ON ALMOST-ANALYTIC FUNCTIONS, TENSORS AND INVARIANT SUBSPACES

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0. Introduction. In previous papers [2], [6],[1] we introduced the notion of Φ-tensors as a generalization of analytic tensors. The main purpose of this paper is to introduce the notion of almost-analytic function and almost-analytic submanifold and to define an operator acting on tensor fields defined on an invariant subspace.

In §1 and §2 we define almost-analytic functions and almost-analytic submanifolds in an almost-complex space. A new Φ-operator on an invariant subspace of the manifold admitting a tensor fields of type (1, 1) is defined in §4, after preliminary facts are given in §3. Some formulas about the Φ-operator are given in §5 and we devote §6 to Kählerian spaces. The results of all sections except §6 are independent from connection or metric.

1. Almost-analytic functions. Consider an n-dimensional differentiable manifold admitting a tensor field whose components with respect to local coordinates $\{x^\lambda\}$ are $\varphi_\mu{}^\lambda$ [2]. A tensor field $T_{(\mu)}{}^{(\lambda)} \equiv T_{\mu_q\ldots\mu_1}{}^{\lambda_p\ldots\lambda_1}$ is called to be pure in λ_i, μ_j or briefly to be $\mathfrak{p}(\lambda_i, \mu_j)$ if it commutes with φ_μ^λ in λ_i, μ_j i.e. it holds that

$$\varphi_{\mu_j}{}^\sigma T_{\mu_q\ldots\sigma\ldots\mu_1}{}^{(\lambda)} = \varphi_\sigma{}^{\lambda_i} T_{(\mu)}{}^{\lambda_p\ldots\sigma\ldots\lambda_1}.$$

Analogously $T_{(\mu)}{}^{(\lambda)}$ is called to be $\mathfrak{p}(\lambda_i, \lambda_j)$ or $\mathfrak{p}(\mu_i, \mu_j)$ if

$$\varphi_\sigma{}^{\lambda_i} T_{(\mu)}{}^{\lambda_p\ldots\sigma\ldots\lambda_j\ldots\lambda_1} = \varphi_\sigma{}^{\lambda_j} T_{(\mu)}{}^{\lambda_p\ldots\lambda_i\ldots\sigma\ldots\lambda_1},$$

or

$$\varphi_{\mu_i}{}^\sigma T_{\mu_q,\ldots\sigma\ldots\mu_j\ldots\mu_1}{}^{(\lambda)} = \varphi_{\mu_j}{}^\sigma T_{\mu_q\ldots\mu_i\ldots\sigma\ldots\mu_1}{}^{(\lambda)}$$

is valid respectively. If a tensor is pure in every pair of its indices, then it is called a pure tensor. We consider for convenience sake the covariant vector and the contravariant vector as pure tensors.

A tensor field $T_{(\mu)}{}^{(\lambda)}$ is called to be hybrid in λ_i, μ_j or briefly to be $\mathfrak{h}(\lambda_i,\mu_j)$ if it anticommutes with $\varphi_\mu{}^\lambda$ in λ_i, μ_j i.e. it holds that

$$\varphi_{\mu_j}{}^\sigma T_{\mu_q\ldots\sigma\ldots\mu_1}{}^{(\lambda)} = -\varphi_\sigma{}^{\lambda_i} T_{(\mu)}{}^{\lambda_p\ldots\sigma\ldots\lambda_1}.$$

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1) The number in brackets refers to the the Bibliography at the end of the paper.
2) $\lambda, \mu, \nu, \cdots = 1, 2, \cdots, n$. As to the notations we follow Tachibana, S., [6].

Analogously $\mathfrak{h}(\lambda_i, \lambda_j)$ and $\mathfrak{h}(\mu_i, \mu_j)$ are defined.

The Φ-operator is defined for pure tensors as follows,

$$\begin{aligned}\Phi_\nu T_{(\mu)}{}^{(\lambda)} = \varphi_\nu{}^\sigma \partial_\sigma T_{(\mu)}{}^{(\lambda)} - \partial_\nu \overset{*}{T}_{(\mu)}{}^{(\lambda)} \\ + \sum_{i=1}^{q} (\partial_{\mu_i}\varphi_\nu{}^\sigma)\, T_{\mu_q \ldots \sigma \ldots \mu_1}{}^{(\lambda)} \\ + \sum_{j=1}^{p} (\partial_\nu \varphi_\sigma{}^{\lambda_j} - \partial_\sigma \varphi_\nu{}^{\lambda_j})\, T_{(\mu)}{}^{\lambda_p \ldots \sigma \ldots \lambda_1},\end{aligned}$$

where $\partial_\sigma = \partial/\partial x^\sigma$ and

$$\overset{*}{T}_{(\mu)}{}^{(\lambda)} = \varphi_{\mu_i}{}^{\varepsilon} T_{\mu_q \ldots \varepsilon \ldots \mu_1}{}^{(\lambda)} = \varphi_\varepsilon{}^{\lambda_j} T_{(\mu)}{}^{\lambda_p \ldots \varepsilon \ldots \lambda_1}.$$

This operator yields from a pure tensor of type (p, q) a tensor of type $(p, q+1)$. The definition is independent from connection or metric, so it is defined on differentiable manifolds admitting a tensor field of type (1, 1).

