j \phi^i_h) - \phi^h_j \partial_h \xi^i, \\ N_j &\equiv N^\infty_{\infty j} = \xi^i(\partial_i \eta_j - \partial_j \eta_i). \end{aligned}\right. \tag{2.6}$$

Now we suppose that an affine connection Γ^i_{jk} is given on the manifold M^{2n+1}. We denote the torsion tensor of the connection by

$$S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}), \tag{2.7}$$

and denote the covariant differentiation by a comma, then we can easily see that the four sets of components N^i_{jk}, N^i_j, N_{jk} and N_j can be written as follows:

$$\left\{\begin{aligned} N^i_{jk} &= \phi^h_k(\phi^i_{j,h} - \phi^i_{h,j}) - \phi^h_j(\phi^i_{k,h} - \phi^i_{h,k}) + \xi^i_{,j}\eta_k - \xi^i_{,k}\eta_j \\ &\qquad - 2S^i_{mh}\phi^m_j\phi^h_k + 2\phi^i_m(S^m_{jh}\phi^h_k - S^m_{kh}\phi^h_j) + 2S^i_{jk}, \\ N^i_j &= \xi^h(\phi^i_{j,h} - \phi^i_{h,j}) - \phi^h_j\xi^i_{,h} \\ &\qquad + 2S^m_{jh}\phi^i_m\xi^h - 2S^i_{mh}\phi^m_j\xi^h, \\ N_{jk} &= \phi^h_k(\eta_{h,j} - \eta_{j,h}) - \phi^h_j(\eta_{h,k} - \eta_{k,h}) \\ &\qquad + 2\eta_m(S^m_{hj}\phi^h_k - S^m_{hk}\phi^h_j), \\ N_j &= \xi^i(\eta_{j,i} - \eta_{i,j}) - 2S^h_{ij}\xi^i\eta_h. \end{aligned}\right. \tag{2.8}$$

Hence, we get the following

THEOREM 1. *If M^{2n+1} is a manifold with (ϕ, ξ, η)-structure, then the four sets of components N^i_{jk}, N^i_j, N_{jk} and N_j of the Nijenhuis tensor of the almost complex structure on $M^{2n+1} \times R$ give four tensors on the manifold*

M^{2n+1} *which are uniquely determined by the* (ϕ, ξ, η)*-structure.*

Especially, when the connection Γ^i_{jk} *is symmetric,* (2.8) *can be simplified as follows:*

$$
(2.9)\qquad \left\{\begin{aligned}
N^i_{jk} &= \phi^h_k(\phi^i_{j,h} - \phi^i_{h,j}) - \phi^h_j(\phi^i_{k,h} - \phi^i_{h,k}) + \xi^i_{,j}\eta_k - \xi^i_{,k}\eta_j,\\
N^i_j &= \xi^h(\phi^i_{j,h} - \phi^i_{h,j}) - \phi^h_j\xi^i_{,h},\\
N_{jk} &= \phi^h_k(\eta_{h,j} - \eta_{j,h}) - \phi^h_j(\eta_{h,k} - \eta_{k,h}),\\
N_j &= \xi^i(\eta_{j,i} - \eta_{i,j}).
\end{aligned}\right.
$$

3. Some properties of the tensor fields N^i_{jk}, N^i_j, N_{jk} and N_j. In this section we study some properties of the tensors defined in §2. If we calculate the Lie derivatives of η_j and ϕ^i_j with respect to the infinitesimal transformation ξ^i, we get

$$
\begin{aligned}
(\pounds(\xi)\eta)_j &= \xi^k\partial_k\eta_j + \eta_k\partial_j\xi^k\\
&= \xi^k\partial_k\eta_j - \xi^k\partial_j\eta_k = N_j
\end{aligned}
$$

and

$$
\begin{aligned}
(\pounds(\xi)\phi)^i_j &= \xi^k\partial_k\phi^i_j - \phi^k_j\partial_k\xi^i + \phi^i_k\partial_j\xi^k\\
&= \xi^k\partial_k\phi^i_j - \phi^k_j\partial_k\xi^i - \xi^k\partial_j\phi^i_k\\
&= N^i_j.
\end{aligned}
$$

Therefore we have the following

THEOREM 2. (3.1) $N^i_j = (\pounds(\xi)\phi)^i_j$,
(3.2) $N_j = (\pounds(\xi)\eta)_j$.

