ON RIESZ SUMMABILITY FACTORS

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1. In this note we give, in Theorem A, necessary and sufficient conditions for a sequence of real numbers $\{\varepsilon_n\}$ to be such that $\Sigma a_n\varepsilon_n$ is absolutely convergent whenever Σa_n is summable (R, λ, κ), $\kappa \geqq 0$. The case $\kappa = 0$, i.e. $(R, \lambda, 0)$ equivalent to convergence, was dealt with by Fekete [3]. If we take $\lambda_n = n$ in Theorem A, then the known equivalence of (R, n, κ) and (C, κ) summability yields a result due to Bosanquet [1], Theorem 3 (the case $\rho = 0$, $\kappa > 0$).

Jurkat [6], Satz 1, has given a matrix method of summability which, with certain restrictions on λ_n, is equivalent to (R, λ, κ) summability for all $\kappa \geqq 0$. We shall employ this method of summability to give a necessary condition for $\Sigma|a_n\varepsilon_n| < \infty$ whenever Σa_n is summable (R, λ, κ).

In what follows we shall refer to a number of summability methods.

(i) Suppose that $\{\lambda_n\}$ is a sequence of non-negative numbers increasing to infinity. Define for $\kappa \geqq 0$,

$$A^\kappa(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^\kappa a_\nu .$$

If $\omega^{-\kappa}A^\kappa(\omega) \to s\ (\omega \to \infty)$, then we say that Σa_n is summable (R, λ, κ) to s.

(ii) If in (i) above ω takes only the values λ_{n+1}, then we say Σa_n is summable $(\bar{R}, \lambda, \kappa)$ to s. For $0 < \kappa \leqq 1$, Jurkat [4], Satz 2, has shown that (R, λ, κ) and $(\bar{R}, \lambda, \kappa)$ summability are equivalent.

(iii) Let $p \geqq 0$ be an integer and $\kappa = p + \theta$, $0 < \theta \leqq 1$. Define

$$C_n^0 = s_n = \sum_{m=0}^{n} a_m, \quad C^\kappa[s_m] = C_n^\kappa, \quad \text{where}$$

$$C_n^\kappa = \frac{1}{(p+1)\Gamma(\theta+1)} \sum_{m=0}^{n} \Delta_m(\lambda_{n+1} - \lambda_m)^\theta \frac{\lambda_{m+p+1} - \lambda_m}{\lambda_{m+1} - \lambda_m} C_m^p.$$

Successive applications of this last formula with $\theta = 1$ define C_n^p; a further application defines C_n^κ.

If $C^\kappa[s_m]/C^\kappa[1] \to s\ (n \to \infty)$, then we say Σa_n is summable C^κ to s.

(iv) We say Σa_n is summable $|B|$ if $\Sigma|\Delta t_n| < \infty$, where

$$t_n = \sum_{m=0}^{n} b_{n,m} a_m.$$

(v) Let $A^\kappa(\omega)$ be defined as in (i). Then Σa_n is summable $|R, \lambda, \kappa|$, $\kappa \geqq 0$, if

$$\int_{\lambda_0}^{\infty} |d\{\omega^{-\kappa} A^\kappa(\omega)\}| < \infty.$$

By $|R, \lambda, 0|$ summability we mean $\Sigma |a_n| < \infty$.

2. For the proof of Theorem A we shall require some preliminary lemmas.

LEMMA 1. *Let $\kappa = p + \theta$ ($p \geqq 0$ integral, $0 < \theta \leqq 1$), and suppose that $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ increases. For non-integral κ let $|\Delta\lambda_n|$ be monotonic and $n|\Delta\lambda_n|^\theta$ increase. Then the following are equivalent*

$$\frac{A^\kappa(\omega)}{\omega^\kappa} = o(1) \textit{ and } \frac{C^\kappa[s_m]}{C^\kappa[1]} = o(1).$$

This was proved by Jurkat [6], Satz 1.

LEMMA 2. *If $A^\kappa(\omega) = o(\omega^\kappa)$, $\kappa > 0$, then*

$$s_n = \sum_{m=0}^{n} a_m = o(\Lambda_n^\kappa), \textit{ where } \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

Lemma 2 is the limitation theorem for Riesz means (see for example Hardy and Riesz [4], Theorem 22).