The following equations are known[3],

$$\Phi_\nu v^\lambda = -\underset{v}{\mathfrak{L}}\, \varphi_\nu{}^\lambda, \qquad \Phi_\nu \delta_\mu{}^\lambda = 0,$$

$$\Phi_\nu \varphi_\mu{}^\lambda = N_{\nu\mu}{}^\lambda,$$

where $\underset{v}{\mathfrak{L}}$ denotes the operator of Lie derivation with respect to v^λ and $N_{\nu\mu}{}^\lambda$ is the so-called Nijenhuis tensor defined by

$$N_{\nu\mu}{}^\lambda = \varphi_\nu{}^\sigma \partial_\sigma \varphi_\mu{}^\lambda - \varphi_\mu{}^\sigma \partial_\sigma \varphi_\nu{}^\lambda + \varphi_\sigma{}^\lambda (\partial_\mu \varphi_\nu{}^\sigma - \partial_\nu \varphi_\mu{}^\sigma)^{4)}.$$

Now we consider an almost-complex space which is a differentiable manifold admitting a tensor field $\varphi_\mu{}^\lambda$ such that $\varphi_\sigma{}^\lambda \varphi_\mu{}^\sigma = -\delta_\lambda{}^\mu$. In the space a pure tensor $T_{(\mu)}{}^{(\lambda)}$ satisfying $\Phi_\nu T_{(\mu)}{}^{(\lambda)} = 0$ is called to be *almost-analytic*. If there exists a local function g such that $\varphi_\lambda{}^\varepsilon \partial_\varepsilon f = \partial_\lambda g$ for a local function f, then we shall call f an *almost-analytic function* of x^λ or a *locally almost-analytic function* and call g its associated function. If such a function f is defined globally, then we call it an almost-analytic function. In a complex manifold i. e. an almost-complex space with vanishing Nijenhuis tensor, we say "analytic" instead of "almost-analytic".

For a local function f we have

$$\begin{aligned}\Phi_\nu \partial_\lambda f &= \varphi_\nu{}^\varepsilon \partial_\varepsilon \partial_\lambda f - \partial_\nu (\varphi_\lambda{}^\varepsilon \partial_\varepsilon f) + (\partial_\lambda \varphi_\nu{}^\varepsilon) \partial_\varepsilon f \\ &= \partial_\lambda (\varphi_\nu{}^\varepsilon \partial_\varepsilon f) - \partial_\nu (\varphi_\lambda{}^\varepsilon \partial_\varepsilon f),\end{aligned}$$

so $\Phi\, \partial_\lambda f = 0$ is equivalent to the existence of a local function g such that $\partial_\nu g = \varphi_\nu{}^\varepsilon \partial_\varepsilon f$.

3) Tachibana, S., [6].
4) Nijenhuis, A., [4].

Hence a locally almost-analytic function f is characterized by $\Phi_\nu \partial_\lambda f = 0$. For any v^λ and u_λ we have the equation

$$v^\sigma \Phi_\nu u_\sigma + u_\sigma \Phi_\iota v^\sigma = \varphi_\nu{}^\varepsilon \partial_\varepsilon (v^\sigma u_\sigma) - \partial_\nu (\overset{*}{v}{}^\sigma u_\sigma).$$

Hence if v^λ is almost-analytic, then $v^\sigma \partial_\sigma f$ is almost-analytic for any almost-analytic f and if u_λ is almost-analytic, then $u_\sigma v^\sigma$ is almost-analytic for any almost-analytic v^λ.

We have known much about almost-analytic tensors in almost-Hermitian spaces. In § 1 and § 2 we give some theorems about them which are independent from connection or metric.

First we have known the following

LEMMA 1. (Yano, K.) *Let f be a scalar function on a compact almost-complex space. If the form $\eta = \eta_\nu dx^\nu$ defined by*

$$\eta_\nu = \varphi_\nu{}^\varepsilon \partial_\varepsilon f \tag{1. 1}$$

is closed, then it is constant.

PROOF. As such a space always admits a positive definite Riemannian metric tensor $g_{\mu\lambda}$ which is $\mathfrak{h}$ (λ, μ), we denote by ∇_ν the operator of the Riemannian covariant derivation and make use of it. From (1. 1) we have $\nabla_\lambda f = -\varphi_\lambda{}^\varepsilon \eta_\varepsilon$. Applying $g^{\mu\lambda}\nabla_\mu$ to this we have

$$g^{\mu\lambda}\nabla_\mu \nabla_\lambda f = - g^{\mu\lambda}(\nabla_\mu \varphi_\lambda{}^\varepsilon)\eta_\varepsilon.$$

Substituting (1. 1) into the right hand side we get

$$g^{\mu\lambda}\nabla_\mu\nabla_\lambda f + g^{\mu\lambda}(\nabla_\mu \varphi_\lambda{}^\varepsilon)\varphi_\varepsilon{}^\nu \nabla_\nu f = 0,$$

which and Hopf's theorem[5] yield the Lemma. q. e. d.

