HOMOGENEOUS CONTACT TRANSFORMATIONS

SHIGEO SASAKI

(Received July 5, 1962)

Introduction. Let M^n be an n-dimensional differentiable manifold of class C^∞. Take a point x of M^n and consider the set F_x of all non-zero covectors at x. Then, F_x with the natural topology is homeomorphic with $F = E^n - O$, where E^n is a Euclidean space and O is a point of E^n. We can easily see that the set

$$^cT(M^n) = \bigcup_{x \in M^n} F_x$$

with the natural topology is a fibre bundle with the standard fibre F and the structural group $GL(n, R)$. We shall call this fibre bundle as the *cotangent bundle of* M^n.

In this paper, I want so show that cotangent bundles play an important role for the study of homogeneous contact transformations of differentiable manifolds. The classical Laguerre's geometry of $(n-1)$-spheres in E^n can be regarded as a geometry of $^cT(E^n) \approx E^n \times F$ under a certain group of homogeneous contact transformations and the classical Lie's higher $(n-1)$-sphere geometry in E^n can be regarded as a geometry of $^cT(S^n)$ under a certain group of homogeneous contact transformations, where S^n is the n-dimensional sphere. Therefore, it is natural to study $^cT(M^n)$ in connection with homogeneous contact transformations of M^n.

On the background of Lie's works L.P. Eisenhart [4] [5] [6] founded the theory of homogeneous contact transformations of a differentiable manifold M^n in 1929 and T.Hosokawa [7], K.Yano [2] [9] [10], Y. Mutô [8] [9], T.C.Doyle [3], E.T.Davies [1] [2] and others followed him. From our stand point of view, their theories are local theories of $^cT(M^n)$ or tensor calculus of $2n$ dimensional manifolds under local contact coordinate transformations. It seems to me that their theories can be understood the meaning well by studying the cotangent bundle $^cT(M^n)$ globally.

1. Homogeneous contact transformations. Let M^n be a differentiable

About half of this paper was done when the author was a visiting professor of the National Taiwan University and stayed at the Academia Sinica, Nankang, Formosa from Oct. 1961 to March 1962.

manifold of class C^∞ and ${}^cT(M^n)$ be its cotangent bundle. We denote by π the natural projection

$$\pi : {}^cT(M^n) \to M^n.$$

Every point z of ${}^cT(M^n)$ can be expressed as a pair (x, p), where $x = \pi z$ and p is a covector at x. We shall call p as *the covector of* z. We sometimes call the pair (x, p) as an *element* in M^n.

We take an open covering of M^n by coordinate neighborhoods $\{U_\lambda\}$ $(\lambda \in \Lambda)$ and denote local coordinates in U_λ by x^i_λ. If we denote the components of a covector p at any point $x_{(\lambda)}$ in U_λ with respect to the natural frame $\dfrac{\partial}{\partial x^i_{(\lambda)}}$ at $x_{(\lambda)}$ by $p^{(\lambda)}_i$, then the set of all $(x^i_{(\lambda)}, p^{(\lambda)}_i)$ $(x^i_{(\lambda)} \in U_\lambda, p^{(\lambda)}_i \in F)$ constitutes local coordinates in $\pi^{-1}(U_\lambda)$. This mapping

$$\pi^{-1}(U_\lambda) \to U_\lambda \times F$$

is a diffeomorphism and its inverse mapping is usually denoted by ϕ_λ. So, we have

$$z = \phi_\lambda(x_{(\lambda)}, p^{(\lambda)}).$$

We denote the map which transfers z to $p^{(\lambda)}$ by ρ_λ. Then, we have

$$\rho_\lambda : \pi^{-1}(U_\lambda) \to F,$$

$\{\pi^{-1}(U_\lambda)\}(\lambda \in \Lambda)$ is an open covering of ${}^cT(M^n)$ by coordinate neighborhoods.

On every neighborhood $\pi^{-1}(U_\lambda)$ $(\lambda \in \Lambda)$ of ${}^cT(M^n)$ we consider a 1-form

$$\eta_{(\lambda)} \equiv p^{(\lambda)}_i dx^i. \tag{1. 1}$$

As the right hand side is a scalar, it does not depend upon the coordinate transformation. So, the set of all η_λ $(\lambda \in \Lambda)$ constitutes a global 1-form η over ${}^cT(M^n)$. We shall call η as the *homogeneous contact form* of M^n.

A diffeomorphism

$$f : {}^cT(M^n) \to {}^cT(M^n)$$

is said to be a *homogeneous contact transformation* of M^n if and only if f leaves invariant the 1-form η, i.e.

$$f^*\eta = \eta, \tag{1. 2}$$

where f^* is the dual map induced by f on differential forms over ${}^cT(M^n)$. f is denoted by

$$\bar{z} = f(z), \qquad z \in {}^cT(M^n)$$

or by

$$(\bar{x}, \bar{p}) = f(x, p),$$

where $z = (x, p)$ and $\bar{z} = (\bar{x}, \bar{p})$.

From the definition, we can immediately see that the following theorem is true.

THEOREM 1.1 *The totality of homogeneous contact transformations of a differentiable manifold M^n forms a group.*

EXAMPLE. Suppose f_0 be a diffeomorphism of M^n onto itself. Then f_0 naturally induces a diffeomorphism f of the cotangent bundle ${}^cT(M^n)$ onto itself. It is easy to see that f is a homogeneous contact transformation. This map f is called to be an *extension* of the diffeomorphism f_0 of M^n.

THEOREM 1.2 *A homogeneous contact transformation f of ${}^cT(M^n)$ onto itself is an extension of a diffeomorphism of M^n onto itself if and only if f is a fibre preserving map.*

The proof is easy.

2. Analytic expressions of homogeneous contact transformations. Let U be a coordinate neighborhood of M^n with local coordinates x^i. We denote components of a covector p at a point x of U with respect to the natural frame by p_i. For the sake of convenience, we now put

$$x^{n+i} \equiv x^{i^*} \equiv p_i, \qquad i^*=n+i \tag{2. 1}$$

and consider $x^\lambda = (x^i, x^{n+i})$ $(\lambda, \mu = 1,\cdots,2n)$ as local coordinates of ${}^cT(M^n)$ in $\pi^{-1}(U)$. Then the contact form η over ${}^cT(M^n)$ can be written as

$$\eta = \eta_\lambda dx^\lambda \tag{2. 2}$$

in $\pi^{-1}(U)$, where we have put

$$\eta_\lambda = (p_i, 0). \tag{2. 3}$$

η_λ determines a $(2n-1)$-dimensional vector subspace of the tangent space of ${}^cT(M^n)$ at (x, p) containing the tangent space of the fibre at the point.

Now, we consider the exterior differential $d\eta$ of the contact form η. In $\pi^{-1}(U)$, it is given by

$$d\eta = \frac{1}{2}\,\phi_{\lambda\mu} dx^\lambda \wedge dx^\mu, \tag{2. 4}$$

where we have put

$$\phi_{\lambda\mu} = \partial_\lambda \eta_\mu - \partial_\mu \eta_\lambda,\ \partial_\lambda = \frac{\partial}{\partial x^\lambda}\,. \tag{2. 5}$$

We can easily see that $(\phi_{\lambda\mu})$ has the following numerical components:

$$(\phi_{\lambda\mu}) = \begin{pmatrix} 0 & -\delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}. \tag{2. 6}$$

The entity which has components of the form (2. 6) for every coordinate neighborhood $\pi^{-1}(U)$ of an open covering of ${}^cT(M^n)$ is a skew-symmetric tensor field of ${}^cT(M^n)$. Geometrically, it gives a null-system in every tangent space of ${}^cT(M^n)$. We shall call it the *fundamental null-system* of ${}^cT(M^n)$.

