NOTE ON INFINITESIMAL TRANSFORMATIONS OVER CONTACT MANIFOLDS

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Introduction. A contact manifold is an odd dimensional differentiable manifold M^{2n+1} with a 1-form η over M^{2n+1} such that $\eta \wedge (d\eta)^n \neq 0$. If an infinitesimal transformation X satisfies $\mathfrak{L}(X)\eta = \sigma\eta$, where $\mathfrak{L}(X)\eta$ denotes the Lie derivative of η by X and σ is a differentiable function over M^{2n+1}, then X is called an infinitesimal (strict if $\sigma = 0$) contact transformation.

On the other hand, if there exist tensor fields ϕ^i_j, ξ^i and η_j such that

$$\begin{aligned} &\eta_i\xi^i = 1, \\ &\text{rank } |\phi^i_j| = 2n, \\ &\phi^i_j\xi^j = 0, \ \eta_j\phi^j_i = 0, \\ &\phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_k, \end{aligned}$$

then we say that M^{2n+1} has (ϕ,ξ,η)-structure. Moreover, if a positive definite Riemannian metric g satisfies the following conditions,

$$\begin{aligned} \eta_i &= g_{ij}\xi^j, \\ g_{ij}\phi^i_h\phi^j_k &= g_{hk} - \eta_h\eta_k, \end{aligned}$$

then g is called an associated Riemannian metric to the (ϕ,ξ,η)-structure. M^{2n+1} with (ϕ,ξ,η)-structure admitting an associated Riemannian metric is said to have (ϕ,ξ,η,g)-structure.

S. Sasaki [1] and Y. Hatakeyama [2] proved that if $\eta = \eta_j dx^j$ is a 1-form defining a contact structure, then we can find a differentiable (ϕ,ξ,η,g)-structure such that η_i is the one given by the coefficients of the 1-form η and

$$g_{ih}\phi^h_j = \phi_{ij} = \partial_i\eta_j - \partial_j\eta_i.$$

A tensor ϕ^i_j has a similar property to the fundamental tensor F^i_j of a (or an almost) complex structure in some sense.

In a compact Kähler manifold, it is known (e.g. A. Lichnérowicz [3]) that an infinitesimal analytic transformation X ($\mathfrak{L}(X)F^i_j = 0$) is an infinitesimal isometry under the additional condition that X leaves the volume element, or Chern's 2-form invariant.

The main purpose of this report is to show that, in a compact contact manifold, if X leaves ϕ^i_j invariant, then X is an infinitesimal isometry and at the same time an infinitesimal strict contact transformation.

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1. Elementary properties of infinitesimal transformations. In this report, tensors ϕ_{ij}, ϕ^i_j, ξ^i, η_j and g_{ij} are those of the (ϕ,ξ,η,g)-structure associated to a given contact structure.

We begin with some simple Lemmas.

LEMMA 1. *If an infinitesimal transformation X satisfies $[\mathfrak{L}(X)\eta]_j = 0$, then X satisfies $[\mathfrak{L}(X)\phi]_{ij} = 0$.*

PROOF. First we notice that

$$d\eta = \frac{1}{2}\phi_{ij}dx^i \wedge dx^j, \tag{1. 1}$$

and that the Lie differentiation and exterior differentiation are commutative. Then we have easily $[\mathfrak{L}(X)\phi]_{ij} = 0$.

LEMMA 2. *If an infinitesimal transformation X satisfies $[\mathfrak{L}(X)\eta]_j = 0$, then X satisfies $[\mathfrak{L}(X)\xi]^i = 0$.*

PROOF. By taking the Lie derivative of $(\phi_{ij} + \eta_i\eta_j)\xi^j = \eta_i$ we get

$$(\phi_{ij} + \eta_i\eta_j)[\mathfrak{L}(X)\xi]^j = 0.$$

Lemma 2 follows from this, since $(\phi_{ij} + \eta_i\eta_j)$ is regular.

PROPOSITION 1. *Let M^{2n+1} be a contact manifold. If an infinitesimal transformation X over M^{2n+1} satisfies two of the following conditions, then X satisfies also the remainning one:*

$$[\mathfrak{L}(X)\eta]_i = 0, \tag{1. 2}$$

$$[\mathfrak{L}(X)\phi]^i_j = 0, \tag{1. 3}$$

$$X: \textit{infinitesimal isometry (Killing vector)}. \tag{1. 4}$$

PROOF. (i) [(1. 2) and (1. 3) $\rightarrow$ (1. 4)].
Next relations hold good

$$[\mathfrak{L}(X)\eta]_j = [\mathfrak{L}(X)(g\xi)]_j = [\mathfrak{L}(X)g]_{ij}\xi^i + g_{ij}[\mathfrak{L}(X)\xi]^i.$$

So, by Lemma 2, we get

$$[\mathfrak{L}(X)g]_{ij}\xi^i = 0. \tag{1. 5}$$

On the other hand,

$$[\mathfrak{L}(X)\phi]_{ik} = [\mathfrak{L}(X)g]_{ij}\phi^j_k + g_{ij}[\mathfrak{L}(X)\phi]^j_k.$$

So, by Lemma 1, we get

(1. 6) $$[\mathfrak{L}(X)g]_{ij}\phi^j_k = 0.$$

Two relations (1. 5) and (1. 6) yield

$$[\mathfrak{L}(X)g]_{ij}(\phi^j_k + \xi^j\eta_k) = 0.$$

(1. 4) follows from this.