COROLLARY 1. $N_j = 0$ *if and only if* η_j *is invariant under the transformations generated by the infinitesimal transformation* ξ^i.

COROLLARY 2. $N^i_j = 0$ *if and only if* ϕ^i_j *is invariant under the transformations generated by the infinitesimal transformation* ξ^i.

We know that the Nijenhuis tensor N^A_{BC} is hybrid with respect to the indices A and C and pure with respect to the indices B and C. The condition of hybrid is, by definition, given by

$$
(3.3)\qquad N^A_{BE}F^E_C = -N^E_{BC}F^A_E. \qquad \text{(H)}
$$

If we write down the components of both sides of the last equation by grouping their indices in two groups $(1, 2, \dots\dots\dots, 2n+1)$ and ∞, we get the following eight relations

$$
\left\{\begin{aligned}
&\phi^i{}_hN^h_{jk} + N^i_{jh}\phi^h_k + \xi^iN_{jk} - N^i_j\eta_k = 0,\\
&N^i_{jh}\xi^h + \phi^i_hN^h_j - \xi^iN_j = 0,
\end{aligned}\right.
$$

$$
(3.4)\qquad \begin{cases} \eta_h N^h_{jk} - N_{jh}\phi^h_k - N_j\eta_k = 0, \\ \phi^i_h N^h_j + N^i_h\phi^h_j - \xi^i N_j = 0, \\ N^i_h\xi^h = 0, \\ \eta_h N^h_j - N_{jh}\xi^h = 0, \\ \eta_h N^h_j + N_h\phi^h{}_j = 0, \\ N_h\xi^h = 0, \end{cases}
$$

The condition of purity is, by definition, given by

$$
(3.5)\qquad N^A_{BE}F^E_C = N^A_{EC}F^E_B.
$$

Although, this is an immediate consequence of (3. 3), we shall write down the components of both sides as (3. 4) for later use, omitting equations which appear in (3. 4) too[1].

$$
(3.6)\qquad \begin{cases} N^i_{jh}\phi^h_k - N^i_{hk}\phi^h_j - N^i_j\eta_k - N^i_k\eta_j = 0, \\ N^i_{jh}\xi^h - N^i_h\phi^h_j = 0, \\ N_{jh}\phi^h_k - N_{hk}\phi^h_j + N_j\eta_k - N_k\eta_j = 0, \\ N_{jh}\xi^h + N_h\phi^h_j = 0. \end{cases}
$$

THEOREM 3. *For any manifold with (ϕ, ξ, η)-structure, the relations*

$$
(3.7)\qquad \begin{cases} N_j = N_{hk}\phi^h_j\xi^k, \\ N_j = \eta_h N^h_i\phi^i_j, \\ N_j = N^i_{jh}\xi^h\eta_i, \\ N_{jk} = -\eta_i N^i_{jh}\phi^h_k + N^i_j\eta_i\eta_k, \\ N^i_j = \phi^i_h N^h_{jk}\xi^k + \xi^i N_{jk}\xi^k \end{cases}
$$

hold good.

PROOF. We can easily verify that $(3.7)_1$ follows from $(3.6)_4$ and $(3.4)_8$, $(3.7)_2$ follows from $(3.4)_7$ and $(3.4)_8$, $(3.7)_3$ follows from $(3.4)_2$, and $(3.7)_4$, $(3.7)_5$ follow from $(3.4)_1$.

From (3. 7) we can easily see that the following Theorem is true.

