TRANSFORMATIONS OF CONJUGATE FUNCTIONS

MASAKITI KINUKAWA AND SATORU IGARI[1)]

(Received May 16, 1960, Revised December 10, 1960)

1. Throughout this paper we suppose that each function is defined in $(-\pi, \pi)$, integrable and is periodic with period 2π. If a function $W(x)$, which is defined in $(0,\pi)$, is extended to $(-\pi,\pi)$ as an even function, we denote it by $W_c(x)$, and if extended as an odd function, we denote it by $W_s(x)$. For a given function $f(x)$, we shall denote its conjugate function by $\overline{f}(x)$.

We shall be concerned with the following types of transformations T_H and T_H^* of a function $f(x)$;

$$T_H f(x) = \int_x^\pi \frac{f(t)}{2\tan t/2}\, dt \equiv F(x)$$

and

$$T_H^* f(x) = \frac{1}{2\tan x/2} \int_0^x f(t)dt \equiv F^*(x).$$

These transformations were discussed explicitly or implicitly by Bellman [1], Boas and Izumi [2], Hardy [4], Kawata [6], Loo [7] and Sunouchi [8]. Here we have the following results:

(1) *If $g(x)$ is odd and integrable, then*

$$\overline{G_s}(x) = \overline{(T_H g)_s}(x)$$

is also integrable in $(-\pi,\pi)$;

(ii) *If* $\int_0^\pi |g(x)|(\log^+ 1/x)dx < \infty$, *then*

$$\overline{G_s^*}(x) = \overline{(T_H^* g)_s}(x)$$

is integrable.

These results, which are somewhat better than Loo's theorems ([7], Theorems 4 and 7), can be proved easily by using the following lemma due to

1) The work of the first author was supported by the National Science Foundation (in U.S.A.) The authors wish to express their deep appreciation to Prof. G. Sunouchi for his valuable suggestions and encouragement in the preparation of this paper.

Hardy [5]:

If $g(x)$ is odd and is integrable in $(-\pi, \pi)$ and

$$\int_0^\pi \tan\frac{x}{2}\left|d(\cos^2\frac{x}{2}\cdot g(x))\right| < \infty,$$

then $\bar{g}(x)$ is integrable in $(-\pi, \pi)$.

To make statements simple, we denote the class of functions which are even and integrable in $(-\pi, \pi)$ by L_{c1}. Similarly we use L_{s1}, $L_{c\infty}$, and $L_{s\infty}$, by which we mean that classes of odd-integrable functions, even-measurable bounded functions and odd-measurable bounded functions, respectively. For an operator T from one of these spaces to an other, we denote its adjoint operator by T^*. Here we consider an operator T such that $Tg = \overline{G}_s$, where g is odd. Then Goes' general transformation theorem [3] says that if $T \in (L_{s1}, L_{c1})$, then $T^* \in (L_{c\infty}, L_{s\infty})$. However by the result (i), we have $T \in (L_{s1}\ L_{c1})$, and hence we have $T^* \in (L_{c\infty}, L_{s\infty})$. This means that

(iii) *if an even function $f(x)$ belongs to the class L_∞, then*

$$\overline{F_c^*}(x) = \overline{(T_H^* f)}_c(x)$$

belongs to the same class L_∞.

Applying this kind of argument to the result (ii), we may have

(iv) *if an even function $f(x)$ belongs to the class L_∞, then*

$$\overline{F_c}(x) = \overline{(T_H f)_c}(x)$$

satisfies

$$\int^\pi \exp\{\lambda|\overline{F_c}(x)|\}dx < \infty, \textit{ for some } \lambda > 0.$$

The purpose of this paper is to prove the above results (iii) and a more general result than (iv) directly, that is, to prove the following theorems which are more general than Loo's results ([7], Theorems 12 and 17).