(ii) [(1. 3) and (1. 4) → (1. 2)]. From

$$[\mathfrak{L}(X)(\eta\phi)]_j = [\mathfrak{L}(X)\eta]_i\phi^i_j + \eta_i[\mathfrak{L}(X)\phi]^i_j,$$

and (1. 3), we see that

(1. 7) $$[\mathfrak{L}(X)\eta]_i\phi^i_j = 0.$$

So, X is seen to be an infinitesimal contact transformation. Moreover, if we consider $\mathfrak{L}(X)(\eta\xi) = 0$, using (1. 4) we have

(1. 8) $$[\mathfrak{L}(X)\eta]_i\xi^i = 0.$$

Hence, by (1. 7) and the last equation, we have (1. 2).

(iii) [(1. 4) and (1. 2) → (1. 3)].

This is clear. (q.e.d.)

As we see from the proof of [(1. 3) and (1. 4) → (1. 2)], if an infinitesimal contact transformation X is an infinitesimal isometry, then X is necessarily an infinitesimal strict contact transformation.

However, more precisely we have

PROPOSITION 2. *Let M^{2n+1} be a contact manifold. If an infinitesimal contact transformation X preserves the volume element of the associated Riemannian metric, then X is an infinitesimal strict contact transformation.*

PROOF. The volume element of M^{2n+1} (See [1]) is given by

(1. 9) $$dv = (-1)^{\frac{n(n+1)}{2}}\frac{(2n+1)\,!}{2^n n\,!}\eta_{[1}\phi_{23}\phi_{45}\cdots\phi_{2n2n+1]}\,dx^1\wedge dx^2\wedge\cdots\wedge dx^{2n+1}.$$

By the two relations $\mathfrak{L}(X)\eta = \sigma\eta$ and

(1.10) $$[\mathfrak{L}(X)\phi]_{ij} = \partial_i\sigma\eta_j - \partial_j\sigma\eta_i + \sigma\phi_{ij},$$

where the last equation follows from $\mathfrak{L}(X)d\eta = d\mathfrak{L}(X)\eta = d(\sigma\eta)$, we have

$$(\mathfrak{L}(X)\zeta)_{12\cdots 2n+1} = (n+1)\sigma\zeta_{12\cdots 2n+1},$$

where we have put

$$\zeta_{12\cdots 2n+1} = (-1)^{\frac{n(n+1)}{2}}\frac{(2n+1)\,!}{2^n n\,!}\eta_{[1}\phi_{23}\phi_{45}\cdots\phi_{2n2n+1]}.$$

Thus, Proposition 2 is valid.

2. Separation Tensors. In a (or an almost) complex manifold M^{2n} with complex structure F_j^i, if a tensor, for example T_{rs}^l, satisfies

$$O_{ij}^{rs}T_{rs}^l = T_{ij}^l(\text{or} = 0),\ O_{ij}^{rs} = \frac{1}{2}(\delta_i^r\delta_j^s - F_i^rF_j^s),$$

then T_{ij}^l is said to be pure (or hybrid) in two indices i and j (K. Yano [4]). With respect to the adapted frame (A. Lichnérowicz [5]) pure and hybrid tensors are expressed respectively as follows:

$$\begin{pmatrix} T_{\lambda\mu} & 0 \\ 0 & T_{\bar\lambda\bar\mu} \end{pmatrix}, \begin{pmatrix} 0 & T_{\lambda\bar\mu} \\ T_{\bar\lambda\mu} & 0 \end{pmatrix}. \qquad (\lambda,\mu = 1,2,\cdots,n : \bar\lambda = n + \lambda)$$

As for a differentiable manifold M^{2n+1} with (ϕ,ξ,η,g)-structure, M^{2n+1} has an orthonormal adapted frame $(\xi_{(\lambda)},\xi_{(\bar\lambda)},\xi_\Delta\ ;\ \lambda = 1,\cdots,\ n : \bar\lambda = n + \lambda)$ such that

$$\xi_{(\bar\lambda)}^i = \phi_j^i\xi_{(\lambda)}^j,\ \xi_\Delta^i = \xi^i. \qquad (\Delta = 2n + 1)$$

If we operate a linear transformation to this frame given by

$$\begin{pmatrix} \frac{1}{\sqrt{2}}E^n & -\frac{i}{\sqrt{2}}E^n & 0 \\ \frac{1}{\sqrt{2}}E^n & \frac{i}{\sqrt{2}}E^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_\lambda \\ \xi_{\bar\lambda} \\ \xi_\Delta \end{pmatrix} = \begin{pmatrix} e_\lambda \\ e_{\bar\lambda} \\ \xi \end{pmatrix}$$

where E^n denotes a unit matrix of order n, and $i^2 = -1$, then the new frame $(e_\lambda, e_{\bar\lambda}, \xi)$ satisfies the relations

$$\phi e_\lambda = ie_\lambda,\ \phi e_{\bar\lambda} = -ie_{\bar\lambda},\ \phi\xi = 0.$$

Thus, with respect to this frame, ϕ takes the following form:

$$\begin{pmatrix} iE^n & 0 & 0 \\ 0 & -iE^n & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We call a tensor T_{ij} (N_{ij}) pure (resp. hybrid) if it has components of the following form with respect to this frame

$$T_{ij} = \begin{pmatrix} T_{\lambda\mu} & 0 & 0 \\ 0 & T_{\bar\lambda\bar\mu} & 0 \\ 0 & 0 & T_{\Delta\Delta} \end{pmatrix},\ N_{ij} = \begin{pmatrix} 0 & N_{\lambda\bar\mu} & N_{\lambda\Delta} \\ N_{\bar\lambda\mu} & 0 & N_{\bar\lambda\Delta} \\ N_{\Delta\mu} & N_{\Delta\bar\mu} & 0 \end{pmatrix}.$$