This lemma and the definition of almost-analytic function lead us to

THEOREM 1. *In a compact almost-complex space an almost-analytic function is constant.*

The following lemma is also known[6]

LEMMA 2. *If $T_{(\mu)}{}^{(\lambda)} \equiv T_{\mu_q\cdots\mu_1}{}^{\lambda_p\cdots\lambda_1}$ of type (p, q) $(\neq(1,0))$ is almost-analytic, then so is $\overset{*}{T}_{(\mu)}{}^{(\lambda)}$.*

Now let $T_{(\mu)}{}^{(\lambda)}$ be pure, ρ an σ be scalar functions. If we operate Φ_ν to $\rho\, T_{(\mu)}{}^{(\lambda)} + \sigma\, \overset{*}{T}_{(\mu)}{}^{(\lambda)}$, we have easily

$$\begin{aligned}\Phi_\nu(\rho\, T_{(\mu)}{}^{(\lambda)} + \sigma\, \overset{*}{T}_{(\mu)}{}^{(\lambda)}) &= \rho\, \Phi_\nu T_{(\mu)}{}^{(\lambda)} + \sigma\, \Phi_\nu\, \overset{*}{T}_{(\mu)}{}^{(\lambda)} \\ &+ (\varphi_\nu^\varepsilon\, \partial_\varepsilon\, \rho + \partial_\nu\, \sigma) T_{(\mu)}{}^{(\lambda)} + (\varphi_\nu^\varepsilon \partial_\varepsilon\, \sigma - \partial_\nu \rho)\, \overset{*}{T}_{(\mu)}{}^{(\lambda)}.\end{aligned}$$

5) Yano, K. and S. Bochner, [7].
6) Tachibana, S., [6], Kotō, S., [3].

Taking account of the fact that $T_{(\mu)}{}^{(\lambda)}$ and $\overset{*}{T}_{(\mu)}{}^{(\lambda)}$ are linearly independent we get by virtue of Lemma 1 and Lemma 2 the following

THEOREM 2. *Let $T_{(\mu)}{}^{(\lambda)}$ of type (p, q) $(\neq(1,0))$ be almost-analytic in a compact almost-complex space. Then a necessary and sufficient condition for $\rho\, T_{(\mu)}{}^{(\lambda)} + \sigma \overset{*}{T}_{(\mu)}{}^{(\lambda)}$ to be almost-analytic is that ρ and σ are constant.*

It is known that for pure tensors $V_{(\mu)}{}^{(\lambda)}$ and $U_{(\lambda)}{}^{(\mu)}$ we have[7]

$$V_{(\mu)}^{(\lambda)}\,\Phi_\nu U_{(\lambda)}^{(\mu)} + U_{(\lambda)}^{(\mu)}\Phi_\nu V_{(\mu)}^{(\lambda)} = \varphi_\nu{}^\varepsilon\partial_\varepsilon(V_{(\mu)}^{(\lambda)}U_{(\lambda)}^{(\mu)}) - \partial_\nu(\overset{*}{V}_{(\mu)}^{(\lambda)}U_{(\lambda)}^{(\mu)}),$$

so we have

THEOREM 3. *In an almost-complex space, the scalar function defined by the inner product of two almost-analytic tensors is almost-analytic.*

2. Almost-analytic submanifolds. We consider an almost-complex space X_n and an m-dimensional submanifold X_m expressed locally by the equation

$$x^\lambda = x^\lambda(\xi^a), \qquad \text{rank } (\partial_a x^\lambda) = m,$$

where Greek indices take the values 1 to n, Latin indices the values 1 to $m(< n)$ and $\partial_a = \partial/\partial\xi^a$.

Now we assume that X_m is an almost-complex space too and denote its structure tensor by $\varphi_b{}^a$. Then we can define locally almost-analytic functions on X_m by $\varphi_b{}^a$ as analogous as in § 1. An almost-complex space X_m will be called an *almost-analytic submanifold*, if $f(x^\lambda(\xi^a))$ is always almost-analytic of ξ^a for any almost-analytic function f of x^λ. If X_n and X_m are complex manifolds, then we call an almost-analytic submanifold an analytic submanifold.

On the other hand a submanifold X_m is called an invariant subspace,[8] if its m-dimensional tangent plane at any point remains invariant under the transformation $v^\lambda \to \overset{*}{v}{}^\lambda = \varphi_\varepsilon{}^\lambda v^\varepsilon$.

Suppose that X_m be an invariant subspace. Then as m vectors $B_a{}^\lambda = \partial x^\lambda/\partial\xi^a$ $(a = 1, 2, \ldots, m)$ span the tangent plane at each point of X_m, there exists a tensor field $\varphi_b{}^a$ such that

$$\varphi_\varepsilon{}^\lambda B_a{}^\varepsilon = \varphi_a{}^c B_c{}^\lambda.$$

From this equation we can see $\varphi_c{}^a\varphi_b{}^c = -\delta_b{}^a$, so an invariant subspace becomes an almost-complex space by virtue of the induced tensor $\varphi_b{}^a$. Let f and g be a locally almost-analytic function and its associated function, respectively. Putting $F(\xi^a) = f(x^\lambda(\xi^a))$ and $G(\xi^a) = g(x^\lambda(\xi^a))$ we have

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7) Tachibana, S., [6].