We define $\phi^{\lambda\mu}$ for every coordinate neighborhood of the type $\pi^{-1}(U)$ in ${}^cT(M^n)$ by

$$\phi^{\lambda\mu}\phi_{\mu\nu} = -\delta^{\lambda}_{\nu}, \tag{2. 7}$$

then $\phi^{\lambda\mu}$'s are components of a skew-symmetric tensor field over ${}^cT(M^n)$ and are given by

$$(\phi^{\lambda\mu}) = \begin{pmatrix} 0 & -\delta^{ij} \\ \delta^{ij} & 0 \end{pmatrix}. \tag{2. 8}$$

We define also

$$\xi^{\lambda} = \phi^{\lambda\mu}\eta_{\mu} \tag{2. 9}$$

in every $\pi^{-1}(U)$, then ξ^{λ} defines a vector field over ${}^cT(M^n)$. Its components in $\pi^{-1}(U)$ are rewritten as

$$\xi^{\lambda} = (0, p_i). \tag{2.10}$$

We can easily see that (2. 9) is equivalent to

$$\phi_{\lambda\mu}\xi^{\mu} = -\eta_{\lambda}. \tag{2.11}$$

Now, suppose f be a homogeneous contact transformation of M^n. For every pair of coordinate neighborhoods U and $\overline{U}$ with local coordinates x^i and $\overline{x}^a$ such that $f(\pi^{-1}(U)) \cap \pi^{-1}(\overline{U})$ is not empty, the restriction map $f|\pi^{-1}(U) \cap f^{-1}(\pi^{-1}(\overline{U}))$ can be expressed analytically by

$$\overline{x}^{\lambda} = \overline{x}^{\lambda}(x), \tag{2.12}$$

i.e.

$$\overline{x}^a = \overline{x}^a(x, p), \ \overline{p}_a = \overline{p}_a(x, p). \tag{2.13}$$

The condition $f^*\eta = \eta$ i.e.

$$\eta_{\alpha}(\overline{x})\partial_{\lambda}\overline{x}^{\alpha} = \eta_{\lambda}(x) \tag{2.14}$$

can be written as

$$\overline{p}_a\partial_i\overline{x}^a = p_i, \ \overline{p}_a\partial^i\overline{x}^a = 0 \ \left(\partial^i = \frac{\partial}{\partial p_i}\right). \tag{2.15}$$

From (2.14), we get easily

$$\phi_{\alpha\beta}\partial_\lambda \bar{x}^\alpha \partial_\mu \bar{x}^\beta = \phi_{\lambda\mu}. \tag{2.16}$$

Of course $\phi_{\lambda\mu}$ and $\phi_{\alpha\beta}$ are numerical constants. Contracting $\phi^{\mu\nu}\bar{\partial}_\gamma x^\lambda \left(\bar{\partial}_\gamma = \dfrac{\partial}{\partial \bar{x}^\gamma}\right)$ with (2.16) we get

$$\phi_{\gamma\delta}\partial_\mu \bar{x}^\delta \phi^{\mu\nu} = -\bar{\partial}_\gamma x^\nu. \tag{2.17}$$

Contracting $\phi^{\alpha\gamma}\partial_\nu \bar{x}^\beta$ with the last equation we then get

$$\phi^{\alpha\beta} = \phi^{\lambda\mu}\partial_\lambda \bar{x}^\alpha \partial_\mu \bar{x}^\beta. \tag{2.18}$$

From (2.16) we get the following

THEOREM 2.1. *The fundamental null-system of the cotangent bundle ${}^cT(M^n)$ is transformed to itself by homogeneous contact transformations.*

Now, from the definition we have

$$\xi^\alpha(\bar{x}) = \phi^{\alpha\beta}\eta_\beta(\bar{x}).$$

Putting (2.14) and (2.18) into the last equation, and making use of (2. 9) we get

$$\xi^\alpha(x) = \partial_\lambda \bar{x}^\alpha \xi^\lambda(x). \tag{2.19}$$

The last equation gives an important theorem:

THEOREM 2.2. *The functions $\bar{x}^a(x,p)$ and $\bar{p}_a(x,p)$ of a homogeneous contact transformation* (2.13) *are homogoneous of degree* 0 *and* 1 *respectively with respect to p_i.*

PROOF. By virtue of (2.10), (2.19) is easily seen to be equivalent with

$$p_i\partial^i \bar{x}^a = 0, \qquad p_i\partial^i \bar{p}_a = \bar{p}_a, \tag{2.20}$$

which show that $\bar{x}^a$'s and $\bar{p}_a$'s are homogeneous of degree 0 and 1 with respect to p_i.

Two points $z = (x,p)$ and $z' = (x,p')$ on the same fibre F_x of ${}^cT(M^n)$ are said to be equivalent if and only if there exists a constant $\rho \neq 0$ such that

$$p'_i = \rho p_i. \tag{2.21}$$

We call an equivlaence class in F_x as a *coray at x*. The vector ξ^λ defined in (2.10) is geometrically the tangent vector of the coray. Theorem 2.2 can now be expressed geometrically as follows:

THEOREM 2.3. *Every homogeneous contact transformation of M^n is a coray preserving diffeomorphism of ${}^cT(M^n)$.*

THEOREM 2.4. *Let $\bar{x}^a$ and $\bar{p}_a$ in* (2.13) *are functions which define a homogeneous contact transformation. Then, we have*

$$(2.22)\qquad \begin{cases} \partial_i\bar{x}^a = \bar{\partial}^a p_i, & \partial^i\bar{x}^a = -\bar{\partial}^a x^i, \\ \partial_i\bar{p}_a = -\bar{\partial}_a p_i, & \partial^i\bar{p}_a = \bar{\partial}_a x^i. \end{cases}$$

PROOF. By virtue of (2.17), we have

$$(2.23)\qquad \phi^{\mu\lambda}\partial_\mu\bar{x}^\alpha = \phi^{\alpha\gamma}\bar{\partial}_\gamma x^\lambda.$$

We can easily see that the last equation is equivalent with (2.22).

Suppose

$$X = (X^i,X^{i^*}) = (X^i,P_i),$$
$$Y = (Y^i,Y^{i^*}) = (Y^i,Q_i)$$

be components of vector fields over ${}^cT(M^n)$. Then, since the numerical components (2. 8) of the tensor field $\phi_{\lambda\mu}$ are kept invariant under homogeneous contact transformations, we see that

$$(2.24)\qquad \phi_{\lambda\mu}X^\lambda Y^\mu = -X^iQ_i + Y^iP_i$$

is an invariant under homogeneous contact transformations. Especially, if we take ξ^λ and X^λ instead of X^λ and Y^λ, we see that

$$(2.25)\qquad \phi_{\lambda\mu}\xi^\lambda X^\mu = \eta_\mu X^\mu = p_iX^i$$

is an invariant under homogeneous contact transformations.