In order to determine an operator analogous to O_{ij}^{rs} of the complex case, we start to decompose vector fields X, Y as

$$X^i = (\delta^i_k - \xi^i\eta_k)X^k + \xi^i(\eta_k X^k),$$
$$Y^j = (\delta^j_l - \xi^j\eta_l)Y^l + \xi^j(\eta_l Y^l).$$

Then the tensor T_{ij} can be decomposed as

$$\begin{aligned} T_{kl}X^kY^l &= (T'_{kl} + T^*_{kl} + T''_{kl})X^kY^l \\ &= T_{ij}(\delta^i_k - \xi^i\eta_k)(\delta^j_l - \xi^j\eta_l)X^kY^l + T_{ij}S^{ij}_{kl}X^kY^l + T_{ij}\xi^i\eta_k\xi^j\eta_l X^kY^l, \end{aligned}$$

where we have put

$$S^{ij}_{kl} = \xi^i\eta_k\delta^j_l + \delta^i_k\xi^j\eta_l - 2(\xi^i\eta_k)(\xi^j\eta_l).$$

As $2n$-dimensional subspace of the tangent space at arbitrary point of M^{2n+1}, which is orthogonal to ξ, has an induced almost complex structure ϕ^i_j, the pure part of T'_{kl} is easily seen to be

$$\frac{1}{2}(\delta^k_p\delta^l_q - \phi^k_p\phi^l_q)T'_{kl},$$

therefore, the pure part of T_{ij} is

$$\begin{aligned} &\frac{1}{2}[(\delta^k_p\delta^l_q - \phi^k_p\phi^l_q)(\delta^i_k - \xi^i\eta_k)(\delta^j_l - \xi^j\eta_l) + 2\xi^i\eta_p\xi^j\eta_q]T_{ij} \\ &= \frac{1}{2}[(\phi^i_r\phi^r_p)T_{ij}(\phi^j_s\phi^s_q) + 2(\xi^i\eta_p)T_{ij}(\xi^j\eta_q) - \phi^i_pT_{ij}\phi^j_q]. \end{aligned}$$

We define P^{ij}_{pq} and H^{ij}_{pq} as follows and call them the pure part separation tensor and the hybrid part separaton tensor

$$P^{ij}_{pq} = \frac{1}{2}[(\phi^i_r\phi^r_p)(\phi^j_s\phi^s_q) + 2(\xi^i\eta_p)(\xi^j\eta_q) - \phi^i_p\phi^j_q], \tag{2. 1}$$

$$H^{ij}_{pq} = \delta^i_p\delta^j_q - P^{ij}_{pq}. \tag{2. 2}$$

Furthermore, we define

$$R^{ij}_{pq} = \frac{1}{2}[(\phi^i_r\phi^r_p)(\phi^j_s\phi^s_q) + \phi^i_p\phi^j_q], \tag{2. 3}$$

$$S^{ij}_{pq} = \delta^i_p\delta^j_q - (\phi^i_r\phi^r_p)(\phi^j_s\phi^s_q) - (\xi^i\eta_p)(\xi^j\eta_q), \tag{2. 4}$$

then we have

$$R(T) = \begin{pmatrix} 0 & T_{\bar{\lambda}\mu} & 0 \\ T_{\bar{\lambda}\mu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S(T) = \begin{pmatrix} 0 & 0 & T_{\lambda\Delta} \\ 0 & 0 & T_{\bar{\lambda}\Delta} \\ T_{\Delta\mu} & T_{\Delta\bar{\mu}} & 0 \end{pmatrix}.$$

Obviously, following relations hold good

$$
\left\{
\begin{aligned}
&P^2 = P,\ H^2=H,\ R^2=R,\ S^2=S,\\
&P + H = \text{Identity},\ R + S = H,\ RH = R,\ SH = S,\\
&PH = HP = 0,\ RS = SR = 0.
\end{aligned}
\right. \tag{2. 5}
$$

$$P^{ij}_{pi}\phi^p_j = \phi^i_q.\ P^{ij}_{pq}\phi_{ij} = 0. \tag{2. 6}$$

PROPOSITION 3. *For any infinitesimal transformation X over M^{2n+1}, we have* $P^{ij}_{pq}[\mathfrak{L}(X)\phi]^p_j = 0$.

PROOF. This will be seen by simple calculation.

Let's call a differentiable function σ such that $\mathfrak{L}(X)\eta = \sigma\eta$, an associated function to an infinitesimal contact transformation X.

THEOREM 1. *Let M^{2n+1} be a contact manifold and X be an infinitesimal contact transformation. Then, concerning an associated function σ, the following four conditions are mutually equivalent*:

$$\sigma = constant, \tag{2. 7}$$

$$[\mathfrak{L}(X)\phi]_{ij} = \sigma\phi_{ij}, \tag{2. 8}$$

$$[\mathfrak{L}(X)\xi]^i = -\sigma\xi^i, \tag{2. 9}$$

$$S^{ij}_{pq}[\mathfrak{L}(X)\phi]^p_j = 0. \tag{2.10}$$

PROOF. (i) [(2. 7)$\rightarrow$(2. 8)].
From (1.10) and (2. 7) we have (2. 8).

(ii) [(2. 8)$\rightarrow$(2. 9)].