8) Schouten, J. A. and K. Yano, [5].

$$\begin{aligned}\varphi_c{}^b\partial_b F &= \varphi_c{}^b B_b^\varepsilon \partial^\varepsilon f = \varphi_\lambda^\varepsilon B_c{}^\lambda \partial^\varepsilon f \\ &= B_c{}^\lambda \partial_\lambda g = \partial_c G,\end{aligned}$$

which means that F is an almost-analytic function of ξ^a. Thus we get the following

THEOREM 4. *An invariant subspace in an almost-complex space is an almost-analytic submanifold by virtue of the induced almost-complex structure.*

For a locally almost-analytic f and its associated fnuction g we consider the equations

$$f(x^\lambda) = \text{const.}, \qquad g(x^\lambda) = \text{const.}.$$

As f and g are functionally independent, they define locally an $(n-2)$-dimensional subspace. Since $v^\lambda \partial_\lambda f = v^\lambda \partial_\lambda g = 0$ imply $\overset{*}{v}{}^\lambda \partial_\lambda f = \overset{*}{v}{}^\lambda \partial_\lambda g = 0$, the subspace is invariant. Thus we have

THEOREM 5. *In an almost-complex space if an $(n-2)$-dimensional submanifold is representable locally by the equations $f(x^\lambda) =$ const., $g(x^\lambda) =$ const., where f and g be a locally almost-analytic function and its associated function respectively, then it is an almost-analytic submanifold by virtue of the induced almost-complex structure.*

Let v^λ be a (contravariant) vector field, then v^λ and $\overset{*}{v}{}^\lambda$ span an invariant plane at each point such that $v^\lambda \neq 0$. We call this field of 2-planes the distribution associated to v^λ. Now we suppose v^λ is almost-analytic, then we have $\underset{v}{\pounds}\, \varphi_\mu{}^\lambda = 0$ and

$$[v, \overset{*}{v}]^\lambda = \underset{v}{\pounds}\, \overset{*}{v}_\lambda = \underset{v}{\pounds}\, (\varphi_\varepsilon{}^\lambda v^\varepsilon) = \varphi_\varepsilon{}^\lambda \underset{v}{\pounds}\, v^\varepsilon = 0.$$

This equation means that the distribution is involutive.[9] As its plane elements are invariant we get

THEOREM 6. *The 2-dimensional distribution associated to an almost-analytic vector field v^λ is involutive and its integral manifolds are almost-analytic by virtue of the induced structure.*

3. Tensors of mixed kind on subspaces. We consider an n-dimensional differentiable manifold X_n and m-dimensional submanifold X_m expressed locally by $x^\lambda = x^\lambda(\xi^a)$. We assume the existence of $\varphi_\lambda{}^k$ and $\varphi_b{}^a$ which are tensors of type (1, 1) in X_n and X_m respectively. By a tensor of mixed kind $T_{(b)(\mu)}{}^{(a)(\lambda)} \equiv T_{b_q \ldots b_1 \mu_s \ldots \mu_1}{}^{a_p \ldots a_1 \lambda_r \ldots \lambda_1}$ we mean a quantity defined on X_m that it is a tensor field of type (r, s) in X_n for fixed indices (a) and (b) and is a tensor field of type (p, q) in X_m for fixed indices (λ) and (μ). For simplicity we call it a tensor of type $(r, s; p, q)$.

9) Chevally, C., [1].

If $T_{(b)(\mu)}{}^{(a)(\lambda)}$ commutes with $\varphi_\mu{}^\lambda$ and $\varphi_b{}^a$ in its two indices, then we say that it is pure in the indices. More precisely speaking it is pure in b_i, b_j, for simplicity we denote it $\mathfrak{p}(b_i, b_j)$, if

$$\varphi_{b_i}{}^t T_{b_q\ldots b_j\ldots t\ldots b_1(\mu)}{}^{(a)(\lambda)} = \varphi_{b_j}{}^t T_{b_q\ldots t\ldots b_i\ldots b_1(\mu)}{}^{(a)(\lambda)},$$

and is $\mathfrak{p}(b_j, \mu_i)$ if

$$\varphi_{b_j}{}^t T_{b_q\ldots t\ldots b_1(\mu)}{}^{(a)(\lambda)} = \varphi_{\mu_i}{}^\varepsilon T_{(b)\mu_s\ldots\varepsilon\ldots\mu_1}{}^{(a)(\lambda)},$$

and is $\mathfrak{p}(a_j, \mu_i)$ if

$$\varphi_t{}^{a_j} T_{(b)(\mu)}{}^{a_p\ldots t\ldots a_1(\lambda)} = \varphi_{\mu_i}{}^\varepsilon T_{(b)\mu_s\ldots\varepsilon\ldots\mu_1}{}^{(a)(\lambda)},$$

and so on.

If a tensor is pure in all pairs of its indices, then it is called a pure tensor. The vector v^b or u_c on X_m or V^λ or U_μ restricted to X_m is considered to be pure, by convention.