If U is a function defined over ${}^cT(M^n)$, then

$$(2.26)\qquad \partial_\lambda U = (\partial_i U,\ \partial^i U)$$

is a covector field over ${}^cT(M^n)$. However,

$$(2.27)\qquad \phi^{\lambda\mu}\partial_\mu U = (-\partial^i U,\ \partial_i U)$$

is a vector field over ${}^cT(M^n)$.

Let U,V be differentiable functions defined over ${}^cT(M^n)$. We define the so-called Poisson's bracket of U and V by

$$(2.28)\qquad (U,V) = \phi^{\lambda\mu}\partial_\lambda U\partial_\mu V = \partial^i U\partial_i V - \partial_i U\partial^i V.$$

Then, (U,V) is also a function defined over ${}^cT(M^n)$. It is evident that if U,V are invariant under homogeneous contact transformation f, then (U,V) is also invariant under f.

3. Fundamental varieties. Suppose that

$$f:\ {}^cT(M^n) \to {}^cT(M^n)$$

be a homogeneous contact transformation. Denoting the fibre of ${}^cT(M^n)$ at a point x of M^n by F_x, we put

$$(3.\ 1)\qquad S_x = \pi\circ f(F_x),$$
$$(3.\ 2)\qquad \bar{S}_x = \pi\circ f^{-1}(F_x),$$

and call S_x and $\bar{S}_x$ as the *fundamental varieties* at x of f and f^{-1} respectively. If f is an extension of a diffeomorphism f of M^n onto itself, then it is evident that all fundamental varieties of f and f^{-1} reduce to points.

THEOREM 3.1. *If a point y belongs to S_x. then the point x belongs to $\bar{S}_y$. The converse is also true.*

PROOF. $y \in S_x$ means that $y \in \pi\circ f(F_x)$ and it is equivalent to $F_y \cap f(F_x) \neq \phi$. The last equation can be written also as $f^{-1}(F_y) \cap F_x \neq \phi$, and so it is equivalent to $\pi\circ f^{-1}(F_y) \ni x$. Therefore, $x \in \bar{S}_y$. The converse can be proved easily by the process inverse to the above.

COROLLARY. (i) *If $y \in M^n$, then*

$$\bar{S}_y = \{x \mid y \in S_x\}. \tag{3. 3}$$

(ii) *If $x \in M^n$, then*

$$S_x = \{y \mid x \in \bar{S}_y\}. \tag{3. 4}$$

Now, from $(2.15)_2$ we can see that the rank of the matrix $(\partial^i \bar{x}_a)$ is smaller than $n-1$. Geometrically, it is nothing but the number of linearly independent tangent vectors at $\bar{x} = \pi\circ f(x,p)$. We shall call it *rank* of S_x at the point $\bar{x}$. So, it is independent upon the choice of coordinate neighborhoods. The variety S_x may have singularities in the sense that at some points the rank of S_x is less than that of generic points on S_x. We can see that

$$\dim S_x = \max_{\bar{x}\in S_x} \{\text{rank of } S_x \text{ at } \bar{x}\}.$$

For every point $z = (x,p)$ of ${}^cT(M^n)$ we make correspond an integer r_f by

$$r_f(z) = \text{rank of } S_x \text{ at } \bar{x} = \pi\circ f(z). \tag{3. 5}$$

Then, we get an integral valued function r_f over ${}^cT(M^n)$ such that

$$0 \leqq r_f \leqq n-1. \tag{3. 6}$$

We call r_f as the *rank function of the first kind* of the homogeneous contact transformation f.

THEOREM 3.2. *The necessary and sufficient condition that a homogeneous contact transformation f of M^n is an extension of a diffeomorphism of M^n is that the rank function of the first kind r_f of f is identically equal to zero.*

PROOF. Necessity. If f is an extension of a diffeomorphism of M^n, then S_x is a point. So r_f *is equal to zero.*

Sufficiency. As S_x is arcwise connected, if $r_f \equiv 0$, then every S_x reduces to a point. So, f is a fibre-preserving diffeomorphism of ${}^cT(M^n)$. Hence, by Theorem 1.3 we can see that f is an extension of a diffeomorphism of M^n.

In the next place, we fix a point z in ${}^cT(M^n)$ and take a coordinate neighborhood U_λ of $\{U_\lambda\}$ $(\lambda \in \Lambda)$ such that $f(z) \in \pi^{-1}(U_\lambda)$. We denote the subset of indices of Λ which satisfy the last property by Λ_z. For every $\lambda \in \Lambda_z$ we put

$$R_{\lambda,\pi(z)} = \rho_\lambda \{f(F_{\pi(z)}) \cap \pi^{-1}(U_\lambda)\}, \tag{3. 7}$$

$$r'_f(z,U_\lambda) = \text{rank of } R_{\lambda,\pi(z)} \text{ at } \rho_\lambda \circ f(z). \tag{3. 8}$$

Analytically, if we denote the coordinate neighborhood of the point z by $\pi^{-1}(U)$ $(\pi z \in U)$ and denote f restricted to $\pi^{-1}(U) \cap f_{\circ}^{-1}\pi^{-1}(U_\lambda)$ by

$$x^a_{(\lambda)} = x^a_{(\lambda)}(x,p),\ p^{(\lambda)}_a = p^{(\lambda)}_a(x,p),$$

then we see easily that

$$r'_f(z,U) = \text{rank } (\partial^i \overline{p}^{(\lambda)}_a) \text{ at } (x,p).$$

However, contrary to the rank of S_x at a point of $S_x, r'_f(z,U_\lambda)$ depends upon the choice of coordinate neighborhoods. So, we define $r'_f(z)$ by

$$r'_f(z) = \max_{\lambda\in\Lambda} r'_f(z,U_\lambda). \tag{3. 9}$$

If we vary z over ${}^cT(M^n)$, we get again an integral valued function r'_f over ${}^cT(M^n)$ depending upon f and such that

$$0 \leqq r'_f \leqq n. \tag{3.10}$$

The function r'_f generally depends upon the open covering $\{U_\lambda\}$ of M^n. However, if we consider a covering which contains all possible fine neighborhoods and all possible coordinate systems in them, then r'_f is a well defined function over ${}^cT(M^n)$ by the homogeneous contact transformation f. We shall call r'_f the *rank function of the second kind* of the homogeneous contact transformation f.

THEOREM. 3.3 *At every point z of ${}^cT(M^n)$ and for every homogeneous contact transformation f of M^n, we have*

$$r_f(z) + r'_f(z) \geqq n. \tag{3.11}$$

PROOF. We denote the tangent space of ${}^cT(M^n)$ at $f(z)$ by $T_{f(z)}$ and U_λ be a coordinate neighborhood such that $\pi\circ f(z) \in U_\lambda$. Then, the maximal subspace V (vertical space) of $T_{f(z)}$ such that every vector of V is mapped to zero by π and the maximal subspace H_λ of $T_{f(z)}$ such that every vector of H_λ is mapped to zero by ρ_λ are disjoint and complementary.

Now, the dimension of $f(F_{\pi(z)})$ at $f(z)$ is n. If $\dim S_{\pi\circ f(z)}$ is $n-s$ at $\pi\circ f(z)$, then the s-dimensional subspace of $T_{f(z)}$ which is spanned by s independent vectors of $T_{f(z)}$ such that each of them is mapped to zero by π is mapped to s-dimensional subspace of the standard fibre F by ρ_λ. Therefore, the dimension of $R_{\lambda,\pi(z)}$ is at least s. Hence, we get (3.11).