By $\mathfrak{L}(X)(\eta\xi) = 0$, we get

$$\eta_i[\mathfrak{L}(X)\xi]^i = -\sigma, \tag{2.11}$$

and by $[\mathfrak{L}(X)(\phi\xi)]_j = 0$, we get

$$\phi_{ji}[\mathfrak{L}(X)\xi]^i = -[\mathfrak{L}(X)\phi]_{ji}\xi^i = 0. \tag{2.12}$$

(2.11) multiplied by ξ^j and (2.12) yield

$$(\phi^j_i + \xi^j\eta_i)[\mathfrak{L}(X)\xi]^i = -\sigma\xi^j.$$

Therefore, we have (2. 9).

(iii) [(2. 9)$\rightarrow$(2.10)].

As we can easily show that

$$
\begin{aligned}
S^{ij}_{pq} &= \delta^i_p\delta^j_q - (\phi^i_r\phi^r_p)(\phi^j_s\phi^s_q) - (\xi^i\eta_p)(\xi^j\eta_q)\\
&= \xi^i\eta_p\delta^j_q + \delta^i_p\xi^j\eta_q - 2(\xi^i\eta_p)(\xi^j\eta_q),
\end{aligned}
$$

we get

$$S^{ij}_{pq}[\mathfrak{L}(X)\phi]^p_j = [\mathfrak{L}(X)\phi]^i_j\xi^j\eta_q + [\mathfrak{L}(X)\phi]^p_q\xi^i\eta_p - 2[\mathfrak{L}(X)\phi]^p_j\eta_p(\xi^i\xi^j\eta_q).$$

If we notice that

$$\eta_p[\mathfrak{L}(X)\phi]^p_q = [\mathfrak{L}(X)(\eta\phi)]_q - [\mathfrak{L}(X)\eta]_p\phi^p_q = 0,$$

we can see that (2.10) is equivalent to $[\mathfrak{L}(X)\phi]^i_j\xi^j = 0$.

On the other hand, under the condition (2. 9), the relation

$$[\mathfrak{L}(X)(\phi\xi)]^i = [\mathfrak{L}(X)\phi]^i_j\xi^j - \phi^i_j(\sigma\xi^j) = 0$$

reduces to $[\mathfrak{L}(X)\phi]^i_j\xi^j = 0$. This shows (2.10).

(iv) [(2.10) $\to$ (2. 7)].

(2.10) is equivalent to $[\mathfrak{L}(X)\phi]^i_j\xi^j = 0$, and the latter is equivalent to $\phi^i_j[\mathfrak{L}(X)\xi]^j = 0$, and also to $\phi_{ij}[\mathfrak{L}(X)\xi]^j = 0$, furthermore to $[\mathfrak{L}(X)\phi]_{ij}\xi^j = 0$. By virtue of (1.10) and the last relation, we have

$$\partial_i\sigma = (\xi^j\partial_j\sigma)\eta_i = (\mathfrak{L}(\xi)\sigma)\eta_i. \tag{2.13}$$

Therefore, we have $d\sigma \wedge \eta = 0$.

And if we take the exterior derivative, $d\sigma \wedge d\eta = 0$, that is $(\mathfrak{L}(\xi)\sigma)\eta \wedge d\eta = 0$. However, η is a contact form, and so $\eta \wedge (d\eta)^n$ does not vanish. Consequently, $\mathfrak{L}(\xi)\sigma = 0$.

Then (2.13) means that σ is constant. (q.e.d.)

REMARK: (2.10) is equivalent to $R^{ij}_{pq}[\mathfrak{L}(X)\phi]^p_j = [\mathfrak{L}(X)\phi]^i_q$, this follows from Proposition 3 and (2. 5).

EXAMPLE: Let $E^{2n+1} - O$ be a $(2n+1)$-dimensional Euclidean space removed one point (origin O) and $(x^\lambda, y^\lambda, z : \lambda = 1, \cdots, n)$ be its coordinate system, then $\eta = dz - \sum_{\lambda=1}^{n} y^\lambda dx^\lambda$ defines a contact structure. We define an infinitesimal transformation X of the form

$$X = \alpha\sum_{\lambda=1}^{n} x^\lambda \frac{\partial}{\partial x^\lambda} + \beta\sum_{\lambda=1}^{n} y^\lambda \frac{\partial}{\partial y^\lambda} + \sigma z\frac{\partial}{\partial z} \qquad (\alpha,\beta,\sigma : \text{constant},\ \alpha+\beta=\sigma).$$

Then X is an infinitesimal contact transformation such that the associated function is σ (constant).

Just the similar consideration as in the proof of Theorem 1 leads us to the following

PROPOSITION 4. *If an infinitesimal contact transformation X leaves ϕ_{ij} invariant, then X is an infinitesimal strict contact transformation.*

3. Compact case

LEMMA 3. *Let M^{2n+1} be a contact manifold. If an infinitesimal transformation X leaves ϕ^i_j invariant, then X is an infinitesimal contact transformation such that the associated function σ is constant.*

PROOF. By the relation

$$[\mathfrak{L}(X)(\eta\phi)]_j = [\mathfrak{L}(X)\eta]_i\phi_j^i + \eta_i[\mathfrak{L}(X)\phi]_j^i,$$

we have $[\mathfrak{L}(X)\eta]_i\phi_j^i = 0$. This means that X is an infinitesimal contact transformation. And since $S_{pq}^{ij}[\mathfrak{L}(X)\phi]_j^p = 0$, the associated function to X must be constant.

