PROJECTIVE MODULES OVER SEMILOCAL RINGS

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Let R be a commutative ring with a unit element. If there exist no proper ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$, then R is said to be indecomposable. If the number of maximal ideals of R is finite, then R is said to be semilocal. In [6], I. Kaplansky proved that, over a local ring, any projective module is free. Our objective in this paper is to generalize his theorem into

THEOREM. *Over a commutative indecomposable semilocal ring, any projective module is free.*

Every ring considered in this paper has a unit element which acts as unit operator on any module. Λ denotes a ring (not always commutative) and R denotes a commutative ring. Modules are always left modules.

1. Some lemmas on projective modules.

We begin with a trivial

LEMMA 1. *Let L, M, N be modules over a ring Λ such that $L \supset M \supset N$. If N is a direct summand of L, N is a direct summand of M.*

PROOF. Let $L = N \oplus N'$. Then we have $M = N \oplus (N' \cap M)$.

LEMMA 2. *Let P be a projective module over a ring Λ and p an element of P. If $p \notin \mathfrak{m}P$ for any maximal right ideal $\mathfrak{m}$ of Λ, then Λp is a direct summand of P and p is a free basis of Λp, where $\mathfrak{m}P$ is the image of $\mathfrak{m} \otimes_{\Lambda} P \rightarrow P$ by the natural map.*

PROOF. Let F be a free module such that $F = P \oplus Q$, $\{u_i\}$ a basis of F;

$$p = \sum_{i=1}^{n} r_i u_i, \ r_i \in R;$$

$$u_i = p_i + q_i, \quad p_i \in P, \ q_i \in Q.$$

Then we have that the right ideal $(r_1, \ldots\ldots, r_n)$ generated by r_i is equal to Λ, since, if $r_i \in \mathfrak{m}$ for a maximal right ideal $\mathfrak{m}$, we have

$$p = \sum r_i p_i,$$

i. e., $p \in \mathfrak{m}P$. Therefore there exist elements $s_1, \ldots\ldots, s_n$ in Λ such that

$$\sum r_i s_i = 1.$$

We consider a free module $F' = \Lambda v \oplus F$, where v is a variable. Then we have

$$p = v + \sum r_i(u_i - s_i v).$$

Now $\{v, u_1 - s_1 v, \ldots\ldots, u_n - s_n v, u_{n+1}, \ldots\ldots\}$ and $\{p, u_1 - s_1 v, \ldots\ldots, u_n - s_n v, u_{n+1}, \ldots\ldots\}$ are free bases of F'. Thus Λp is a direct summand of F', whence Λp is a direct summand of P by Lemma 1. It is evident that p is a free basis of Λp.

LEMMA 3. *For a projective module $P(\neq(0))$ over a ring Λ, we have $JP \neq P$, where J is the Jacobson radical of Λ.*

PROOF.[1] Let F be a free module such that $F = P \oplus Q$ and x any element of P. Select a basis $\{u_i\}$ of F such that the expression of x in terms of that basis has the smallest possible number of non-zero entries. Assume that $JP = P$, i.e., $JF \supset P$,

$$x = \sum_{i=1}^{n} r_i u_i, \quad r_i \neq 0, \ r_i \in \Lambda;$$

$$u_i = p_i + q_i, \quad p_i \in P, \quad q_i \in Q;$$

$$p_i = \sum_{j=1}^{m} s_{ij} u_j, \quad s_{ij} \in \Lambda; \ i = 1, 2, \ldots\ldots, n.$$

Then we have

$$x = \sum_i r_i u_i = \sum_i r_i p_i = \sum_{i,j} r_i s_{ij} u_j.$$

Thus we have

$$r_1 = \sum_{i=1}^{n} r_i s_{i1}, \text{ i.e., } r_1(1 - s_{11}) = \sum_{i=2}^{n} r_i s_{i1}.$$

By assumption $s_{11} \in J$, whence $1 - s_{11}$ is inversible in Λ. Put $s = 1/(1 - s_{11})$, and we have

$$r_1 = \sum_{i=2}^{n} r_i s_{i1} s.$$

Therefore

$$x = \sum_{i=2}^{n} r_i(u_i + s_{i1} s u_1).$$

This is a shorter expression for x, a contradiction if $x \neq 0$, since $\{u_1, u_2 + s_{21} s u_1,$

1) The method of this proof is due to Kaplansky [6]. This lemma is prop. 2.7 of H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960), pp. 466–488].

$\ldots\ldots, u_n + s_{n1}su_1, u_{n+1}, \ldots\ldots\}$ is a free basis.