THEOREM 1. *Suppose that $f(x)$ is even and periodic with period 2π. If $f(x)$ belongs to the class L_∞, then*

$$\frac{1}{2\tan x/2}\int_0^x \bar{f}(t)dt$$

belongs to the same class.

THEOREM 2. *Under the same assumption of Theorem* 1, *we have that the adjoint transformation*

$$\int_x^{\pi} \frac{\bar{f}(t)}{2\tan t/2}\, dt$$

belongs to the class L_∞.

2. Before proceeding to prove Theorems 1 and 2, we recall the definition of conjugate function $\bar{f}(x)$ of $f(x)$, that is,

$$\begin{aligned}\bar{f}(x) &= -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)-f(x-t)}{2\tan t/2}\, dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2\tan(x-t)/2}\, dt = \lim_{\varepsilon\to+0} \bar{f}_\varepsilon(x),\end{aligned}$$

where

$$\begin{aligned}\bar{f}_\varepsilon(x) &= -\frac{1}{\pi}\int_\varepsilon^{\pi} \frac{f(x+t)-f(x-t)}{2\tan t/2}\, dt \\ &= \frac{1}{\pi}\left\{\int_{-\pi}^{x-\varepsilon} + \int_{x+\varepsilon}^{\pi}\right\} \frac{f(t)}{2\tan(x-t)/2}\, dt \\ &= \int_{-\pi}^{\pi} K(x-t;\, \varepsilon) f(t)\, dt,\end{aligned}$$

where

$$K(u;\, \varepsilon) = \begin{cases} \dfrac{1}{2\pi}\cot\dfrac{u}{2} & \text{for } u \in \{[-\pi,\pi]-[-\varepsilon,\varepsilon]\} \\ 0 & \text{for } u \in [-\varepsilon,\varepsilon]. \end{cases}$$

Now we proceed to prove Theorem 1. Define a function $L(t; x)$ by

$$L(t;\, x) = \begin{cases} 1 & \text{for } t \in [0, x], \\ 0 & \text{for } t \in \{[-\pi,\pi]-(0,x)\}. \end{cases}$$

Then we have, for each $x > 0$,

$$\int_0^x \bar{f}(t)\, dt = \int_{-\pi}^{\pi} L(t;\, x) \lim_{\varepsilon\to 0} \bar{f}_\varepsilon(t)\, dt.$$

By a theorem of Zygmund ([9], vol. 1, p. 279), $\bar{f}_\varepsilon(t)$ has an integrable majorant which is independent of ε, and so by the theorem of Lebesgue we have

$$\int_0^x \bar{f}(t)\, dt = \lim_{\varepsilon\to 0}\int_{-\pi}^{\pi} L(t;\, x)\bar{f}_\varepsilon(t)\, dt$$

$$= \lim_{\varepsilon\to 0} \int_{-\pi}^{\pi} L(t;\, x) \left[\int_{-\pi}^{\pi} K(t-u;\, \varepsilon) f(u) du \right] dt$$

$$= \lim_{\varepsilon\to 0} \int_{-\pi}^{\pi} f(u) \left[\int_{-\pi}^{\pi} K(t-u;\, \varepsilon) L(t;\, x) dt \right] du.$$

Again by the theorem of Zygmund and by the boundedness of $f(u)$, we have

$$\int_0^x \bar{f}(t)dt = \int_{-\pi}^{\pi} f(u) \left[\lim_{\varepsilon\to 0} \int_{-\pi}^{\pi} K(t-u;\, \varepsilon) L(t;\, x) dt \right] du$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \left[\int_0^x \cot \frac{t-u}{2} \right] du$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \log \left| \frac{\sin(x-u)/2}{\sin u/2} \right| du$$

$$= \frac{1}{\pi} \int_0^{\pi} f(u) \log \left| \frac{\sin(x-u)/2 \cdot \sin(x+u)/2}{(\sin u/2)^2} \right| du.$$

Hence we have

$$\left| \int_0^x \bar{f}(t) dt \right| \leqq C \int_0^{\pi} \left| \log \left| \frac{\sin(x-u)/2 \cdot \sin(x+u)/2}{(\sin u/2)^2} \right| \right| du$$

$$= 2C(\sin x/2) \int_0^{1/(\sin x/2)} \frac{1}{\sqrt{1-y^2 \sin^2 x/2}} \left| \log \left| \frac{1-y^2}{y^2} \right| \right| dy^{1)}$$

$$= 2C(\sin x/2) K^*(x), \text{ say.}$$

So we have to prove $K^*(x) = O(1)$ as $x \to 0$.