LEMMA 4. *Let M^{2n+1} be a compact contact manifold. For every infinitesimal contact transformation X, the associated function σ to X satisfies*

$$\int_{M^{2n+1}} \sigma \, dv = 0.$$

PROOF. We denote by ∇_k the covariant derivative with respect to the Riemannian connection defined by g, then we have

$$\begin{aligned}\sigma &= \xi^k[\mathfrak{L}(X)\eta]_k \\ &= \xi^k(\nabla_l\eta_k X^l + \nabla_k X^l\eta_l) \\ &= \xi^k\nabla_l\eta_k X^l + \nabla_k(\xi^k X^l\eta_l) - \nabla_k\xi^k X^l\eta_l - \xi^k X^l\nabla_k\eta_l \\ &= \xi^k(\partial_l\eta_k - \partial_k\eta_l)X^l + \nabla_k(\xi^k X^l\eta_l) \\ &= \nabla_k(\xi^k X^l\eta_l),\end{aligned}$$

because, we have $\nabla_k\xi^k = 0$, by virtue of $\mathfrak{L}(\xi)\eta = 0$, Lemma 1 and (1. 9). Since a contact manifold is always orientable, we see by Green's Theorem that

$$\int_{M^{2n+1}} \sigma \, dv = 0.$$

Lemma 4 tells us that the differentiable function σ over a compact M^{2n+1} vanishes on some compact subset. Especially, σ cannot be a non-zero constant. Thus, we have

PROPOSITION 5. *Let M^{2n+1} be a compact contact manifold. If an infinitesimal contact transformation X satisfies one of the following conditions, then X is an infinitesimal strict contact transformation:*

$$[\mathfrak{L}(X)\phi]_{ij} = \sigma\phi_{ij}, \tag{2. 8}$$

$$[\mathfrak{L}(X)\xi]^i = -\sigma\xi^i, \tag{2, 9}$$

$$S_{pq}^{ij}[\mathfrak{L}(X)\phi]_j^p = 0. \tag{2.10}$$

THEOREM 2. *Let M^{2n+1} be a compact contact manifold. In order that an infinitesimal transformation X leaves ϕ_j^i invariant, it is necessary and sufficient that X is an infinitesimal isometry and at the same time an infinitesimal strict contact transformation.*

PROOF. (Necessity). By virtue of Lemma 3, X is an infinitesimal contact transformation such that the associated function σ is constant. Lemma 4 shows that σ must be zero over M^{2n+1}, so X is an infinitesimal strict contact transformation. Consequently, by Proposition 1, X is an infinitesimal isometry.

(Sufficiency). If X is an infinitesimal isometry and strict contact transformation, then by Proposition 1 again, X leaves ϕ^i_j invariant. (q.e.d.)

COROLLARY. *Let M^{2n+1} be a (compact) contact manifold. Any non-zero vector field X which belongs to the $2n$-dimensional distribution orthogonal to ξ with respect to the associated Riemannian metric g cannot satisfiy the relation* $[\mathfrak{L}(X)\phi]^i_j = 0$.

Here we notice that compactness in Theorem 2 cannot be omitted. To see this, we need only to consider an infinitesimal transformotion X over $E^{2n+1} - O$ (former example) which is given as follows:

$$X = \frac{\sigma}{2}\sum_{\lambda=1}^{n} x^\lambda \frac{\partial}{\partial x^\lambda} + \frac{\sigma}{2}\sum_{\lambda=1}^{n} y^\lambda \frac{\partial}{\partial y^\lambda} + \sigma z \frac{\partial}{\partial z} \qquad (\sigma : \text{const.}).$$

This X satisfies $[\mathfrak{L}(X)\eta]_i = \sigma\eta_i$,

$$[\mathfrak{L}(X)\phi]^i_j = 0,$$

where

$$\phi^i_j = \begin{pmatrix} 0 & \delta^\lambda_\mu & 0 \\ -\delta^\lambda_\mu & 0 & 0 \\ 0 & y^\mu & 0 \end{pmatrix}$$

with respect to the coordinate system $(x^\lambda, y^\lambda, z)$.

4. Linear transformation ψ^i_j. The sets of all infinitesimal isometries and of all infinitesimal (strict) contact transformations over any contact manifold constitute Lie algebras with respect to the usual bracket operation. We give them the notations L_i and L_c (${}^sL_c$) respectively. This is the same about the set of all infinitesimal transformations (L_ϕ) whose element leaves ϕ^i_j invariant, because

$$\mathfrak{L}([X,Y])\phi = \{\mathfrak{L}(X)\mathfrak{L}(Y) - \mathfrak{L}(Y)\mathfrak{L}(X)\}\phi = 0 \qquad X,Y \in L_\phi.$$

Then, Theorem 2 says that, if M^{2n+1} is compact

$$L_\phi = {}^sL_c \cap L_i.$$

The linear transformation ψ^i_j defined bellow, is regular:

$$\psi^i_j = \phi^i_j + \xi^i\eta_j. \tag{4. 1}$$

LEMMA 5. *An infinitesimal transformation X of the form $\alpha\xi$ is an infinitesimal contact transformation, if and only if α is constant.*

PROOF. As we can easily see

$$[\mathfrak{L}(\alpha\xi)\eta]_i = \partial_i\alpha + \alpha[\mathfrak{L}(\xi)\eta]_i.$$

The second term of the right hand side vanishes, and we have $[\mathfrak{L}(\alpha\xi)\eta]_i = \partial_i\alpha$. If $[\mathfrak{L}(\alpha\xi)\eta]_i$ is proportional to η_i, then $\phi^i_j\partial_i\alpha = 0$. By the same argument as in the proof of Theorem 1 [(2.10) $\to$ (2.7)], we can see that α is constant.