Now let R be a commutative ring, S a multiplicatively closed set not containing 0 of R. As usual we denote by R_S the ring of quotient with respect to S, and if $S = R - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, we write $R_\mathfrak{p}$ for $R_{R-\mathfrak{p}}$. Similary, for an R-module M, we denote by M_S the module of quotient with respect to S, and we write $M_\mathfrak{p}$ for $M_{R-\mathfrak{p}}$, if $\mathfrak{p}$ is a prime ideal. We know that $M_S = M \underset{R}{\otimes} R_S$ and that there exists a canonical map φ: $M \rightarrow M_S$ and the kernel of this map is the S-component of (0) in M: Ker $\varphi = \{m \in M |$ there exists $s \in S$ such that $sm = 0\}$.

Now we have the following result which states that, over a commutative indecomposable semilocal ring, any non-zero projective module is faithfully flat[2)].

LEMMA 4. *Let R be a commutative indecomposable semilocal ring, and $P(\neq (0))$ a projective module. Then we have that $\mathfrak{m}P \neq P$ for any maximal ideal $\mathfrak{m}$ of R.*

PROOF. First we notice that $P_\mathfrak{m}$ is a projective module over a local ring $R_\mathfrak{m}$. Therefore, if $P_\mathfrak{m} \neq (0)$, we have that $\mathfrak{m}R_\mathfrak{m}P_\mathfrak{m} \neq P_\mathfrak{m}$ by Lemma 3. But $\mathfrak{m}P = P$ implies that $\mathfrak{m}R_\mathfrak{m}P_\mathfrak{m} = P_\mathfrak{m}$ .Therefore we have $P_\mathfrak{m} = (0)$ if $\mathfrak{m}P = P$. Now $P_\mathfrak{m} = (0)$ implies that, for any element p of P, there exists an element s of R, $s \notin \mathfrak{m}$, such that $sp = 0$. Thus the fact that $\mathfrak{m}P = P$ for each maximal ideal $\mathfrak{m}$ of R implies that $(0 : p) = R$ for any element p of P, i.e., $P = (0)$. Thus there must exist a maximal ideal $\mathfrak{n}$ of R such that $\mathfrak{n}P \neq P$. Now let $\{\mathfrak{m}_1, \ldots\ldots, \mathfrak{m}_m; \mathfrak{n}_1, \ldots\ldots, \mathfrak{n}_n\}$ be the set of all maximal ideals of R such that $\mathfrak{m}_iP = P$, $\mathfrak{n}_jP \neq P$ for $i = 1, \ldots \ldots, m$; $j = 1, \ldots\ldots, n$. We shall prove that $m = 0$, by proving that, if $m \neq 0$, R is not indecomposable.

Now $P_{\mathfrak{n}_j}$ is a projective module over $R_{\mathfrak{n}_j}$. Therefore there exists an element $p_j \in P_{\mathfrak{n}_j}$, $\notin \mathfrak{n}_jP_{\mathfrak{n}_j}$. Let φ_j be the canonical map of $P \rightarrow P_{\mathfrak{n}_j}$. Then we may assume that $\bar{p}_j \in \varphi_jP$. Put $\varphi_j(p_j) = \bar{p}_j$. Since $P_{\mathfrak{m}_i} = (0)$, there exists an element s_{ij} of R such that $s_{ij} \notin \mathfrak{m}_i$, $s_{ij}p_j = 0$, whence $s_{ij}\bar{p}_j = 0$. By Lemma 2, $R_{\mathfrak{n}_j}\bar{p}_j$ is a direct summand of $P_{\mathfrak{n}_j}$ and $\bar{p}_j$ is a free basis of $R_{\mathfrak{n}_j}\bar{p}_j$. Therefore we have $\varphi_j(s_{ij}) = 0$ in $R_{\mathfrak{n}_j}$, i.e., there exists an element s'_{ij} in R such that $s'_{ij} \notin \mathfrak{n}_j$, $s_{ij}s'_{ij} = 0$. Let $\mathfrak{a}_i$ be the principal ideal generated by $s_{i1}s_{i2}\ldots\ldots s_{in}$ and $\mathfrak{b}_i$ the ideal generated by $(s'_{i1}, s'_{i2}, \ldots\ldots, s'_{in})$. Then we have that $\mathfrak{a}_i \nsubseteq \mathfrak{m}_i$, $\mathfrak{b}_i \nsubseteq \mathfrak{n}_j$ for $j = 1,2,\ldots\ldots, n$ and that $\mathfrak{a}_i\mathfrak{b}_i = (0)$. Put $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \ldots\ldots, \mathfrak{a}_m)$, $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2\cdots\cdots\mathfrak{b}_m$, and we have that $\mathfrak{a} \nsubseteq \mathfrak{m}_i$, $\mathfrak{b} \nsubseteq \mathfrak{n}_j$ for $i = 1, 2, \ldots\ldots, m$; $j = 1, 2, \ldots\ldots, n$ and that $\mathfrak{a}\mathfrak{b} = (0)$. Therefore we have $R = \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{a} \neq (0)$, $\mathfrak{b} \neq (0)$. This is a contradiction if $m \neq 0$. Thus $m = 0$ and we have $\mathfrak{m}P \neq P$ for any maximal ideal $\mathfrak{m}$ of R.