The tensor fields δ_μ^λ, φ_μ^λ, $\delta_b{}^a$ and $\varphi_b{}^a$ are pure. If $T_{(b)(\mu)}{}^{(a)(\lambda)}$ anti-commutes with $\varphi_\lambda{}^\mu$ and $\varphi_b{}^a$ in its two indices, then we say it is hybrid in the indices. For instance if

$$\varphi_{b_j}{}^t T_{b_q\ldots t\ldots b_1(\mu)}{}^{(a)(\lambda)} = -\varphi_\varepsilon^{\lambda_l} T_{(b)(\mu)}{}^{(a)\lambda_r\ldots\varepsilon\ldots\lambda_1}$$

hold good, then it is hybrid in b_j, λ_l. We denote this fact by $\mathfrak{h}(b_j, \lambda_l)$.

The following facts are easily seen.[10)]

If $T_{b\mu}{}^{a\lambda}$ is $\mathfrak{p}(a, b)$ (or $\mathfrak{h}(a, b)$) and also $\mathfrak{p}(\lambda, b)$ (or $\mathfrak{h}(\lambda, b)$), then it is $\mathfrak{p}(a, \lambda)$ etc..

If $T_{b\mu}{}^{a\lambda}$ is $\mathfrak{p}(a, b)$ and $\mathfrak{h}(\lambda, b)$, then it is $\mathfrak{h}(a, \lambda)$, etc..

If $T_{b\mu}{}^{a\lambda}$ and $U_{a\lambda}{}^{c\kappa}$ are $\mathfrak{p}(a, \lambda)$ (or $\mathfrak{h}(a, \lambda)$), then $T_{b\iota}{}^a U_{a\nu}{}^{c\kappa}$ is $\mathfrak{p}(\lambda, u)$, etc..

If $T_{b\iota}{}^{a\lambda}$ is $\mathfrak{p}(a,\lambda)$ and $U_{a\lambda}{}^{c\kappa}$ is $\mathfrak{h}(a, \lambda)$, then $T_{b\mu}{}^{a\lambda} U_{a\nu}{}^{c\kappa}$ is $\mathfrak{h}(\lambda, \nu)$, etc..

If $T_{b\mu}{}^{a\lambda}$ is pure in some indices, then so is $\overset{*}{T}_{b\mu}{}^{a\lambda} = T_{\iota\mu}{}^{a\lambda}\varphi_b{}^t$, and so on.

4. Φ-operator. Let X_n be a differentiable manifold admitting a tensor field $\varphi_\mu{}^\lambda$ and X_m be a submanifold represented locally by $x^\lambda = x^\lambda(\xi^a)$. An m-plane in the tangent space at a point of X_n is called invariant if it contains its image under the transformation $v^\lambda \to \overset{*}{v}{}^\lambda = \varphi_\varepsilon{}^\lambda v^\varepsilon$. If the tangent m-plane at any point of X_m is invariant, we call X_m an invariant subspace.

Let X_m be an invariant subspace. As m vectors $B_a{}^\lambda = \partial x^\lambda/\partial\xi^a$ $(a=1, 2, \ldots\ldots, m)$ in X_n span the tangent m-plane of X_m, their images are linear combinations of themselves. Hence there exists a tensor field $\varphi_b{}^a$ on X_m such that

$$\varphi_\gamma{}^\lambda B_a^\gamma = \varphi_a{}^c B_c{}^\lambda. \tag{4. 1}$$

Conversely if there exists a tensor field $\varphi_b{}^a$ satisfying (4. 1), then X_m is invariant. We shall call $\varphi_b{}^a$ satisfying (4. 1) the induced $\varphi_b{}^a$ or the induced structure.

10) cf. Tachibana, S., [6]. Kotō, S., [3].

The equation (4. 1) means that $B_a{}^\lambda$ of an invariant subspace is pure with respect to $\varphi_\mu{}^\lambda$ and the induced $\varphi_b{}^a$.

In the following we introduce an operator Φ on an invariant X_m which is associated to $\varphi_\mu{}^\lambda$ and the induced $\varphi_b{}^a$. This operator will yield from a pure tensor of type $(r, s; p, q)$ a new tensor of type $(r, s; p, q+1)$.

Now we define an operator Φ acting on a pure tensor $T_{(b)(\mu)}{}^{(a)(\lambda)}$ by the following equation,

$$
\begin{aligned}
(4.\ 2) \qquad \Phi_c T_{(b)(\mu)}{}^{(a)(\lambda)} = {} & \varphi_c{}^l \partial_l T_{(b)(\mu)}{}^{(a)(\lambda)} - \partial_c \overset{*}{T}_{(b)(\mu)}{}^{(a)(\lambda)} \\
& + \sum_{j=1}^{q} (\partial_{b_j} \varphi_c{}^l) T_{b_q \ldots l \ldots b_1 (\mu)}{}^{(a)(\lambda)} + \sum_{i=1}^{p} (\partial_c \varphi_l{}^{a_i} - \partial_l \varphi_c{}^{a_i}) T_{(b)(\mu)}{}^{a_p \ldots l \ldots a_1 (\lambda)} \\
& + \sum_{j=1}^{s} (\partial_{\mu_j} \varphi_\varepsilon{}^\sigma) B_c^\varepsilon T_{(b)\mu_s \ldots \sigma \ldots \mu_1}{}^{(a)(\lambda)} \\
& + \sum_{i=1}^{r} (\partial_\varepsilon \varphi_\sigma{}^{\lambda_i} - \partial_\sigma \varphi_\varepsilon{}^{\lambda_i}) B_c^\varepsilon T_{(b)(\mu)}{}^{(a)\lambda_r \ldots \sigma \ldots \lambda_1},
\end{aligned}
$$

where we put

$$
\overset{*}{T}_{(a)(\mu)}{}^{(b)(\lambda)} = \varphi_{b_j}{}^l T_{b_q \ldots l \ldots b_1 (\mu)}{}^{(a)(\lambda)} = \varphi_\varepsilon{}^{\lambda_j} T_{(b)(\mu)}{}^{(a)\lambda_r \ldots \varepsilon \ldots \lambda_1},
$$

etc., on taking account of the purity.