As we have proved it in Theorem 3.1, if a point $\overline{x}$ belongs to S_x, then the point x belong to $\bar{S}_{\bar{x}}$. Any pair of points x and $\bar{x}$ which are in such relation is called *to be in the relation S*.

Now, suppose $\bar{M}^n$ is a copy of M^n and we consider the point $\bar{x}$ as a point in $\bar{M}^n$. Then, the set Σ of all pairs $(x,\bar{x})$ in the relation S can be regarded as a submanifold of $M^n \times \bar{M}^n$. It may have some singularities. It is clear that

$$\Sigma = \bigcup_{x \in M^n} \{x, S_x\} = \bigcup_{\bar{x} \in \bar{M}^n} \{\bar{S}_{\bar{x}}, \bar{x}\}. \tag{3.12}$$

When $(x_0,\bar{x}_0)$ belongs to Σ, we take coordinate neighborhoods U of x_0 and $\bar{U}$ of $\bar{x}_0$ and we express the homogeneous contact tranrformation f restricted to $\pi^{-1}(U) \cap f^{-1}(\pi^{-1}(\bar{U}))$ by

$$\bar{x}^a = \bar{x}^a(x,p), \quad \bar{p}_a = \bar{p}_a(x,p). \tag{3.13}$$

If $S_x(x \in U)$ has a constant rank for every $(x,\bar{x})$ over a subdomain of Σ which contains (x_0,p_0), then equations of Σ in a neighborhood of $(x_0, \bar{x}_0)$ are given by

$$F_\sigma(x^1,\cdots,x^n ; \bar{x}^1,\cdots,\bar{x}^n) = 0 \qquad (\sigma = 1,\cdots,s) \tag{3.14}$$

which are obtained by eliminating p_a's from the first equation of (3.13). It is evident that the number s is equal to n minus the constant rank of S_x, $x \in U$.

THEOREM 3.4. *If $(x,\bar{x})$ be a pair of points in the relation S, then the rank of S_x at $\bar{x}$ is equal to the rank of $\bar{S}_{\bar{x}}$ at x.*

PROOF. We consider the rank of Σ at the point $(x,\bar{x})$. Then, we can easily see that it is equal to (rank of S_x at $\bar{x}$) + n and (rank of $\bar{S}_{\bar{x}}$ at x) + n. So we have

$$\text{rank of } S_x \text{ at } \bar{x} = \text{rank of } \bar{S}_{\bar{x}} \text{ at } x.$$

4. Integral submanifolds. The homogeneous contact form η of M^n determines an $(2n-1)$-dimensional distribution defined by

$$\eta = 0. \tag{4. 1}$$

We shall call it the fundamental distribution of the cotangent bundle ${}^cT(M^n)$.

Suppose N be a differentiable submanifold of ${}^cT(M^n)$ and

$$\iota : N \rightarrow {}^cT(M^n)$$

be the injection map. If

$$\iota^*\eta = 0, \tag{4. 2}$$

then N is said to be an *integral submanifold* of the fundamental distribution of an integral submanifold for brevity.

THEOREM 4.1. *A submanifold N of ${}^cT(M^n)$ is an integral submanifold if and only if every point z_0 of N has the property that its covector p_0 is tangent to πN at $x_0 = \pi z_0$.*

PROOF. Take a coordinate neighborhood U of M^n with coordinates x^i so that $\pi^{-1}(U)$ contains the point z_0 of N. We also take a coordinate neighborhood V of N with coordinates $(u^1,\cdots,u^r)$, $r = \dim N$, so that V contains the point z_0. Then, a sufficiently small neighborhood of z_0 with respect to N can be expessed analytically as

$$x^i = x^i(u^1,\cdots,u^r),\ p_i = p_i(u^1,\cdots,u^r). \tag{4. 3}$$

The condition (4.2) can now be written as

$$p_i(u)\frac{\partial x^i}{\partial u^\lambda} = 0 \quad (\lambda = 1,\cdots,r). \tag{4. 4}$$

As $(4.\ 3)_1$ is the equation of πN in the neighborhood of x_0, the last equation shows that p_0 is tangent to πN at x_0.

Conversely, if p_0 is tangent to πN at x_0 for every point z_0 of N, then we have (4. 4) identically. So, we see that (4. 2) is true. Hence, N is an integral submanifold.

EXAMPLE 1. Every point of ${}^cT(M^n)$ is a zero dimensional integral submanifold.

EXAMPLE 2. Every fibre $F_x(x \in M^n)$ of ${}^cT(M^n)$ is an n-dimensional integral submanifold.

THEOREM 4.2. *Let N be an integral submanifold in ${}^cT(M^n)$. If f is a homogeneous contact transformation of M^n, then fN is also an integral submanifold in ${}^cT(M^n)$.*

PROOF. We denote the injection map of N into ${}^cT(M^n)$ by ι. Then, the injection map of fN into ${}^cT(M^n)$ is given by $f\circ\iota$.
As

$$\begin{aligned}(f\circ\iota)^*\eta &= \iota^*\circ f^*\eta \\ &= \iota^*\eta = 0,\end{aligned}$$

we can see that fN is an integral submanifold.

COROLLARY. *If f is a homogeneous contact transformation of M^n, then the images $f(F_x)$ and $f^{-1}(F_x)$ of a fibre F_x at $x \in M^n$ are integral submanifolds.*

An important consequence of the last corollary is the following

THEOREM 4.3. *Let z be a point of ${}^cT(M^n)$ and $\bar{z}$ be the image of it under a homogeneous contact transformation f. Then the covector $\bar{p}$ of z is* *tangent to* S_x *at* $\bar{x} = \pi\bar{z}$ *and the covector* p *of* z *is tangent to* $\bar{S}_{\bar{x}}$ *at* $x = \pi z$.

PROOF. As a point of ${}^cT(M^n)$, $z = (x,p)$ belongs to F_x and so $\bar{z} = (\bar{x},\bar{p})$ belongs to $f(F_x)$. However, by virtue of the last Corollary, $f(F_x)$ is an integral submanifold in ${}^cT(M^n)$. Therefore, $\bar{p}$ is tangent to $\pi\circ f(F_x) = S_x$.

In the same way, p is tangent to $\pi\circ f^{-1}(F_{\bar{x}}) = \bar{S}_{\bar{x}}$.

THEOREM 4.4. *The dimensions of integral submanifolds of the homogeneous contact form* η *of a differentiable manifold* M^n *can not be greater than* n.

PROOF. Let N be an integral submanifold and $z \in N$. We denote the rank of πN at πz by r. Then the dimension of the set of covectors which are tangent to πN at z is clearly $n - r$. Hence, the dimension of N is at most $r + (n-r)$, which is to be proved.

Now, we define $F_{x,\bar{x}}$ by

$$F_{x,\bar{x}} = \{z \mid z \in F_x,\ f(z) \in F_{\bar{x}}\}.$$

Then, we get the following

THEOREM 4.5. *Suppose* f *is a homogeneous contact transformation. Then, in order that a covector* $\bar{p}$ *at a point* $\bar{x}$ *be tangent to* $S_x = \pi\circ f(F_x)$, *it is necessary and sufficient that* $\bar{z} = (\bar{x},\bar{p})$ *is the image under* f *of an element of* $F_{x,\bar{x}}$.