THEOREM 4. *If any one of N_{jk}, N^i_j and N^i_{jk} vanishes, then N_j vanishes. If N^i_j and N^i_{jk} vanish, then N_{jk} vanishes. If N_{jk} and N^i_{jk} vanish, then N^i_j vanishes.*

Now, we put

$$
(3.8)\qquad a_{jk} = \partial_j\eta_k - \partial_k\eta_j,
$$

1) We derived several of (3. 4) and (3. 6) by direct calculations. The usefulness of purity and hybrid is remarked by S. Tachibana.

then $\frac{1}{2} a_{jk} dx^j \wedge dx^k$ is the exterior derivative of $\eta = \eta_j dx^j$ and we have the following

THEOREM 5. *$N_{jk} = 0$, if and only if*

$$a_{jk}\phi^j_l\phi^k_m = a_{lm}, \tag{3.9}$$

i.e., if and only if $d\eta$ is invariant under ϕ^i_j.

PROOF. Necessity. Since $N_{jk} = 0$, we have

$$\phi^h_k a_{jh} = \phi^h_j a_{kh}.$$

If we multiply with the last equation by ϕ^j_l and sum for j, we get

$$\begin{aligned} a_{jh}\phi^h_k\phi^j_l &= (- \delta^h_l + \xi^h\eta_l)a_{kh} \\ &= - a_{kl} + a_{kh}\xi^h\eta_l. \end{aligned}$$

On the other hand, by virtue of the definition of N_j and Theorem 3, we have

$$a_{kh}\xi^h = - N_k = 0.$$

Therefore

$$a_{jk}\phi^j_l\phi^k_m = a_{lm}.$$

Sufficiency. From

$$a_{jk}\phi^j_l\phi^k_m = a_{lm},$$

we get

$$a_{lm}\xi^m = a_{jk}\phi^j_l\phi^k_m\xi^m = 0.$$

So we have

$$\begin{aligned} a_{lm}\phi^m_p &= a_{jk}\phi^j_l\phi^k_m\phi^m_p = a_{jk}\phi^j_l(- \delta^k_p + \xi^k\eta_p) \\ &= - a_{jp}\phi^j_l. \end{aligned}$$

Hence

$$N_{jk} = \phi^h_k a_{jh} - \phi^h_j a_{kh} = 0.$$

Q. E. D.

COROLLARY 1. *$N_{jk} = 0$ if and only if*

$$a_{jk}\psi^j_l\psi^k_m = a_{lm}, \tag{3.10}$$

where we have put

$$\psi^j_l = \phi^j_l + \xi^j\eta_l. \tag{3.11}$$

PROOF. The necessity is easily seen, since

$$a_{jk}\phi^j_l\phi^k_m = a_{lm}, \qquad a_{jk}\xi^j = 0.$$

To prove the sufficiency, we multiply (3.10) by ξ^m and sum for m, then we get

$$a_{lm}\xi^m = a_{jk}\xi^k\psi^j_l.$$

So $a_{lm}\xi^m$ is the characteristic covector of ψ^j_l corresponding to the characteristic value 1, therefore

$$a_{lm}\xi^m = \lambda\eta_l.$$

And from this we get

$$\lambda = a_{lm}\xi^m\xi^l = 0.$$

So we get

$$a_{jk}\xi^k = 0.$$

Making use of this and the above condition, we have

$$a_{jk}\phi^j_l\phi^k_m = a_{lm}.$$

Therefore by virtue of Theorem 4, we get

$$N_{jk} = 0. \qquad \text{Q. E. D}$$

COROLLARY 2. *If the (ϕ, ξ, η)-structure is the one induced from a contact structure, then N_{jk} and N_j vanish identically.*

PROOF. Since the (ϕ, ξ, η)-structure is given by a contact structure we have

$$a_{ij} = \phi_{ij}.$$

On the other hand, we may easily show that

$$\phi_{ij}\phi^i_k\phi^j_l = \phi_{kl}. \tag{3.12}$$

Therefore, from the last Theorem, we see that

$$N_{jk} = 0.$$

$N_j = 0$ follows from Theorem 4. Q. E. D.