We must prove the tensor property of the operator. To do it we introduce any affine connections $\Gamma_{\mu\nu}{}^\lambda$ and $\Gamma_{bc}{}^a$ in X_n and X_m respectively and denote by ∇_ν and ∇_c the corresponding operators of covariant derivation. Of course ∇_c operates on, for instance, as follows,

$$
\begin{aligned}
\nabla_c T_b{}^a &= \partial_c T_b{}^a + T_b{}^l \Gamma_{cl}{}^a - T_l{}^a \Gamma_{cb}{}^l, \\
\nabla_c T_{b\mu}{}^\lambda &= \partial_c T_{b\mu}{}^\lambda - T_{l\mu}{}^\lambda \Gamma_{cb}{}^l + B_c^\varepsilon (T_{b\sigma}{}^\lambda \Gamma_{\varepsilon\mu}{}^\sigma - T_{b\mu}{}^\sigma \Gamma_{\varepsilon\sigma}{}^\lambda).
\end{aligned}
$$

Denoting the torsion tensors by

$$
S_{\nu\mu}{}^\lambda = (1/2)(\Gamma_{\nu\mu}{}^\lambda - \Gamma_{\mu\nu}{}^\lambda), \qquad S_{cb}{}^a = (1/2)(\Gamma_{cb}{}^a - \Gamma_{bc}{}^a)
$$

we have

$$
\begin{aligned}
\partial_{b_j} \varphi_c{}^l &= \nabla_{b_j} \varphi_c{}^l - \varphi_c{}^t \Gamma_{b_j t}{}^l + \varphi_t{}^l \Gamma_{b_j c}{}^t, \\
\partial_c \varphi_l{}^{a_i} - \partial_l \varphi_c{}^{a_i} &= \nabla_c \varphi_l{}^{a_i} - \nabla_l \varphi_c{}^{a_i} - \varphi_l{}^t \Gamma_{ct}{}^{a_i} \\
& \quad + \varphi_c{}^t \Gamma_{lt}{}^{a_i} + 2\varphi_t{}^{a_i} S_{cl}{}^t, \\
\partial_{\mu_j} \varphi_\varepsilon{}^\sigma &= \nabla_{\mu_j} \varphi_\varepsilon{}^\sigma + \varphi_\tau{}^\sigma \Gamma_{\mu_j \varepsilon}{}^\tau - \varphi_\varepsilon{}^\tau \Gamma_{\mu_j \tau}{}^\sigma, \\
\partial_\varepsilon \varphi_\sigma{}^{\lambda_i} - \partial_\sigma \varphi_\varepsilon{}^{\lambda_i} &= \nabla_\varepsilon \varphi_\sigma{}^{\lambda_i} - \nabla_\sigma \varphi_\varepsilon{}^{\lambda_i} - \varphi_\sigma{}^\tau \Gamma_{\varepsilon\tau}{}^{\lambda_i} \\
& \quad + \varphi_\varepsilon{}^\tau \Gamma_{\sigma\tau}{}^{\lambda_i} + 2\varphi_\tau{}^{\lambda_i} S_{\varepsilon\sigma}{}^\tau.
\end{aligned}
$$

Substituting these relations into (4. 2) and taking account of the purity of $B_a{}^\lambda$

and $T_{(b)(\mu)}{}^{(a)(\lambda)}$ we have

$$
(4.\ 3) \qquad \Phi_c T_{(b)(\mu)}{}^{(a)(\lambda)} = \varphi_c{}^l \Delta_l T_{(b)(\mu)}{}^{(a)(\lambda)} - \nabla_c \overset{*}{T}_{(b)(\mu)}{}^{(a)(\lambda)}
$$

$$
+ \sum_{j=1}^{q} [\nabla_{b_j}\varphi_c{}^l + 2(S_{b_j c}{}^t \varphi_t{}^l - S_{b_j}{}^{\cdot l}{}_{\cdot}\varphi_c{}^t)]\, T_{b_q \ldots l \ldots b_1 (\mu)}{}^{(a)(\lambda)}
$$

$$
+ \sum_{i=1}^{p} [\nabla_c \varphi_l{}^{a_i} - \nabla_l \varphi_c{}^{a_i} + 2(S_{cl}{}^t \varphi_t{}^{a_i} - S_{tl}{}^{a_i}\varphi_c{}^t)]\, T_{(b)(\mu)}{}^{a_p \ldots l \ldots a_1 (\lambda)}
$$