PROOF. Necessity. If $\bar{p}$ is tangent to S_x at $\bar{x}$, then

$$\bar{z} = (\bar{x},\bar{p}) \in F_{\bar{x}} \cap f(F_x).$$

Therefore,

$$z = (x,p) = f^{-1}(\bar{z}) \in F_x$$

Hence,

$$z \in F_{x,\bar{x}}$$

Sufficiency. If $\bar{z} = f(z)$, $z \in F_{x,\bar{x}}$, then $\bar{z} \in f(F_x)$. As $f(F_x)$ is an integral submanifold, $\bar{p}$ is tangent to S_x at $\bar{x}$.

Suppose N^{n-1} be an $(n-1)$-dimensional orientable submanifold of M^n. At every point of N^{n-1} we take a unit tangent covector of N^{n-1} with respect to an arbitrary but fixed Riemannian metric of M^n. Then, all such unit covectors constitute a differentiable field over N^{n-1} and the set of elements (x,p_x), where $x \in N^{n-1}$ and p_x is the unit tangent covector at x defined above, determines an $(n-1)$-dimensional submanifold in ${}^cT(M^n)$. We shall call it the *lift* of N^{n-1} and denote it by lN^{n-1}. l may be regarded as a diffeomorphism

$$l: N^{n-1} \to l\, N^{n-1}$$

whose inverse is the restriction map $\pi | lN^{n-1}$. It is evident that lN^{n-1} is an integral submanifold of M^n.

Now, suppose that f is a homogeneous contact transformation. Then, $f \circ lN^{n-1}$ is also an $(n-1)$-dimensional integral submanifold. However,

$$\bar{N} = \pi \circ f \circ lN^{n-1}$$

is not necessarily $(n-1)$-dimensional. $\bar{N}$ is said to be the *image of* N^{n-1} under f.

EXAMPLE. Consider a dilatation f in Euclidean space E^n. Then, for any point $y \in E^n, \bar{S}_y = \pi \circ f^{-1}(F_y)$ is an $(n-1)$-dimensional sphere in E^n. If we take $\bar{S}_y$ with unit tangent covectors as N^{n-1}

$$\begin{aligned} \bar{N} &= \pi \circ f \circ l \circ N^{n-1} \\ &= \pi \circ f \circ l \circ \pi \circ f^{-1} F_y = y. \end{aligned}$$

So, $\bar{N}$ is a point. Therefore, $\bar{N}$ is 0-dimensional.

Now, if we put

$$(\bar{x}, \bar{p}_{\bar{x}}) = f(x, p_x)$$

then $\bar{p}_{\bar{x}}$ is tangent to $\bar{N}$ at $\bar{x}$, as $f \circ lN^{n-1}$ is an integral submanifold in ${}^cT(M^n)$.

Suppose N_1^{n-1}, N_2^{n-1} be two $(n-1)$-dimensional orientable submanifolds in M^n such that they are tangent at a point x_0. Then, we may construct unit covector fields over N_1^{n-1} and N_2^{n-1} so that they have (x_0, p_{x_0}) in common. If we construct lN_1^{n-1}, lN_2^{n-1}, then they have a point in common and so $f \circ lN_1^{n-1}$, $f \circ lN_2^{n-1}$ have a point in common too. Therefore,

$$\bar{N}_1 = \pi \circ f \circ lN_1^{n-1}, \ \bar{N}_2 = \pi \circ f \circ lN_2^{n-1}$$

have a common tangent covector at the point $\bar{x}_0 = \pi f(x_0, p_0)$. Hence, we get the following

THEOREM 4.6. *Let N_1^{n-1} and N_2^{n-1} be two $(n-1)$-dimensional orientable submanifolds in M^n such that they are tangent at a point. Then, the images of N_1^{n-1} and N_2^{n-1} under a homogeneous contact transformation have a tangent covector in common.*

If the images $\bar{N}_1$ and $\bar{N}_2$ are both $(n-1)$-dimensional at $\pi \circ f(x_0, p_0)$, then they are tangent to each other in the proper sense and this is the reason why our diffeomorphism of ${}^cT(M^n)$ is called to be a (homogeneous) contact transformation.

In the above argument, the fact that N_1^{n-1} and N_2^{n-1} are submanifolds of M^n in the proper sense is not essential. To get the same result, it is essential that lN_1^{n-1} and lN_2^{n-1} have only a point in common. So, instead of lN_1^{n-1} and

lN_2^{n-1} we may take lN_1^{n-1} and $F^*_{x_0}$ as they have just a point in common, where $F^*_{x_0}$ is the submanifold of F_{x_0} whose points consist of units covectors. This leads us to the following

THEOREM 4.7. *Let N^{n-1} be an $(n-1)$-dimensional orientable submanifold in M^n. If $x_0 \in N^{n-1}$, then the image of N^{n-1} under a homogeneous contact transformation has a tangent covector in common with S_{x_0}.*

Therefore, in the favourable case when the image $\overline{N}$ of N^{n-1} and $S_x (x \in N^{n-1})$ are all $(n-1)$-dimensional, $\overline{N}$ is an envelope of S_x's $x \in N^{n-1}$.

5. Lie algebra of infinitesimal homogeneous contact transformations.

A vector field $X^\lambda = (X^i, P_i)$ over ${}^cT(M^n)$ is said to be an *infinitesimal homogeneous contact transformation* if it satisfies

$$\mathfrak{L}(X)\eta_\lambda = 0, \tag{5. 1}$$

where $\mathfrak{L}(X)$ means the operator of Lie derivation with respect to the vector field X.

THEOREM 5.1. *The set L of all infinitesimal homogeneous contact transformations of M^n constitutes a Lie algebra with respect to the usual bracket operation.*

PROOF. By virtue of the property of the Lie derivative

$$\mathfrak{L}(X)\mathfrak{L}(Y) - \mathfrak{L}(Y)\mathfrak{L}(X) = \mathfrak{L}([X,Y]), \tag{5. 2}$$

it is clear that if X and Y are infinitesimal homogeneous contact transformation, then $[X,Y]$ is also an infinitesinal homogeneous contact transformation. Therefore, we can easily see that our theorem is true.

The equation (5. 1) is equivalent to

$$p_i\partial_j X^i = -P_j, \quad p_i\partial^j X^i = 0. \tag{5. 3}$$

If we put

$$U = \eta_\lambda X^\lambda = p_i X^i, \tag{5. 4}$$

then we have

$$\partial_j U = p_i\partial_j X^i = -P_j,$$
$$\partial^j U = X^j + p_i\partial^j X^i = X^j$$

and

$$p_j\partial^j U = p_j X^j = U. \tag{5. 5}$$

So, U is a coray function of degree 1 over ${}^cT(M^n)$ and X^λ can be written as

$$(5.\ 6)\qquad X^\lambda = -\phi^{\lambda\mu}\partial_\mu U = (\partial^i U, -\partial_i U).$$

Conversely, every vector field over ${}^cT(M^n)$ of the form (5. 6), where U is a coray function of degree 1 over ${}^cT(M^n)$ is easily seen to be an infinitesimal homogeneous contact transformation. Hence, we get the

THEOREM 5.2. *Every infinitesimal homogeneous contact transformation X of a differentiable manifold M^n can be written as* (5. 6), *where U is a coray function of degree* 1. *The converse is also true.*

The function U is said to be the *characteristic function* of the infinitesimal homogeneous contact transformation X.