LEMMA 3. *Let $\{u_n\}$ converge, $\Sigma |\alpha_n| < \infty$ and*

$$\alpha_n = \sum_{m=0}^{n} s_{n,m} u_m \quad (n=0,1,\cdots).$$

Then $\sum_{n=0}^{\infty} |s_{n,n}| < \infty$.

This follows from a result due to Chow [2], Lemma 6.

LEMMA 4. *Let A and B be normal matrices. If $\Sigma a_n \varepsilon_n$ is summable $|B|$, whenever Σa_n is summable A, then*

$$\Sigma |b_{n,n} a_{n,n}^{-1} \varepsilon_n| < \infty.$$

PROOF. Let $u_n = \sum_{m=0}^{n} a_{n,m} a_m$, $t_n = \sum_{m=0}^{n} b_{n,m} a_m \varepsilon_m$, and $\alpha_n = t_n - t_{n-1}$. Then $\{u_n\}$

is convergent, and

$$\alpha_n = b_{n,n}a_n\varepsilon_n + \sum_{m=0}^{n-1}(b_{n,m} - b_{n-1,m})a_m\varepsilon_m$$

$$\equiv \sum_{m=0}^{n} s_{n,m}u_m,$$

where

$$s_{n,n} = b_{n,n}a'_{n,n}\varepsilon_n,$$

$$s_{n,m} = b_{n,n}a'_{n,m}\varepsilon_n + \sum_{r=m}^{n-1}\varepsilon_r(b_{n,r} - b_{n-1,r})a'_{r,m}$$

for $0 \leqq m \leqq n-1$. Here $(a'_{r,m})$ is the reciprocal matrix obtained by solving for a_n in terms of u_n. Since $a'_{n,n} = a_{n,n}^{-1}$, we have on applying Lemma 3,

$$\sum_{n=0}^{\infty} |b_{n,n}a_{n,n}^{-1}\varepsilon_n| = \sum_{n=0}^{\infty} |s_{n,n}| < \infty.$$

This proves the lemma.

3. We now prove the main result.

THEOREM A. *Suppose that the conditions of Lemma* 1 *are satisfied. Then $\Sigma a_n\varepsilon_n$ is absolutely convergent whenever Σa_n is summable (R, λ, κ), $\kappa \geqq 0$, if and only if*

$$\Sigma \Lambda_n^{\kappa}|\varepsilon_n| < \infty, \quad \textit{where } \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

PROOF. *Necessity.* By Lemma 1 we may employ C^{κ} summability in place of (R, λ, κ) summability. For positive integers p, we have

$$C_n^p = \frac{1}{p}\sum_{m=0}^{n}(\lambda_{m+p} - \lambda_m)C_m^{p-1}.$$

Hence

$$C_n^1 = \sum_{m=0}^{n}(\lambda_{m+1} - \lambda_m)s_m = \sum_{m=0}^{n}(\lambda_{n+1} - \lambda_m)a_m.$$

It is readily shown by induction on r, that for integers $r \geqq 1$,

$$C_n^r = \sum_{m=0}^{n} c_{n,m}^r a_m,$$

where

(1) $$c_{n,n}^r = (\lambda_{n+r} - \lambda_n)(\lambda_{n+r-1} - \lambda_n)\cdots(\lambda_{n+1} - \lambda_n)/r!.$$

Thus it follows from (1) that

$$C_n^\kappa = \sum_{m=0}^{n} d_{n,m} a_m,$$

where

$$d_{n,m} = \frac{1}{(p+1)\Gamma(\theta+1)} \sum_{q=m}^{n} \Delta_q(\lambda_{n+1} - \lambda_q)^\theta \frac{(\lambda_{q+p+1} - \lambda_q)}{(\lambda_{q+1} - \lambda_q)} c_{q,m}^p.$$

Hence we have $d_{n,n} = (\lambda_{n+1} - \lambda_n)^\kappa / \Gamma(\kappa+1)$ if $0 < \kappa \leqq 1$, and $d_{n,n} = (\lambda_{n+1} - \lambda_n)^\theta (\lambda_{n+p+1} - \lambda_n)(\lambda_{n+p} - \lambda_n) \cdots (\lambda_{n+2} - \lambda_n)/(p+1)!\ \Gamma(\theta+1)$ if $\kappa > 1$.