$$
+ \sum_{j=1}^{s} [\nabla_{\mu_j}\varphi^{\sigma} + 2(S_{\mu_j \varepsilon}{}^{\tau}\varphi_{\tau}{}^{\sigma} - S_{\mu_j \tau}{}^{\sigma}\varphi_{\varepsilon}{}^{\tau})] B_c{}^{\varepsilon}\, T_{(b)\mu_s \ldots \sigma \ldots \mu_1}{}^{(a)(\lambda)}
$$

$$
+ \sum_{i=1}^{r} [\nabla_{\varepsilon}\varphi_{\sigma}{}^{\lambda_i} - \nabla_{\sigma}\varphi_{\varepsilon}{}^{\lambda_i} + 2(S_{\varepsilon\tau}{}^{\tau}\varphi_{\tau}{}^{\lambda_i} - S_{\tau\sigma}{}^{\lambda_i}\varphi_{\varepsilon}{}^{\tau})]\, B_c{}^{\varepsilon} T_{(b\,(\mu)}{}^{(a)\lambda_r \ldots \sigma \ldots \lambda_1}.
$$

This shows that $\Phi_c T_{(b)(\mu)}{}^{(a)(\lambda)}$ is a tensor and hence the Φ-operator has the tensor property.

5. Some formulas. From (4. 2) we have

$$
\Phi_c T_{(b)}{}^{(a)} = \varphi_c{}^l \partial_l T_{(b)}{}^{(a)} - \partial_c \overset{*}{T}_{(b)}{}^{(a)}
$$

$$
+ \sum_{j=1}^{q} (\partial_{b_j}\varphi_c{}^l) T_{b_q \ldots l \ldots b_1}{}^{(a)} + \sum_{i=1}^{p} (\partial_c \varphi_l{}^{a_i} - \partial_l \varphi_c{}^{a_i}) T_{(b)}{}^{a_p \ldots l \ldots a_1}
$$

for a pure tensor $T_{(b)}{}^{(a)}$ in X_m, which corresponds to the Φ_ν-operator in X_n in § 1. If we consider a pure tensor $T_{(\mu)}{}^{(\lambda)}$ of X_n on X_m and operate Φ_c to it, then we have

$$
\Phi_c T_{(\mu)}{}^{(\lambda)} = B_c{}^{\varepsilon} \Phi_{\varepsilon} T_{(\mu)}{}^{(\lambda)}.
$$

We can also prove the following formula

$$
\Phi_c (V_b{}^{\varepsilon} U_{\varepsilon}{}^{\lambda}) = (\Phi_c V_b{}^{\varepsilon}) U_{\varepsilon}{}^{\lambda} + V_b{}^{\varepsilon} \Phi_c U_{\varepsilon}{}^{\lambda},
$$

whose analogous formula is valid for pure tensors of the most general type too.

As $B_a{}^{\lambda}$ is pure we can operate Φ_c to it and then we have

$$
\begin{aligned}
\Phi_c B_a{}^{\lambda} &= \varphi_c{}^l \partial_l B_a{}^{\lambda} - \partial_c(\varphi_a{}^l B_l{}^{\lambda}) + (\partial_a \varphi_c{}^l) B_l{}^{\lambda} \\
&\quad + (\partial_{\varepsilon}\varphi_{\sigma}{}^{\lambda} - \partial_{\sigma}\varphi_{\varepsilon}{}^{\lambda}) B_c{}^{\varepsilon} B_a{}^{\sigma} \\
&= \partial_a(\varphi_c{}^l B_l{}^{\lambda}) - \partial_c(\varphi_a{}^l B_l{}^{\lambda}) + (\partial_{\varepsilon}\varphi_{\sigma}{}^{\lambda} - \partial_{\sigma}\varphi_{\varepsilon}{}^{\lambda}) B_c{}^{\varepsilon} B_a{}^{\sigma}.
\end{aligned}
$$

If we differentiate $\varphi_c{}^l B_l{}^{\lambda} = \varphi_{\varepsilon}{}^{\lambda} B_c{}^{\varepsilon}$ with respect to ξ^a, then we have

$$
(5.\ 1) \qquad \partial_a(\varphi_c{}^l B_l{}^{\lambda}) - (\partial_{\sigma}\varphi_{\varepsilon}{}^{\lambda}) B_c{}^{\varepsilon} B_a{}^{\sigma} = \varphi_{\varepsilon}{}^{\lambda} \partial_a B_c{}^{\varepsilon}.
$$

Interchanging a and c in (5. 1) and substracting the equation thus obtained from (5.1) we get

$$\Phi_c B_a{}^\lambda = \Phi_c\left(\frac{\partial x^\lambda}{\partial \xi^a}\right) = 0.$$

This corresponds to $\Phi_\nu \delta_\mu{}^\lambda = 0$ in § 1.