N.B. We can easily verify that (5. 3) is equivalent to any one of the three equations

$$(5.\ 7)\qquad \mathfrak{L}(X)\xi^\lambda = 0,\ [\xi,X] = 0,\ \mathfrak{L}(\xi)X^\lambda = 0$$

and

$$(5.\ 8)\qquad \mathfrak{L}(X)\phi^{\lambda\mu} = 0.$$

THEOREM 5.3. *Let $X^\lambda = (X^i,P_i)$, $Y^\lambda = (Y^i,Q_i)$ be infinitetimal homogeneous contact transformations and U,V be characteristic functions of them. Then, the characteristic function of the infinitesimal homogeneous contact transformation $[X,Y]$ is given by the Poisson bracket*

$$(5.\ 9)\qquad (U,V) = \varphi^{\lambda\mu}\partial_\lambda U\partial_\mu V.$$

PROOF. By virtue of (5. 6), we can easily verify that

$$[X,Y]^\lambda = -\phi^{\lambda\mu}\partial_\mu(\phi^{\alpha\beta}\partial_\alpha U\partial_\beta V),$$

which shows that our assertion is true.

THEOREM 5.4. *The set C of all coray functions of degree* 1 *over the cotangent bundle ${}^cT(M^n)$ constitutes a Lie algebra with respect to the natural addition and the bracket operation* (5. 9).

PROOF. As we can easily see that

$$(U,V) = -(V,U),$$
$$(U,(V+W)) = (U,V) + (U,W)$$

hold good, we only need to show that the Jacobi identity

$$(5.10)\qquad ((U,V),W) + ((V,W),U) + ((W,U),V) = 0$$

holds good. However, as

$$((U,V),W) = \phi^{\lambda\mu}\phi^{\rho\nu}(\partial_\lambda\partial_\rho U\partial_\mu V + \partial_\lambda U\partial_\mu\partial_\rho V)\partial_\nu W,$$

adding other two similar terms, we can easily see that (5.10) is true. Hence, the theorem is proved.

THEOREM 5.5. *If we define the map*

$$h: C \to L$$

by

$$U \to -\phi^{\lambda\mu}\partial_\mu U, \qquad U \in C, \tag{5.11}$$

then h is an isomorphism of C onto L.

PROOF. First it is clear that h is an homomorphism of C onto L if we regard them merely as additive groups. So, to prove that h is a homomorphism of the Lie algebra C onto the Lie algebra L, it is sufficient to show

$$h(U,V) = [hU,hV]. \tag{5.12}$$

However, the last equation can be written as

$$-\phi^{\lambda\mu}\partial_\mu(\phi^{\alpha\beta}\partial_\alpha U \partial_\beta V) = [\phi^{\lambda\alpha}\partial_\alpha U, \phi^{\lambda\beta}\partial_\beta V]$$

and its equality is already verified in the proof of Theorem 5.2. So, h is a homomorphism.

Now, the kernel of h is equal to zero, because if

$$\phi^{\lambda\mu}\partial_\mu U = 0$$

we have $U =$ const. and hence U has to be equal to zero.

COROLLARY 1. *If we have k parametric Lie group G_k of homogeneous contact transformations of a differentiable manifold M^n, we denote k infinitesimal homogeneous contact transformations which generate G_k by X_p ($p=1 \cdots, k$) and their characteristic functions by U_p. Then, U_p's are linearly independent with respect to constant coefficients and satisfy the relation*

$$(U_p,U_q) = c_{pq}{}^r U_r \ (p,q,r = 1,\cdots,k) \tag{5.13}$$

where $c_{pq}{}^r$ are constant.

N.B. $(U_p,U_q) = 0$ is the necessary and sufficient condition for the commutativity of the group generated by $U_1,\cdots,U_k$.

COROLLARY 2. *The Lie algebra L of all infinitesimal homogeneous contact transformations of M^n is infinite dimensional.*

PROOF. As the Lie algebra L and C are isomorphic and $\dim C$ is infinity, so $\dim L$ is equal to infinity.

Now, we shall prove the

THEOREM 5.6. *If a differentiable manifold M^n is compact, then every* *infinitesimal homogeneous contact transformation X generates a global one parameter group of global homogeneous contact transformations of M^n.*

PROOF. We take a point $z_0 \in {}^cT(M^n)$ and a coordinate neighborhood $U(x^i)$ of πz_0. In $\pi^{-1}(U)$, we consider the set of differential equations of the type

$$\frac{dx^\lambda}{dt} = X^\lambda. \tag{5.14}$$

Then, by virtue of the classical existence theorem on ordinary differential equations we can find a neighborhood $V(z_0)$ in $\pi^{-1}(U)$ and a positive constant $\varepsilon(z_0)$ so that

(a) (5.14) admits a solution

$$x^\lambda = f_t^\lambda(z_*) \qquad |t| < \varepsilon(z_0) \tag{5.15}$$

with the initial condition $f_0(z_*) = z_*$ for every point z_* of $V(z_0)$ and

(b) f_t for every $|t| < \varepsilon(z_0)$ is a diffeomorphism of $V(z_0)$ onto its image under f_t and

(c) if t,t' and $t + t'$ belong to the interval $(-\varepsilon(z_0), +\varepsilon(z_0))$, then

$$f_t \circ f_{t'} = f_{t+t'} \tag{5.16}$$

holds good.

The number $\varepsilon(z_0)$ generally depends upon the choice of z_0. On account of this fact, an infinitesimal homogeneous contact transformation generally may not generate a group of global homogeneous contact transformations. However, it is known that if we can choose $\varepsilon(z_0)$ so that it does not depend upon the choice of z_0, then the infinitesimal homogeneous contact transformation generates a global one parameter group G_1 of global homogeneous contact transformations.

Now, we define a transformation T_c by

$$T_c(x,p) = (x,cp), \tag{5.17}$$

where c is a positive constant. T_c for $0 < c < \infty$ is the one parametric multiplicative group generated by ξ^λ. So by (5. 7), X is invariant under T_c and hence we may take $\varepsilon(z_0)$ as $\varepsilon(T_c z_0)$. Accordingly $\varepsilon(z_0)$ depends only upon the coray on which z_0 lies.

Therefore, it is clear that $\varepsilon(z_0)$, $z_0 \in {}^cT(M^n)$ has a positive greatest lower bound if M^n is compact. Hence, our theorem is proved.

Now, let us introduce a positive definite Riemannian metric g over M^n. Then, the set of all unit covariant vectors of M^n constitutes a submanifold of ${}^cT(M^n)$, which we denote by ${}^cT_1(M^n)$. Any differentiable function $W(x,p_1)$ defined over $T_1(M^n)$ such that $W(x, -p_1) = W(x,p_1)$, where $p_1 \in {}^cT(M^n)$, can be easily extended to a coray function of degree 1 over ${}^cT(M^n)$.

Hence, by virtue of the last theorem, we get the following

THEOREM 5.7. *If M^n is a compact differentiable manifold, then there always exist homogeneous contact transformations.*

6. Contact dstribution of the first kind. The tangent n-space to the fibre at a point $z=(x,p)$ of ${}^cT(M^n)$ is called to be the *vertical space* at z. We consider an n-space which is disjoint and complementary to the vertical space at z and call it as a *transversal space* to the vertical space at z.