Moreover, relative to the tensor N^i_{jk}, we have the following

THEOREM 6. *If the tensor N^i_{jk} vanishes, then other three tensors N_j, N_{jk}* and *N^i_j vanish.*

PROOF. As $N^i_{jk} = 0$, $N_j = 0$ by Theorem 4. Hence, we get from $(3.4)_3$ and

$(3.6)_4$

$$N_{jh}\phi^h_k = 0,$$
$$N_{jh}\xi^h = 0.$$

Therefore, we get

$$N_{jh} = 0.$$

As N^i_{jk} and N_{jk} vanish, we see by virtue of Theorem 4 that

$$N^i_j = 0. \qquad \text{Q. E. D.}$$

4. An affine connetion which leaves the tensor ϕ^i_j covariant constant. On an almost complex manifold, we can always find affine connections which leave the fundamental collineation covariant constant. In this section, we shall find such a connection on a manifold with (ϕ, ξ, η)-structure and study some properties of this connection. We begin with the following

THEOREM 7. *Let $\overset{0}{\Gamma}{}^i_{jk}$ be an arbitrary affine connection on a manifold with (ϕ, ξ, η)-structure, and put*

$$T^i_{jk} = -\frac{1}{2}\phi^m_{j,k}\phi^i_m - \frac{1}{2}\xi^i_{,k}\eta_j + \xi^i\eta_{j,k}, \tag{4.1}$$

where the comma is the covariant differentiation with respect to the connection $\overset{0}{\Gamma}{}^i_{jk}$. Then ϕ^i_j is covariant constant with respect to the connection defined by

$$\Gamma^i_{jk} = \overset{0}{\Gamma}{}^i_{jk} + T^i_{jk}. \tag{4.2}$$

PROOF. If we denote the covatiant differentiation with respect to the latter connection by ;, then

$$\begin{aligned}
\phi^i_{j;k} &= \phi^i_{j,k} + T^i_{hk}\phi^h_j - T^h_{jk}\phi^i_h \\
&= \phi^i_{j,k} + \Big(-\frac{1}{2}\phi^m_{h,k}\phi^i_m - \frac{1}{2}\xi^i_{,k}\eta_h + \xi^i\eta_{h,k}\Big)\phi^h_j \\
&\qquad -\Big(-\frac{1}{2}\phi^m_{j,k}\phi^h_m - \frac{1}{2}\xi^h_{,k}\eta_j + \xi^h\eta_{j,k}\Big)\phi^i_h \\
&= \phi^i_{j,k} - \frac{1}{2}\phi^m_{h,k}\phi^i_m\phi^h_j + \xi^i\eta_{h,k}\phi^h_j \\
&\qquad + \frac{1}{2}\phi^m_{j,k}(-\delta^i_m + \xi^i\eta_m) + \frac{1}{2}\xi^h_{,k}\eta_j\phi^i_h \\
&= \frac{1}{2}\phi^i_{j,k} - \frac{1}{2}[(\phi^m_h\phi^h_j)_{,k} - \phi^m_h\phi^h_{j,k}]\phi^i_m
\end{aligned}$$

$$+ \xi^i \eta_{h,k} \phi^h_j + \frac{1}{2} \phi^m_{j,k} \xi^i \eta_m + \frac{1}{2} \xi^h_{,k} \eta_j \phi^i_h$$

$$= \frac{1}{2} \phi^i_{j,k} - \frac{1}{2} \xi^m_{,k} \eta_j \phi^i_m - \frac{1}{2} \xi^m \eta_{j,k} \phi^i_m$$

$$+ \frac{1}{2} \phi^h_{j,k}(-\delta^i_h + \xi^i \eta_h) + \xi^i \eta_{h,k} \phi^h_j$$

$$+ \frac{1}{2} \phi^m_{j,k} \xi^i \eta_m + \frac{1}{2} \xi^h_{,k} \eta_j \phi^i_h$$

$$= \phi^h_{j,k} \xi^i \eta_h + \xi^i \eta_{h,k} \phi^h_j = \xi^i (\phi^h_j \eta_h)_{,k} = 0. \qquad \text{Q.E.D.}$$

THEOREM 8 *If Γ^i_{jk} is a connection which leaves ϕ^i_j covariant constant, then with respect to this connection*

$$\xi^i_{;k} = \lambda_k \xi^i, \quad \eta_{j;k} = -\lambda_k \eta_j \tag{4.3}$$

hold good, where λ_k is a covariant vector.