Conversely if α is constant, then $\alpha\xi$ is an infinitesimal (strict) contact transformation.

THEOREM 3. *Let M^{2n+1} be a contact manifold and X be an element of L_c. In order that ψX belongs to L_c again, it is necessary and sufficient that X is of the form $\alpha\xi$ where α is constant.*

PROOF. (Necessity). By assumption, there exist differentiable functions σ,ρ over M^{2n+1} such that

$$[\mathfrak{L}(X)\eta]_k = \sigma\eta_k, \ [\mathfrak{L}(\psi X)\eta]_k = \rho\eta_k.$$

From the first relation, it follows

$$\eta_j\partial_k X^j = \sigma\eta_k - \partial_j\eta_k X^j. \tag{4. 2}$$

By (4. 1), the second one is rewritten as

$$[\mathfrak{L}(\phi X)\eta]_k + [\mathfrak{L}\{\xi(\eta X)\}\eta]_k = \rho\eta_k. \tag{4. 3}$$

And

$$\begin{aligned}
[\mathfrak{L}(\phi X)\eta]_k &= \partial_r\eta_k\phi^r_j X^j + \partial_k(\phi^r_j X^j)\eta_r \\
&= \partial_r\eta_k\phi^r_j X^j - (\phi^r_j X^j)\partial_k\eta_r \\
&= (\partial_r\eta_k - \partial_k\eta_r)\phi^r_j X^j \\
&= \phi_{rk}\phi^r_j X^j = (g_{kj} - \eta_k\eta_j)X^j. \\
[\mathfrak{L}\{\xi(\eta X)\}\eta]_k &= \partial_r\eta_k\xi^r\eta_j X^j + \partial_k(\xi^r\eta_j X^j)\eta_r \\
&= \partial_k\eta_j X^j + \eta_j\partial_k X^j \\
&= \partial_k\eta_j X^j + \sigma\eta_k - \partial_j\eta_k X^j \quad (\text{by}(4.2)) \\
&= \phi_{kj}X^j + \sigma\eta_k.
\end{aligned}$$

Putting these two relations into (4. 3)

$$[\mathfrak{L}(\psi X)\eta]_k = (g_{kj} - \eta_k\eta_j)X^j + \phi_{kj}X^j + \sigma\eta_k.$$

Contracting with g^{ki}, we get

$$\rho\xi^i = X^i - \xi^i\eta_j X^j + \phi^i_j X^j + \sigma\xi^i,$$

therefore

$$X^i = -\phi^i_j X^j + \xi^i(\eta_j X^j + \rho - \sigma). \tag{4. 4}$$

If we operate ψ^k_i to (4. 4), we have

$$\phi^k_i X^i + \xi^k\eta_i X^i = (\delta^k_j - \xi^k\eta_j)X^j + \xi^k(\eta_j X^j + \rho - \sigma)$$

$$= X^k + \xi^k(\rho - \sigma).$$

Eliminating $\phi^i_j X^j$ from (4. 4) by the last equation, we have

$$X^i = \xi^i(\eta_l X^l).$$

By Lemma 5, $\alpha = \eta_l X^l$ must be constant, this completes the proof of necessity. (Sufficiency) This is clear. (q.e.d.)

COROLLARY. $\quad {}^sL_c \cap \psi^s L_c = L_c \cap \psi L_c.$

THEOREM 4. *Let M^{2n+1} be a contact manifold. If ξ is an infinitesimal isometry, then $L_\phi \cap \psi L_\phi = (\alpha\xi : \alpha \in R)$ where R is a real line). If ξ is not so, then $L_\phi \cap \psi L_\phi = (0)$.*

PROOF. It is easy to see that every element X of L_ϕ is an infinitesimal contact transformation. If ψX belongs to L_ϕ, then ψX is also an infinitesimal contact transformation. Therefore, by the last Theorem, X must be of the form $\alpha\xi$ (α: constant), this means

$$L_\phi \cap \psi L_\phi \subset (\alpha\xi : \alpha \in R).$$

Now, we suppose that ξ is an infinitesimal isometry, then $\alpha\xi$ (α: const.) is also an infinitesimal isometry, and also an infinitesimal strict contact transformation. Hence, by Proposition 1, $\alpha\xi \in L_\phi$. Here, if we notice that $\alpha\xi$ is invariant under the linear transformation ψ^i_j, we have

$$L_\phi \cap \psi L_\phi = (\alpha\xi : \alpha \in R).$$

Next we suppose that ξ is not so, then for any real number ($\alpha \neq 0$), $\alpha\xi$ does not belong to L_ϕ. Thus, we have

$$L_\phi \cap \psi L_\phi = (0). \quad \text{(q.e.d.)}$$

LEMMA 6. *Suppose ξ be an infinitesimal isometry. An infinitesimal transformation X of the form $\alpha\xi$ belongs to L_ϕ, if and only if α is constant.*

PROOF. Because, by $[\mathfrak{L}(\xi)\phi]^i_j = 0$, we get

$$[\mathfrak{L}(\alpha\xi)\phi]^i_j = -\partial_k\alpha\phi^k_j\xi^i.$$

From $\phi^k_j\partial_k\alpha = 0$ it follows that α is constant. (q.e.d.)