In a coordinate neighborhood $\pi^{-1}(U)$ with coordinates (x^i,p_i), we put

$$\partial_i = \frac{\partial}{\partial x^i}, \ \partial^i = \frac{\partial}{\partial p_i}. \tag{6. 1}$$

If $\lambda_j{}^i\partial_i + \mu_{ji}\partial^i$ are n vectors which span the transversal n-space, then their natural projections $\lambda_j{}^i\partial_i$ have to be linearly independent, so we have $|\lambda^i{}_j| \neq 0$. Therefore, we may assume that n-vectors which span the transversal n-space have the form

$$e_i = \partial_i + \Gamma_{ij}\partial^j. \tag{6. 2}$$

We assume that

$$\Gamma_{ij} = \Gamma_{ji}, \tag{6. 3}$$

then we can see that it is independent upon the choice of local coordinates. To show it, let

$$\bar{x}^a = \bar{x}^a(x), \ \bar{p}_a = p_i\bar{\partial}_a x^i \tag{6. 4}$$

be a coordinate transformation of local coordinates and its extension, then we can easily verify that

$$\bar{\partial}_a + \bar{\Gamma}_{ab}\bar{\partial}^b = \bar{\partial}_a x^i(\partial_i + \Gamma_{ij}\partial^j),$$

where we have put

$$\bar{\Gamma}_{ab} = p_k\bar{\partial}_a\bar{\partial}_b x^k + \bar{\partial}_a x^i\bar{\partial}_b x^j\Gamma_{ij}.$$

Therefore, we have $\bar{\Gamma}_{ab} = \bar{\Gamma}_{ba}$, which shows that our assertion is true.

Hereafter we consider a distribution D_1 of transversal n-spaces such that the symmetry condition (6. 3) is satisfied at every point of ${}^cT(M^n)$. We call such distribution as *contact distributions of the first kind* and each of the set of n-vectors e_i as *contact frame of the first kind* belonging to it and corresponding to the coordinate neighborhood in consideration. We say that Γ_{ij}'s are parameters of the contact frame.

EXAMPLE. Let Γ^k_{ij} be a symmetric affine connection defined over M^n. Then, we can easily verify that

(6. 5) $$\Gamma_{ij} = \Gamma^k_{ij} p_k$$

defines a contact distribution of the first kind.

Now, we consider the vectors e_i as operators in the sense

$$e_i f = \partial_i f + \Gamma_{ij} \partial^j f$$

for any function f over ${}^cT(M^n)$ and define a quantity defined by

(6. 6) $$R_{ijk} = e_k \Gamma_{ij} - e_j \Gamma_{ik}.$$

Then we get the following

THEOREM 6.1. *The contact distribution D_1 of the first kind is completely integrable if and only if*

(6. 7) $$R_{ijk} = 0.$$

PROOF. n-planes of D_1 are spanned by vectors e_i. As e_j for fixed j has components $(\delta_j^i, \Gamma_{ji})$ with respect to natural frames, we can easily see that

$$[e_j, e_k]^i = 0,$$
$$[e_j, e_k]^{n+i} = - R_{ijk}.$$

So, $[e_j, e_k]$ is a linear combination of e_r if and only if $R_{ijk} = 0$. Hence, the theorem follows.

COROLLARY. *If the contact distribution D_1 of the first kind is the one which is associated to a symmetric affine connection by* (6. 5). *then D_1 is completely integrable if and only if the affine connection is flat.*

PROOF. We can easily verify that

(6. 8) $$R_{ijk} = R^h{}_{ijk} p_h,$$

where $R^h{}_{ijk}$'s are components of the curvature tensor of the affine connection. As p_h's are arbitrary, we have $R^h{}_{ijk} = 0$ if $R_{ijk} = 0$. Hence, the theorem is proved.

THEOREM 6.2. *The contact distribution D_1 of the first kind is invariant under the transformation T_c if and only if $\Gamma_{ij}(x,p)$'s are coray functions of degree* 1.

PROOF. As the n-space of the contact distribution of the first kind at $z = (x,p)$ is spanned by n-vectors with components $(\delta^i_a, \Gamma_{ai}(x,p))$ $(a = 1, \cdots, n)$ we can easily see that it is defined by equations $\omega_{n+i} = 0$, where we have put

(6. 9) $$\omega_{n+i} = dp_i - \Gamma_{ij}(x,p) dx^j.$$

The equations $\omega_{n+i} = 0$ at $'z = T_c z$ are satisfied by vectors of the n-space which is the image of the n-space of D_1 at z under T_c if and only if

$\Gamma_{ij}(x,p)$'s are coray functions of degree 1. Hence, the theorem is proved.

Let $U_\lambda(\lambda \in \Lambda)$ be an open covering of M^n and f be a homogeneous contact transformation of M^n. Suppose $f\circ\pi^{-1}(U_\lambda) \cap \pi^{-1}(U_\mu)$ is not empty, then the restriction map

$$f\colon \pi^{-1}(U_\lambda) \cap f^{-1}\circ\pi^{-1}(U_\mu) \to f\circ\pi^{-1}(U_\lambda) \cap \pi^{-1}(U_\mu)$$

can be expressed by

$$\bar{x}^a = \bar{x}^a(x,p), \ \bar{p}_a = \bar{p}_a(x,p), \tag{6.10}$$

where (x^i,p_i) are coordinates in $\pi^{-1}(U_\lambda)$ and $(\bar{x}^a,\bar{p}_a)$ are coordinates in $\pi^{-1}(U_\mu)$.

Now, in order to get good insight of the complicated calculations, we introduce matrix notation

$$\left\{\begin{aligned} A &= (\partial_i\bar{x}^a), & B &= (\partial^i\bar{x}^a), \\ C &= (\partial_i\bar{p}_a), & D &= (\partial^i\bar{p}_a) \end{aligned}\right. \tag{6.11}$$

and

$$\left\{\begin{aligned} \bar{A} &= (\bar{\partial}_a x^i), & \bar{B} &= (\bar{\partial}^a x^i), \\ \bar{C} &= (\bar{\partial}_a p_i), & \bar{D} &= (\bar{\partial}^a p_i). \end{aligned}\right. \tag{6.12}$$

Then, by virtue of (2.22) and (2.18), we have

$$\left\{\begin{aligned} A &= {}^t\bar{D}, & B &= -\,{}^t\bar{B}, \\ C &= -\,{}^t\bar{C}, & D &= {}^t\bar{A} \end{aligned}\right. \tag{6.13}$$

and

$$\left\{\begin{aligned} &B\,{}^tA = A\,{}^tB, \qquad D\,{}^tC = C\,{}^tD, \\ &B\,{}^tC - A\,{}^tD = -\,E, \end{aligned}\right. \tag{6.14}$$

where t's on the left shoulders of matrices mean their transposes and E is the unit matrix. It is evident that we have also the identities

$$\left\{\begin{aligned} A\bar{A} + B\bar{C} &= E, & A\bar{B} + B\bar{D} &= 0, \\ C\bar{A} + D\bar{C} &= 0, & C\bar{B} + D\bar{D} &= E. \end{aligned}\right. \tag{6.15}$$

Now, if

$$|A + B\Gamma| \neq 0 \tag{6.16}$$

at a point $z = (x,p)$ or $\pi^{-1}(U_\lambda) \cap f^{-1}\circ\pi^{-1}(U_\mu)$, we say that f is *regular at* z with respect to the contact distribution D_1. And if f is regular at every point of $T(M^n)$, we say that f is *regular* with respect to D_1. The independence of the notion of regularity upon coordinate neighborhood comes from the following

THEOREM 6.3. *If a homogeneous contact transformation f of M^n is* *regular with respect to a contact distribution of the first kind D_1 at a point $z=(x,p)$ of ${}^cT(M^n)$, then the image of the n-space of D_1 at the point z is also transversal to the vertical space at the point $f(z)$. The converse is also true.*

PROOF. The transversal n-space of D_1 at the point z is spanned by n vectors e_i.