PROOF. From the fact that

$$\phi^i_j \xi^j = 0,$$

we get

$$\phi^i_j \xi^j_{;k} = 0..$$

While the rank $|\phi^i_j| = 2n$, and ξ^j is a characteristic vector corresponding to the characteristic value 0, we have

$$\xi^j_{;k} = \lambda_k \xi^j.$$

Similarly, we get

$$\eta_{j;k} = \mu_k \eta_j.$$

Since $\xi^i \eta_i = 1$, we have

$$\xi^j_{;k} \eta_j + \xi^j \eta_{j;k} = 0.$$

Therefore

$$\lambda_k + \mu_k = 0. \qquad \text{Q.E.D.}$$

N.B. We can easily see that

$$\lambda_k = \eta_j \xi^j_{;k}. \tag{4.4}$$

In the same way as the above proof, we can prove the following

THEOREM 9. *With respect to the connection which leaves ψ^i_j covariant constant, we have*

$$\xi^{i}{}_{;k} = \nu_k \xi^i, \quad \eta_{j;k} = -\nu_k \eta_j, \tag{4.5}$$

where ν_k is a covariant vector.

From Theorems 8 and 9, we get

COROLLARY. $\phi^i_{j;k} = 0$, if and only if $\psi^i_{j;k} = 0$.

Next if we calculate the covariant derivative of ξ^i with respect to the connection stated in Theorem 7, we get

$$\begin{aligned}
\xi^i{}_{;k} &= \xi^i_{,k} + T^{\;i}_{hk}\xi^h \\
&= \xi^i_{,k} - \frac{1}{2}\phi^m_{h,k}\phi^i_m\xi^h - \frac{1}{2}\xi^i_{,k}\eta_h\xi^h + \xi^i\eta_{h,k}\xi^h \\
&= \frac{1}{2}\xi^i_{,k} + \frac{1}{2}\xi^h_{,k}\phi^m_h\phi^i_m + \xi^i\eta_{h,k}\xi^h \\
&= \frac{1}{2}\xi^i_{,k} + \frac{1}{2}\xi^h_{,k}(-\delta^i_h + \xi^i\eta_h) - \xi^i\eta_h\xi^h_{,k}
\end{aligned}$$

i. e.,

$$\xi_{;k} = -\frac{1}{2}\xi^h_{,k}\eta_h\xi^i. \tag{4.6}$$

And so

$$\eta_{j;k} = \frac{1}{2}\xi^h_{,k}\eta_h\eta_j. \tag{4.7}$$

On the other hand, according to a theorem of Ishihara and Obata in [1], we can find a symmetric affine connection which leaves ξ^i covariant constant. So if we take this connection as $\overset{0}{\Gamma}{}^i_{jk}$ in Theorem 7, we have

$$\phi^i_{j;k} = 0, \quad \xi^i{}_{;k} = 0, \quad \eta_{j;k} = 0. \tag{4.8}$$

Therefore we get

THEOREM 10. *On a manifold with a (ϕ, ξ, η)-structure we can find an affine connection which leaves ϕ^i_j, ξ^i and η_j covariant constant.*

N. B. We call the connection stated in the last theorem a (ϕ, ξ, η)-connection.

Next, we consider a manifold with (ϕ, ξ, η, g)-structure. If we take the Christoffel's symbol $\left\{ {i \atop jk} \right\}$ with respect to g_{ij} as $\overset{0}{\Gamma}{}^i_{jk}$, we get

$$g_{ij;k} = g_{ij,k} - \left(-\frac{1}{2}\phi^m_{i,k}\phi^h_m - \frac{1}{2}\xi^h_{,k}\eta_i + \xi^h\eta_{i,k}\right)g_{hj}$$

$$-\left(-\frac{1}{2}\phi^m_{j,k}\phi^h_m-\frac{1}{2}\xi^h_{,k}\eta_j+\xi^h\eta_{j,k}\right)g_{ih}$$

$$=\frac{1}{2}\phi^m_{i,k}\phi^h_m g_{hj}+\frac{1}{2}\eta_{j,k}\eta_i-\eta_j\eta_{i,k}$$

$$+\frac{1}{2}\phi^m_{j,k}\phi^h_m g_{ih}+\frac{1}{2}\eta_{i,k}\eta_j-\eta_i\eta_{j,k}$$

$$=\frac{1}{2}\phi^m_{i,k}\phi^h_m g_{jh}+\frac{1}{2}\phi^m_{j,k}\phi^h_m g_{ih}-\frac{1}{2}(\eta_i\eta_j)_{,k}.$$