Combining this with Lemma 2, we have

COROLLARY 5. *Over a commutative indecomposable semilocal ring, any finitely generated projective module is free.*

2) Cf. § 6.4, p.57 of [4].

REMARK. If R is an indecomposable commutative ring and P is a finitely generated projective module over R, then the $\mathfrak{p}$-rank of P (i.e., the number of free generators of $P_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$) is independent of the prime ideal $\mathfrak{p}$ of R[3]. Therefore the above Corollary 5 is contained in Theorems 3, 4 of [1] since R is semilocal if and only if $X = m\text{-spec}(R)$ is a decomposition space and dim $X = 0$[4].

LEMMA 6. *Let P be a projective module over a commutative indecomposable semilocal ring R. If there exists a maximal ideal $\mathfrak{m}$ of R such that $P/\mathfrak{m}P$ is finitely generated over $R/\mathfrak{m}$. Then P is finitely generated over R.*

PROOF. Suppose P is not finitely generated and $\dim(P/\mathfrak{m}P\colon R/\mathfrak{m}) = s$. By Lemma 4, there exist elements $p_j \in P$, $\notin \mathfrak{m}_j P$ for $j = 1, 2, \ldots\ldots, m$ where $\{\mathfrak{m}_1, \ldots\ldots, \mathfrak{m}_m\}$ is the set of all maximal ideals of R. Let e_i be elements of R such that $e_i \in \bigcap_{j \neq i} \mathfrak{m}_j$, $e_i \notin \mathfrak{m}_i$. Then $p_1' = \sum_{j=1}^{m} e_j p_j \notin \mathfrak{m}_i P$ for $i = 1, 2, \ldots\ldots, m$. Thus Rp_1' is a direct summand of P, i.e., there exists a submodule P_1 of P such that $P = Rp_1' \oplus P_1$. By assumption, P_1 is not finitely generated. Repeating this process, we can select $s+1$ elemens $p_1', \ldots\ldots, p_{s+1}'$ from P such that $P = Rp_1' \oplus \ldots\ldots \oplus Rp_{s+1}' \oplus P_{s+1}$. Then we have that $P/\mathfrak{m}P = Rp_1'/\mathfrak{m}Rp_1' \oplus \ldots\ldots \oplus Rp_{s+1}'/\mathfrak{m}Rp_{s+1}' \oplus P_{s+1}/\mathfrak{m}P_{s+1}$. Thus we have $\dim(P/\mathfrak{m}P\colon R/\mathfrak{m}) \geqq s+1$. This contradiction completes the proof.

2. Proof of Theorem. The following lemma is essential in the proof of our theorem.

LEMMA 7. (Eilenberg). *Let P be a projective module. Then there exists a free module F such that $F \oplus P$ is free.*

PROOF.[5] Suppose $P \oplus Q$ is free. Define F to be the direct sum of an infinite number of copies of $P \oplus Q$. Then F is free and it is evident that $F \oplus P \cong F$.

Before proving our theorem we must state two lemmas without proof[6].

LEMMA 8 (Kaplansky). *Any projective module over a ring is a direct sum of countably generated projective modules.*

LEMMA 9 (Kaplansky). *Let Λ be any ring, M a countably generated Λ-module. Assume that any direct summand N of M has the following property: any element of N can be embedded in a free direct summand of N. Then M is free.*

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3) See [3].
4) Cf. [1], [2], [5] and [8].
5) This proof is the same as in [7].
6) See Kaplansky [6].