Putting $X = 1/\sin \dfrac{x}{2}$, we have

$$K^*(x) = \int_0^X \frac{X}{\sqrt{X^2-y^2}} \left| \log \left| \frac{1-y^2}{y^2} \right| \right| dy$$

$$= \int_0^1 + \int_1^X = K_1 + K_2,$$

say, where

$$K_1 \leqq \frac{X}{\sqrt{X^2-1}} \int_0^1 \left| \log \left| \frac{1-y^2}{y^2} \right| \right| dy = O(1), \text{ as } X \to +\infty,$$

and

$$K_2 = O(1) \int_1^X \frac{X}{y^2 \sqrt{X^2-y^2}} dy$$

1) Note $[\sin(x-u)/2)] \cdot [\sin(x+u)/2] = \sin^2 x/2 - \sin^2 u/2$ and transform the variable u by $y = (\sin u/2)/(\sin x/2)$.

$$= O(1)\int_{1}^{X/2}\frac{dy}{y^2} + O(1)\int_{X/2}^{X}\frac{dt}{\sqrt{1-t^2}} = O(1),$$

which completes the proof of Theorem 1.

By the same way as in the proof of Theorem 1, we can prove Theorem 2. Let us define a function $M(t; x)$ by

$$M(t;\,x) = \begin{cases} \dfrac{1}{2}\cot\dfrac{t}{2} & \text{for } t \in [x, \pi] \\ 0 & \text{for } t \in \{[-\pi, \pi] - [x, \pi]\}. \end{cases}$$

Then we have

$$\begin{aligned}\int_{x}^{\pi}\frac{\bar{f}(t)}{2\tan t/2}\,dt &= \int_{-\pi}^{\pi} M(t;\,x)\left[\lim_{\epsilon\to 0}\int_{-\pi}^{\pi} K(t-u;\,\varepsilon)f(u)du\right]dt \\ &= \lim_{\epsilon\to 0}\int_{-\pi}^{\pi} M(t;\,x)dt\int_{-\pi}^{\pi} K(t-u;\,\varepsilon)f(u)du \\ &= \lim_{\epsilon\to 0}\int_{-\pi}^{\pi} f(u)du\int_{-\pi}^{\pi} K(t-u;\,\varepsilon)M(t;\,x)dt \\ &= \int_{-\pi}^{\pi} f(u)\,du\left[\lim_{\epsilon\to 0}\int_{-\pi}^{\pi} K(t-u;\,\varepsilon)M(t;\,x)dt\right],\end{aligned}$$

Here we have

$$\begin{aligned}&\int_{-\pi}^{\pi} K(t-u;\,\varepsilon)M(t;\,x)dt \\ &\quad = \frac{1}{4\pi}\int_{(x,\pi)-(u-\epsilon,u+\epsilon)}\frac{dt}{\tan(t-u)/2\cdot\tan t/2} \\ &\quad = \frac{1}{4\pi}\int_{(x,\pi)-(u-\epsilon,u+\epsilon)}\left[\frac{1}{\tan u/2}\left(\frac{1}{\tan\,(t-u)/2} - \frac{1}{\tan\,t/2}\right) - 1\right]dt\end{aligned}$$

which tends to, as $\varepsilon \to 0$, for $u \neq x$ and $\neq \pi$,

$$\frac{1}{2\pi\,\tan\,u/2}\log\frac{|\cos\,u/2\cdot\sin\,u/2|}{|\sin\,(x-u)/2|} - \frac{\pi - x}{4\pi}.$$