By virtue of (1.8), we have

$$g_{ij;k}=-\frac{1}{2}g_{hm}\phi^h_j\phi^m_{i,k}-\frac{1}{2}g_{hm}\phi^h_i\phi^m_{j,k}-\frac{1}{2}(\eta_i\eta_j)_{,k}$$

$$=-\frac{1}{2}(g_{hm}\phi^h_j\phi^m_i+\eta_j\eta_i)_{,k}=-\frac{1}{2}g_{ji,k}=0.$$

Then, as ξ^i is a unit vector field, we get

$$0=(g_{ij}\xi^i\xi^j)_{;k}=g_{ij}\xi^i_{;k}\xi^j+g_{ij}\xi^i\xi^j_{;k}$$
$$=2\lambda_k.$$

So we have

$$\xi^i{}_{;k}=0,\qquad \eta_{j;k}=0.$$

Hence, we get the following

THEOREM 11. *On a manifold with (ϕ,ξ,η,g)-structure we can find an affine connection which leaves ϕ^i_j, ξ^i, η_j and g_{ij} covariant constant.*

5. Symmetric (ϕ,ξ,η)-connections. In this section, we study the condition for the existence of symmetric (ϕ,ξ,η)-connections. We begin with the following lemma.

LEMMA 2. *On the manifold admitting vector fields ξ^i, η_j which satisfy the condition*

$$\xi^i\eta_i=1, \tag{5.1}$$

there exists a symmetric affine connection which leaves ξ^i and η_j covariant constant, if and only if η_j is a gradient (i.e., η is closed).

PROOF. The necessity is trivial. So we prove the sufficiency. By virtue of Ishihara and Obata's theorem, we can find a symmetric affine connection which leaves ξ^i covariant constant. We denote the coefficients of this connection by $\overset{0}{\Gamma}{}^i_{jk}$ and the operation of covariant differentiation by a vertical line | respectively. If we set

$$\Gamma^i_{jk} = \overset{0}{\Gamma}{}^i_{jk} + \xi^i \eta_{j|k}, \tag{5.2}$$

then Γ^i_{jk} defines a symmetric affine connection, as η_j is a gradient by assumption. If we denote the covariant differentiation with respect to the latter connection by a comma, we get

$$\xi^i{}_{,k} = \xi^i_{|k} + \xi^i \eta_{j|k} \xi^j = - \xi^i \eta_j \xi^j_{|k} = 0,$$
$$\eta_{j,k} = \eta_{j|k} - \xi^i \eta_{j|k} \eta_i = 0.$$

So the connection defined by Γ^i_{jk} satisfies the condition stated above. Q. E, D.

By virtue of Lemma 2, we have the following

THEOREM 12. *Let M^{2n+1} be a manifold with (ϕ, ξ, η)-structure. If η_j is a gradient and $N^i_j = 0$, then we can find a (ϕ, ξ, η)-connection whose torsion tensor is equal to $\frac{1}{8} N^i{}_{jk}$.*

PROOF. Since η_j is a gradient, we can find a symmetric affine connection such that

$$\xi^i{}_{,k} = 0, \qquad \eta_{j,k} = 0. \tag{5.3}$$

We denote its coefficients and covariant differentiation by Γ^i_{jk} and a comma respectively. Since $N^i_j = 0$ can be written as

$$\xi^k \phi^i_{j,k} - \phi^k_j \xi^i{}_{,k} - \xi^k \phi^i_{k,j} = 0,$$

it is transformed to

$$\xi^k \phi^i_{j,k} = 0. \tag{5.4}$$

On the other hand, from (1, 3), (1. 4) and (1. 5) we get the following relations.