Let us now consider the C^κ transform of Σa_n:

$$C_n^\kappa / C^\kappa[1] = \sum_{m=0}^{n} a_{n,m} a_m, \text{ where } a_{n,n} = d_{n,n}/C^\kappa[1].$$

Since $\Sigma |a_n \varepsilon_n| < \infty$, whenever Σa_n is summable C^κ, we have by Lemma 4 that

$$\Sigma |a_{n,n}^{-1} \varepsilon_n| < \infty. \tag{2}$$

Now it was shown by Jurkat in the proof of Lemma 1 that

$$0 < a \leqq \frac{C^\kappa[1]}{\lambda_{n+1}^\kappa} \leqq A, \qquad a, A \text{ constants.} \tag{3}$$

Also since Λ_n increases we have $|\Delta\lambda_n| = O(1)|\Delta\lambda_{n-1}|$. Hence for $r = 2, 3, \cdots, p+1$,

$$\frac{\lambda_{n+r} - \lambda_n}{\lambda_{n+1} - \lambda_n} = 1 + \frac{|\Delta\lambda_{n+1}|}{|\Delta\lambda_n|} + \cdots + \frac{|\Delta\lambda_{n+r-1}|}{|\Delta\lambda_n|} = O(1),$$

$$(\lambda_{n+p+1} - \lambda_n) \cdots (\lambda_{n+2} - \lambda_n) = O(1)(\lambda_{n+1} - \lambda_n)^p. \tag{4}$$

Thus by (2), (3) and (4) we have

$$\begin{aligned}
\Sigma \Lambda_n^\kappa |\varepsilon_n| &= \Sigma \Lambda_n^\kappa \frac{C^\kappa[1]}{d_{n,n}} \cdot \frac{d_{n,n}}{C^\kappa[1]} |\varepsilon_n| \\
&= \Sigma |a_{n,n}^{-1} \varepsilon_n| \frac{\lambda_{n+1}^\kappa}{C^\kappa[1]} \frac{d_{n,n}}{(\lambda_{n+1} - \lambda_n)^\kappa} \\
&\leqq a^{-1} \Sigma |a_{n,n}^{-1} \varepsilon_n| (\lambda_{n+1} - \lambda_n)^\theta O(1)(\lambda_{n+1} - \lambda_n)^{p-\kappa} \\
&= O(1) \Sigma |a_{n,n}^{-1} \varepsilon_n| < \infty.
\end{aligned}$$

This proves the necessity. We note that the above proof can be used to establish a rather stronger result. For let us take, in Lemma 4, $b_{n,m} = (1 - \lambda_m/\lambda_{n+1})^\mu$, $\mu > 0$. Then if $\Sigma a_n \varepsilon_n$ is summable $|R, \lambda, \mu|$ it is summable $|B|$. Hence by Lemma 4,

$$\Sigma \Lambda_n^{-\mu} |a_{n,n}^{-1} \varepsilon_n| < \infty. \tag{5}$$

By (2), (3),(4) and (5) we then have

$$\Sigma \Lambda_n^{\kappa-\mu} |\varepsilon_n| < \infty. \tag{6}$$

Thus (6) is necessary for $\Sigma a_n \varepsilon_n$ to be summable $|R, \lambda, \mu|$, $\mu \geqq 0$, whenever Σa_n is summable C^κ.

We note that when $0 < \kappa \leqq 1$, no restriction on λ_n is required, since C^κ is equivalent to $(\bar{R}, \lambda, \kappa)$, which is equivalent to (R, λ, κ) by an earlier remark (Section 1,(ii)).

Sufficiency. Suppose that $\Sigma \Lambda_n^\kappa |\varepsilon_n| < \infty$. Since Σa_n is summable (R, λ, κ) and Λ_n increases, we have by Lemma 2,

$$a_n = o(\Lambda_n^\kappa + \Lambda_{n-1}^\kappa) = o(\Lambda_n^\kappa).$$

Hence

$$\Sigma |a_n \varepsilon_n| = O(1) \quad \Sigma \Lambda_n^\kappa |\varepsilon_n| < \infty.$$

This proves the theorem.

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