Hence we have the following formula

$$\begin{aligned}&\int_{x}^{\pi}\frac{\bar{f}(t)}{2\tan t/2}\,dt \\ &\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{\tan\,u/2}\log\frac{|\cos\,u/2\cdot\sin\,u/2|}{|\sin(x-u)/2|}\,du \\ &\qquad\qquad - \frac{\pi - x}{4\pi}\int_{-\pi}^{\pi} f(u)du\end{aligned}$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{f(u)}{2\tan u/2} \log\left|\frac{\sin(x+u)/2}{\sin(x-u)/2}\right| du$$

$$- \frac{\pi - x}{2\pi}\int_0^{\pi} f(u)du.$$

For our purpose, we have only to prove that

$$K(x) = \int_0^{\pi} \frac{1}{2\tan u/2}\left|\log\left|\frac{\sin(x+u)/2}{\sin(x-u)/2}\right|\right| du = O(1).$$

We have

$$\frac{\sin(x+u)/2}{\sin(x-u)/2} = \frac{1+\cot x/2\cdot\tan u/2}{1-\cot x/2\cdot\tan u/2}$$

and so transforming the variable u by $y = (\cot x/2)(\tan u/2)$, we get

$$K(x) = \int_0^{\infty} \frac{\cot x/2}{2y}\left|\log\left|\frac{1+y}{1-y}\right|\right| \frac{2\cot x/2}{(\cot^2 x/2)+y^2}\, dy$$

$$\leqq \int_0^{\infty} \frac{1}{y}\left|\log\left|\frac{1+y}{1-y}\right|\right| dy = \int_0^1 + \int_1^3 + \int_3^{\infty}$$

$$= K_1 + K_2 + K_3,$$

say, where

$$K_1 = \int_0^1 \frac{1}{y}\log\left[\frac{1+y}{1-y}\right] dy < \infty,$$

$$K_2 = \int_1^3 \frac{1}{y}\log\left[\frac{y+1}{y-1}\right] dy < \infty$$

and

$$K_3 = \int_3^{\infty} \frac{1}{y}\log\left[\frac{y+1}{y-1}\right] dy = \int_3^{\infty}\frac{1}{y}\log\left[1+\frac{2}{y-1}\right] dy$$

$$= O\left(\int_3^{\infty} \frac{1}{y^2}\, dy\right) < \infty,$$

which completes the proof of Theorem 2.

REFERENCES

[1] R. BELLMAN, A note on a theorem of Hardy on Fourier constants, Bull. Amer. Math. Soc., 50 (1944), 741-744.

[2] R. P. BOAS AND S. IZUMI, Absolute convergence of some trigonometric series I, Journ. Indian Math. Soc., (to appear).

[3] G. GOES, Complementary spaces of Fourier coefficients, convolutions, generalized matrix transformations and operators between BK-spaces, Journ. of Math. and

Mechanics, 10 (1961), 135-157.
[4] G. H. HARDY, The arithmetic mean of a Fourier constant, Mess. of Math. 58 (1928), 50-52.
[5] G. H. HARDY, On a theorem of Paley and Wiener, Proc. Cambridge Phil. Soc., 33 (1937), 1-5.
[6] T. KAWATA, On Fourier constants, Proc. Imp. Acad. Tokyo, 20 (1944), 218-222.
[7] C. T. LOO, Note on the properties of Fourier coefficients, Amer. Journ. of Math., 71 (1949), 269-282.
[8] G. SUNOUCHI, On Fourier constants, Proc. Imp. Acad. Tokyo, 20 (1944), 542-544.
[9] A. ZYGMUND, Trigonometric series, Cambridge (1959).

NORTHWESTERN UNIVERSITY, EVANSTON, Ill., U. S. A.,

AND

TÔHOKU UNIVRSITY.