Next as the both hand members of $\varphi_c{}^a B_a{}^\lambda = \varphi_\varepsilon{}^\lambda B_c{}^\varepsilon$ are pure, we have $B_a{}^\lambda \Phi_b \varphi_c{}^a = B_c{}^\varepsilon B_b{}^\sigma \Phi_\sigma \varphi_\varepsilon{}^\lambda$. From which we get

$$N_{bc}{}^a B_a{}^\lambda = N_{\sigma\varepsilon}{}^\lambda B_b{}^\sigma B_c{}^\varepsilon,$$

where $N_{bc}{}^a$ and $N_{\sigma\varepsilon}{}^\lambda$ are the Nijenhuis tensors of $\varphi_b{}^a$ and $\varphi_\mu{}^\lambda$ respectively. Thus we have

THEOREM 7.[11] *In a differentiable manifold admitting a tensor field $\varphi_\mu{}^\lambda$, let $\varphi_b{}^a$ be the induced structure on an invariant subspace. Then if the Nijenhuis tensor of $\varphi_\mu{}^\lambda$ vanishes, the Nijenhuis tensor of $\varphi_b{}^a$ vanishes too.*

REMARK. In the definition of Φ-operator we need not assume that $m < n$. In the case $n = m$, $x^\lambda = x^\lambda(\xi^a)$ is considered as a transformation of local coordinates.

6. Kählerian spaces. An almost-complex space is called a complex manifold if the Nijenhuis tensor vanishes identically. About such a space we get from Theorem 4 and Theorem 7 the following

THEOREM 8. *An invariant subspace in a complex manifold is an analytic submanifold by virtue of the induced structure.*

An almost-complex space is called almost-Hermitian if it is a Riemannian space and that the Riemannian metric $g_{\mu\lambda}$ is hybrid with respect to the almost-complex structure $\varphi_\mu{}^\lambda$. In such a space since the tensor field $\varphi_{\mu\lambda} = \varphi_\mu{}^\varepsilon g_{\varepsilon\lambda}$ is anti-symmetric we have the so-called fundamental form $\varphi = \varphi_{\mu\lambda} dx^\mu \wedge dx^\lambda$. A Kählerian space is an almost-Hermitian space in which $\nabla_\nu \varphi_\mu{}^\lambda = 0$ is valid, where ∇_ν denotes the operator of the Riemannian covariant derivation. It is characterized as the almost-Hermitian space such that it is a complex manifold and that the fundamental form is closed.[12]

In the following we shall only consider a Kählerian space X_n and its invariant subspace X_m. We denote by g_{cb} the induced Riemannian metric defined by $g_{cb} = g_{\mu\lambda} B_b{}^\mu B_c{}^\lambda$. As $g_{\mu\lambda}$ is hybrid and $B_b{}^\mu$ is pure, we see that g_{cb} is hybrid, on account of the arguments in § 3. Hence X_m becomes almost-Hermitian. By the relation $\varphi_{cb} = \varphi_c{}^r g_{rb} = \varphi_{\mu\lambda} B_c{}^\mu B_b{}^\lambda$ we know that the fundamental form of X_m is closed. Thus if we take account of Theorem 7 we get the well known

THEOREM 9.[13] *An invariant subspace in a Kählerian space is itself Kählerian by virtue of the induced structure.*

11) cf. Schouten, J. A. and K. Yano, [5].
12) Schouten, J. A. and K. Yano, [5], Kotō, S., [2].
13) Schouten, J. A. and K. Yano, [5].

As we have $\nabla_b \varphi_c{}^a = 0$ in an invariant X_m in a Kählerian X_n, the equation (4. 3) turns into the following simple form,

$$\Phi_c T_{(b)(\mu)}{}^{(a)(\lambda)} = \varphi_c{}^l \nabla_l T_{(b)(\mu)}{}^{(a)(\lambda)} - \nabla_c \overset{*}{T}_{(b)(\mu)}{}^{(a)(\lambda)}.$$

Hence $\Phi_c T_{(b)(\mu)}{}^{(a)(\lambda)} = 0$ for a pure tensor is equivalent to that $\nabla_c T_{(b)(\mu)}{}^{(a)(\lambda)}$ is also pure.

An infinitesimal conformal transformation v^λ in X_n is a vector field such that $\underset{v}{\mathcal{L}}\, g_{\mu\lambda} = 2\rho g_{\mu\lambda}$, where ρ is a scalar function. An infinitesimal projective transformation is a vector field v^λ such that $\underset{v}{\mathcal{L}} \left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\} = \rho_\mu \delta_\nu^\lambda + \rho_\nu \delta_\mu^\lambda$, where $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ means the Christoffel symbols and ρ_μ is necessarily gradient. Recently Y.Tashiro[14] showed that $\rho_\nu = \partial_\nu \rho$ is (covariant) almost-analytic for these transformations. This means by our terminology that ρ is an almost analytic function. So there exists a family of its associated functions $\{\sigma\}$ and if two functions of the family have a common domain of definition, then their difference is constant. Now we assume that the transformation in consideration is not homothetic i.e. that ρ is not constant. We consider a family of local submanifolds $\sigma=$const., so they define globally a family of (n$-$1)-dimensional submanifolds. The intersection of $\rho =$ const. and a submanifold of the family is invariant by accordance of the argument in § 2. Thus we have

THEOREM 10. *If a Kählerian space admits a non-homothetic infinitesimal conformal (or projective) transformation, then there exist ∞^2 $(n-2)$-dimensional invariant subspaces, each of which is an analytic submanifold and Kählerian by virtue of the induced structure.*

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14) Personal communication.