Of course, Lemma 6 follows also from Lemma 5 or Theorem 4. Above Lemma 6 and subsequent part of this section are appendices.

The most specialized contact manifold may be a normal contact manifold which is characterized by

(4. 5) $\qquad \xi$: infinitesimal isometry,

(4. 6) $$P_{pq}^{ij}\nabla_k\phi_{ij}=0 \quad (\text{i.e.}\nabla_k\phi_{ij} \text{ is hybrid w.r.t. } i \text{ and } j).$$

LEMMA 7. *If ξ is an infinitesimal isometry, then* $\nabla_k\eta_l=\frac{1}{2}\phi_{kl}$.

PROOF. By assumption $\nabla_k\eta_l+\nabla_l\eta_k=0$ and $\phi_{kl}=\nabla_k\eta_l-\nabla_l\eta_k$, then Lemma 7 is clear.

PROPOSITION 6. *Let M^{2n+1} be a normal contact manifold. Then the following important formula holds good:* [6]

(4. 7) $$\nabla_k\phi_{ij}=\frac{1}{2}(\eta_i g_{jk}-\eta_j g_{ik}).$$

PROOF. By the condition (4. 6),

$$(\phi_r^i\phi_p^r)\nabla_k\phi_{ij}(\phi_s^j\phi_q^s)+2(\xi^i\eta_p)\nabla_k\phi_{ij}(\xi^j\eta_q)-\phi_p^i\nabla_k\phi_{ij}\phi_q^j=0.$$

Evidently, the second term vanishes, and the third term is equal to the first term, because

$$\begin{aligned}-\phi_p^i\nabla_k\phi_{ij}\phi_q^j &= -\phi_p^i\nabla_k(\phi_{ij}\phi_q^j)+\phi_p^i\phi_{ij}\nabla_k\phi_q^j\\ &= -\phi_p^i\nabla_k(-g_{iq}+\eta_i\eta_q)+(g_{pj}-\eta_p\eta_j)\nabla_k\phi_q^j\\ &= -\phi_p^i\nabla_k\eta_i\eta_q+\nabla_k\phi_{pq}-\eta_p\eta_j\nabla_k\phi_q^j\\ &= \nabla_k\phi_{pq}-\nabla_k\phi_{pi}\xi^i\eta_q-\nabla_k\phi_{iq}\eta_p\xi^i\\ &= (-\delta_p^i+\xi^i\eta_p)\nabla_k\phi_{ij}(-\delta_q^j+\xi^j\eta_q).\end{aligned}$$

Namely, (4. 6) is eqivalent to

(4. 8) $$\nabla_k\phi_{ij}(\delta_p^i-\xi^i\eta_p)(\delta_q^j-\xi^j\eta_q)=0.$$

$$\begin{aligned}\nabla_k\phi_{pq}-\nabla_k\phi_{pi}\xi^i\eta_q-\nabla_k\phi_{iq}\eta_p\xi^i &= \nabla_k\phi_{pq}+\phi_{pi}\nabla_k\xi^i\eta_q+\phi_{iq}\eta_p\nabla_k\xi^i\\ &= \nabla_k\phi_{pq}-\frac{1}{2}\phi_p^i\phi_{ki}\eta_q+\frac{1}{2}\phi_q^i\eta_p\phi_{ki} \qquad (\text{Lemma 7})\\ &= \nabla_k\phi_{pq}-\frac{1}{2}(\eta_p g_{qk}-\eta_q g_{pk}).\end{aligned}$$

Therefore, (4. 8) is equivalent to (4. 7), under the condition (4. 5). (q.e.d.)

As an application we reprove directly the following:

Let M^{2n+1} be a normal contact manifold. Then, $L_\phi\cap\psi L_\phi=(\alpha\xi:\alpha\in R)$.

PROOF. The condition $X\in L_\phi$ is written as

(4. 9) $$\nabla_k X^m\phi_l^k=X^k\nabla_k\phi_l^m+\nabla_l X^k\phi_k^m.$$

Straightforward calculation of $\mathfrak{L}_{(\phi X+\xi(\eta X))}\phi$ yields

$$(-\phi_m^k\nabla_j X^m\phi_l^j+\phi_m^j\nabla_l X^m\phi_j^k-\xi^k\eta_m\nabla_j X^m\phi_l^j)$$

$$+ (\nabla_j \phi^k_l \phi^j_m - \nabla_j \phi^k_m \phi^j_l - \xi^k \nabla_j \eta_m \phi^j_l + \nabla_l \phi^j_m \phi^k_j) X^m.$$

In the first bracket, $\nabla_j X^m$ can be eliminated by (4. 9), and after arranging by virtue of Lemma 7 and (4. 7), we have

$$\phi_{lm} X^m - g_{lm} X^m + \eta_l \eta_m X^m = 0.$$

This means

$$X^i = (\phi^i_m + \xi^i \eta_m) X^m.$$

Therefore, X is invariant by ψ, and of the form $\alpha\xi$ $(\alpha = (\eta X))$. Furthermore, α is constant by Lemma 6, Thus we have

$$L_\phi \cap \psi L_\phi \subset (\alpha\xi : \alpha \in R).$$

Since the converse is known by Lemma 6, Proof is over.

5. Conformal and projective Killing vectors. An infinitesimal transformation X is said to be a conformal Killing vector, if X satisfies

(5. 1) $$[\mathfrak{L}(X)g]_{ij} = 2\lambda g_{ij}$$

where λ is a differentiable function over M^{2n+1}.