By virtue of Lemmas 8, 9 the following Lemma 10 suffices to complete the proof of our theorem.[7)]

LEMMA 10. *Let P be a projective module over a commutative indecomposable semilocal ring R. Then an element p of P can be embedded in a free direct summand of P.*

PROOF. By Corollary 5, we may assume that P is not finitely generated. By virtue of Lemma 7, there exist free modules U, F such that $U = F \oplus P$. Let

$\{u_i\}$ be a free basis of U,
$\{f_i\}$ a free basis of F,
π the projection from U to F,

(i. e., if $u \in U$, $u = f + p'$, $f \in F, p' \in P$, then $\pi(u) = f$),

$$p = \sum_{i=1}^{n} r_i u_i, \quad r_i \in R,$$

$$\pi u_i = \sum_{j=1}^{m} s_{ij} f_j, \quad s_{ij} \in R, \quad i = 1, 2, \ldots\ldots, n.$$

Put

$$F' = \sum_{i=1}^{m} \oplus R f_i, \quad P' = F' \oplus P, \quad U' = \sum_{i=1}^{n} \oplus R u_i.$$

Then $U' \subset P' \subset U$ and U' is a direct summand of U, hence U' is a direct summand of P', by Lemma 1. Now we have $P' = U' \oplus U''$ and $p \in P \cap U'$. Put

$$\left(\sum_{i=2}^{m} \oplus \; R f_i\right) \oplus P = P''$$

and we have

$$P' = R f_1 \oplus P'' = U' \oplus U''.$$

Let π' be the projection from P' ot U'', and

$$\pi' f_1 \in \bigcap_{j=1}^{s} \mathfrak{m}_{ij} U'', \notin \bigcup_{j=s+1}^{t} \mathfrak{m}_{ij} U'',$$

where t is the number of maximal ideals of R and $\bigcup$ denotes the set theoretical

7) This method of proof is the same as in [6].

union. Now assume that $\pi' P'' \subset \mathfrak{m} U''$ for a maximal ideal $\mathfrak{m}$ of R. Then we have that

$$P'/\mathfrak{m}P' = Rf_1/\mathfrak{m}f_1 \oplus P''/\mathfrak{m}P'' = U'/\mathfrak{m}U' \oplus U''/\mathfrak{m}U''$$

and that

$$\pi'(Rf_1/\mathfrak{m}f_1) = U''/\mathfrak{m}U''.$$

Therefore $U''/\mathfrak{m}U''$ is finitely generated, whence so is $P'/\mathfrak{m}P'$. Thus by Lemma 6 P' is finitely generated and this is a contradiction since P is not finitely generated by assumption. Thus we conclude that, for any maximal ideal $\mathfrak{m}$, $\pi' P'' \not\subseteq \mathfrak{m} U''$. Therefore there exists an element $p'' \in P''$ such that

$$\pi' p'' \notin \bigcup_{j=1}^{s} \mathfrak{m}_{ij} U'', \quad \in \bigcap_{j=s+1}^{t} \mathfrak{m}_{ij} U''.$$

Then $\pi' f_1 + \pi' p'' \notin \mathfrak{m}_i U''$ for each i. Therefore $R\pi'(f_1 + p'')$ is a direct summand of U'', by Lemma 2, i.e., if we put $\pi'(f_1 + p'') = u''$, $U'' = Ru'' \oplus U'''$.

Now we have

$$P' = Rf_1 \oplus P'' = R(f_1 + p'') \oplus P'' = U' \oplus Ru'' \oplus U'''.$$

Let π'' be the projection from P' to U'''. Then we have $\pi'' P'' = U'''$. For: let u be any element of U''', $u = r(f_1 + p'') + q$, $r \in R$, $q \in P''$, then $u = \pi' r(f_1 + p'') + \pi' q = ru'' + \pi' q = \pi'' r u'' + \pi'' \pi' q = \pi'' q$. Therefore we have an exact sequence

$$0 \to K_1 \to P'' \xrightarrow{\pi''} U''' \to 0.$$

This sequence splits since U''' is projective. $K_1 = P'' \cap (U' \oplus Ru'')$ since $K_1 = \{p'' \in P'' \mid \pi'' p'' = 0\}$ and $\pi'' p'' = 0$ if and only if $p'' \in U' \oplus Ru''$. Now K_1 is a direct summand of P' and contained in $U' \oplus Ru''$ which is finitely generated. Thus K_1 is a direct summand of $U' \oplus Ru''$, hence K_1 is a finitely generated free module by Corollary 5. Therefore we have that

$$P'' = Rf_2 \oplus \cdots\cdots \oplus Rf_m \oplus P = K_1 \oplus K_1'$$

and that $p \in K_1 \cap P$ since $p \in U'$. Inductively we have

$$\begin{aligned} P''' &= Rf_3 \oplus \cdots\cdots \oplus Rf_m \oplus P \\ &= K_2 \oplus K_2', \ K_2 \cap P \ni p; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}$$

Lastly we have a finitely generated free module $\bar{K}$ which is a direct summand of P and contains p. Thus we have completed the proof.

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