$$\phi^i_{j,k} \xi^j = 0, \ \phi^i_{j,k} \eta_i = 0, \ (\phi^i_j \phi^j_l)_{,k} = 0. \tag{5,5}$$

If we set

$$\overset{1}{\Gamma}{}^i_{jk} = \Gamma^i_{jk} + T^i_{jk} \tag{5.6}$$

where

$$T^i_{jk} = - \frac{1}{4} \{ \phi^l_j (\phi^i_{k,l} - \phi^i_{l,k}) + \phi^i_l (\phi^l_{k,j} + \phi^l_{j,k}) \} \tag{5.7}$$

and denote the covariant differentiation with respect to the connection defined by $\overset{1}{\Gamma}{}^i_{jk}$ by ; , then we get

$$\phi^i_{j;k} = \phi^i_{j,k} - \frac{1}{4}\{\phi^l_h(\phi^i_{k,l} - \phi^i_{l,k}) + \phi^i_l(\phi^l_{k,h} + \phi^l_{h,k})\}\phi^h_j$$

$$+ \frac{1}{4}\{\phi^l_j(\phi^h_{l,k} - \phi^h_{l,k}) + \phi^h_l(\phi^l_{k,j} + \phi^l_{j,k})\}\phi^i_h$$

$$= \phi^i_{j,k} - \frac{1}{4}(\phi^i_{k,l} - \phi^i_{l,k})(-\delta^l_j + \xi^l\eta_j)$$

$$+ \frac{1}{4}\phi^h_j(\phi^i_{l,h}\phi^l_k + \phi^i_{l,k}\phi^l_h)$$

$$+ \frac{1}{4}\phi^h_j(-\phi^h_k\phi^i_{h,l} + \phi^h_l\phi^i_{h,k})$$

$$+ \frac{1}{4}(\phi^l_{k,j} + \phi^l_{j,k})(-\delta^i_l + \xi^i\eta_l)$$

$$= \frac{1}{2}\phi^i_{j,k} + \frac{1}{4}\phi^h_j\phi^i_{l,h}\phi^l_k + \frac{1}{4}\phi^h_j\phi^i_{l,k}\phi^l_h$$

$$- \frac{1}{4}\phi^l_j\phi^h_k\phi^i_{h,l} + \frac{1}{4}\phi^l_j\phi^h_l\phi^i_{h,k}$$

$$= \frac{1}{2}\phi^i_{j,k} + \frac{1}{4}\phi^i_{l,k}(-\delta^l_j + \xi^l\eta_j) + \frac{1}{4}\phi^i_{h,k}(-\delta^h_j + \xi^h\eta_j)$$

$$= 0,$$

$$\xi^i{}_{;k} = \xi^i{}_{,k} - \frac{1}{4}\{\phi^l_h(\phi^i_{k,l} - \phi^i_{l,k}) + \phi^i_l(\phi^l_{k,h} + \phi^l_{h,k})\}\xi^h$$

$$= 0,$$

and so by Theorem 8

$$\eta_{j;k} = 0$$

The torsion tensor of this connection is easily seen to be given by

$$S^i_{jk} = \frac{1}{2}(T^i_{jk} - T^i_{kj})$$

$$= \frac{1}{8}\{\phi^l_k(\phi^i_{j,l} - \phi^i_{l,j}) - \phi^l_j(\phi^i_{k,l} - \phi^i_{l,k})\}$$

$$S^i_{jk} = \frac{1}{8}N^i_{jk} \tag{5.8}$$

Q.E.D.

By virtue of Theorem 12 and Theorem 6, we get

COROLLARY. *On a manifold with (ϕ,ξ,η)-structure, we can find a symmetric (ϕ,ξ,η)-connection if and only if the following conditions are satisfied*:

i) η_j is a gradient,

ii) $N^i_{jk} = 0$.

REMARK, If a (ϕ, ξ, η)-structre is the one defined by a contact strcture, then η_j is not a gradient. So, in this case, there exists no symmetric (ϕ, ξ, η)-connection.

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