THEOREM 5. *A conformal Killing vector X over a contact manifold M^{2n+1} is an infinitesimal isometry, if X leaves invariant any one of the four tensors η_i, ξ^j, ϕ^i_j and ϕ_{jk}.*

PROOF. (i) $[[\mathfrak{L}(X)\eta]_i = 0 \rightarrow \lambda = 0]$.

By assumption, we have

(5. 2) $$\eta_i [\mathfrak{L}(X)\xi]^i = 0.$$

And by the relation

$$[\mathfrak{L}(X)\eta]_i = [\mathfrak{L}(X)(g\xi)]_i = 2\lambda g_{ij}\xi^j + g_{ij}[\mathfrak{L}(X)\xi]^j$$

we have

$$[\mathfrak{L}(X)\xi]^j = -2\lambda\xi^j.$$

(5. 2) and the last relation shows $\lambda = 0$.

(ii) $[[\mathfrak{L}(X)\xi]^j = 0 \rightarrow \lambda = 0]$.

Proof for this will be done by similar way.

(iii) $[[\mathfrak{L}(X)\phi]^i_j = \rightarrow \lambda = 0]$.

By Lemma 3, there exists a constant σ such that

$$[\mathfrak{L}(X)\eta]_i = \sigma\eta_i, \ [\mathfrak{L}(X)\xi]^j = -\sigma\xi^j.$$

By taking the Lie derivative of $\eta_i = g_{ij}\xi^j$, we have $\lambda = \sigma$.

On the other hand, we have

$$[\mathfrak{L}(X)\phi]_{ij} = [\mathfrak{L}(X)g]_{ik}\phi_j^k + g_{ik}[\mathfrak{L}(X)\phi]_j^k = 2\lambda\phi_{ij}.$$

Therefore, by (1.10) $2\lambda\phi_{ij} = \sigma\phi_{ij}$, that is $\lambda = 0$.

(iv) $[[\mathfrak{L}(X)\phi_{ij} = 0 \rightarrow \lambda = 0]$.

We can see easily that X is an infinitesimal contact transformation. And so, by Proposition 4, X is an infintesimal strict contact transformation. Consequently, this case is reduced to (i). (q.e.d.)

LEMMA 8. *In a contact manifold, we have*

$$\nabla_i\phi_j^i = -n\eta_j. \tag{5. 3}$$

PROOF. Easily we have

$$(\nabla_i\phi_j^i + n\eta_j)\xi^j = -\phi_j^i\nabla_i\xi^j + n = -\frac{1}{2}\phi^{ij}(\nabla_i\eta_j - \nabla_j\eta_i) + n \tag{5. 4}$$

$$= -\frac{1}{2}\phi^{ij}\phi_{ij} + n = 0,$$

and

$$(\nabla_i\phi_j^i + n\eta_j)\phi_l^j = \nabla_i(-\delta_l^i + \xi^i\eta_l) - \phi_j^i\nabla_i\phi_l^j = -\phi^{ij}\nabla_i\phi_{jl},$$

Using $\nabla_i\phi_{jl} + \nabla_j\phi_{li} + \nabla_l\phi_{ij} = 0$, we have

$$-\phi^{ij}\nabla_i\phi_{jl} = \phi^{ij}\nabla_j\phi_{li} + \phi^{ij}\nabla_l\phi_{ij} = -\phi^{ij}\nabla_j\phi_{il} = \phi^{ij}\nabla_i\phi_{jl} = 0.$$

Thus, we have

$$(\nabla_i\phi_j^i + n\eta_j)\phi_l^j = 0. \tag{5. 5}$$

From (5. 4) and (5. 5), Lemma 8 follows.

An infinitesimal transformation X is said to be a projective Killing vector, if it satisfies

$$[\mathfrak{L}(X)\Gamma]_{jk}^i = \delta_j^i\mu_k + \delta_k^i\mu_j, \tag{5. 6}$$

where Γ_{jk}^i is the Christoffel's symbol and

$$\mu_k = \frac{1}{2n+2}\partial_k(\nabla_r X^r).$$

THEOREM 6. *If a projective Killing vector X over a contact manifold M^{2n+1} leaves ϕ_j^i invariant, then X is an infinitesimal strict contact transformation and therefore an infinitesimal isometry.*

PROOF. By the well known formula (e.g.K.Yano [4]), we get

$$[\mathfrak{L}(X)(\nabla\phi)]_{jh}^i - \nabla_j[\mathfrak{L}(X)\phi]_h^i = [\mathfrak{L}(X)\Gamma]_{jr}^i\phi_h^r - [\mathfrak{L}(X)\Gamma]_{jh}^r\phi_r^i.$$

The second term of the left hand side vanishes, and contracting with respect to i and j, we have

$$(5.\ 7)\qquad [\mathfrak{L}(X)(\nabla\phi)]^{i}_{ih} = [\mathfrak{L}(X)\Gamma]^{i}_{ir}\phi^{r}_{h} - [\mathfrak{L}(X)\Gamma]^{r}_{ih}\phi^{i}_{r}.$$

By Lemma 8, the left hand side is equal to $-n[\mathfrak{L}(X)\eta]_h$, moreover by Lemma 3 the latter is equal to $-n\sigma\eta_h$ for some constant σ. Then, by virtue of (5. 6), (5. 7) reduces to

$$-n\sigma\eta_h = (2n+1)\mu_r\phi^{r}_{h}.$$

That is, $\sigma = 0$. So, X is an infinitesimal strict contact transformation and, by Proposition 1, also an infinitesimal isometry.

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