If we fix i, the components of the vector e_i with respect to the natural frame (∂_j,∂^j) are (δ_i^j,Γ_{ij}). So, the components of the image of the vector under the transformation f are easily seen to be given by the i-th columns of the set of matrices $(A+B\Gamma,\ C+D\Gamma)$. Therefore, the π-image of the vector in consideration has as its components the i-th column of $A+B\Gamma$. Hence, the condition (6.16) is equivalent to the fact that the $\pi\circ f$ image of the transversal n-space at the point z spanned by n vectors e_i $(i=1,\cdots,n)$ coincides with the tangent space of M^n at the point $\pi\circ f(z)$. Hence, the condition (6.16) is also equivalent to the fact that the f-image of the n-space of D_1 at the point z is again a transversal n-space at $f(z)$.

It is evident that the converse is also true.

Now, assuming that D_1 is a contact distribution of the first kind and f is a homogeneous contact transformation we consider equations

$$\underset{f}{\overline{\Gamma}}(A+B\Gamma)=C+D\Gamma \quad (6.17)$$

[1)]

for unknowns $\underset{f}{\overline{\Gamma}}_{ab}$.

LEMMA 6.1. *In order that* (6.17) *admits a set of solutions $\underset{f}{\overline{\Gamma}}_{ab}$ at a point $f(z)\in f\circ\pi^{-1}(U_\lambda)\cap\pi^{-1}(U_\mu)$, it is necessary and sufficient that f is regular with respect to D_1 at the point $z\in\pi^{-1}(U_\lambda)\cap f^{-1}\circ\pi^{-1}(U_\mu)$.*

PROOF. Sufficiency is evident.

Necessity. By virtue of (2.22), we can write (6.17) as

$$\underset{f}{\overline{\Gamma}}\,({}^t\overline{D}-{}^t\overline{B}\Gamma)=-\,{}^t\overline{C}+{}^t\overline{A}\Gamma.$$

So we have

$$\Gamma(\overline{A}+\overline{B}\,{}^t\underset{f}{\overline{\Gamma}})=\overline{C}+\overline{D}\,{}^t\underset{f}{\overline{\Gamma}} \quad (6.18)$$

1) If we consider homogeneous contact transformation (2. 13) as a coordinate transformation and introduce an object Γ which is transformed by

$$(*)\qquad \overline{\Gamma}(A+B\Gamma)=C+\overline{D}\Gamma$$

under such coordinate transformation, then we have

$$dp-\overline{\Gamma}dx=(D-\overline{\Gamma}B)(dp-\Gamma dx).$$

So we may define a parallel displacement of p_i which is invariant under homogeneous contact transformations by $dp-\Gamma dx=0$. The object Γ and the equation (*) were first introduced by T. Hosokawa [7] in this way. The word contact frame was first used by L. P. Eisenhart [6] without explicit mention of the vectors $\partial_i+\Gamma_{ij}\partial^j$.

which is similar to (6.17).

On the other hand, we have

$$(A + B\Gamma)(\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}})$$
$$= A\bar{A} + A\bar{B}{}^t\underset{f}{\bar{\Gamma}} + B\Gamma(\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}}).$$

The right hand side is easily transformed to

$$E - B\{\bar{C} + \bar{D}{}^t\underset{f}{\bar{\Gamma}} - \Gamma(\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}})\}.$$

Therefore, by (6.18), we get

$$(A + B\Gamma)(\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}}) = E. \tag{6.19}$$

Hence, we have (6.16) which shows that f is regular at the point $z = (x,p)$.

LEMMA 6.2. *If* (6.17) *admits a set of solutions* $\underset{f}{\bar{\Gamma}}_{ab}$, *then* $\underset{f}{\bar{\Gamma}}_{ab}$*'s are symmetric.*

PROOF. We multiply $\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}}$ to both sides of (6.17). Then, we see first that the left hand side reduces to $\underset{f}{\bar{\Gamma}}$ by virtue of (6.19). Secondly, the right hand side is transformed to

$$(C + D\Gamma)(\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}})$$
$$= (C + D\Gamma)\bar{A} + C\bar{B}{}^t\underset{f}{\bar{\Gamma}} + D\Gamma\bar{B}{}^t\underset{f}{\bar{\Gamma}}.$$

Putting (6.18) into the last term of the right hand side of the last eqation, we see that the right hand side reduces to $\underset{f}{\bar{\Gamma}}$. Hence, replacing c by b, we get

$$\underset{f}{\bar{\Gamma}}_{ab} = \underset{f}{\bar{\Gamma}}_{ba}, \tag{6.20}$$

which is to be proved.

THEOREM 6.4. *If a homogeneous contact transformation f of M^n is regular at a point z of ${}^cT(M^n)$ with respect to a contact distribution D_1 of the first kind determined by Γ, then the image of the n-space of D_1 at z under f is the n-space determined by $\underset{f}{\bar{\Gamma}}$ at $\bar{z} = f(z)$*

PROOF. As the vector e_i for fixed i has components $(\delta_i^j, \Gamma_{ij})$ with respect to the natural frame (∂_j, ∂^j), we can easily see that the image of this vector has as its components the i-th columns of the set of matrices $(A + B\Gamma,\ C + D\Gamma)$ with respect to the natrual frame $(\bar{\partial}_a, \bar{\partial}^a)$. We consider n such vectors, and take a linear combinations of these vectors by multiplying $\bar{A} + \bar{B}{}^t\underset{f}{\bar{\Gamma}}$, then we get $(\delta_b^a, \underset{f}{\bar{\Gamma}}_{ab})$, i.e. we have n vectors

$$\bar{e}_b = \bar{\partial}_b + \underset{f}{\bar{\Gamma}}_{bc}\bar{\partial}^c. \tag{6.21}$$

Hence, the image of the transversal n-space determined by Γ at z is the transversal n-space determined by $\underset{f}{\bar{\Gamma}}$ at $f(z)$.

From the proof of Theorem 6.3, we can see that

$$\bar{e}_b = (\bar{\partial}_b x^i + \bar{\partial}^c x^i \underset{f}{\Gamma}_{bc}) f(e_i), \tag{6.22}$$

where $f(e_i)$ is the vector which is the image of e_i under f.

From Theorems 6.3 and 6.4 we get the following

THEOREM 6.5. *Let D_1 be a contact distribution of the first kind of ${}^cT(M^n)$ determined by Γ and f be a homogeneous contact transformation of M^n. If f is regular at every point of ${}^cT(M^n)$ with respect to D_1, then f induces a new contact distribution fD_1 of the first kind in ${}^cT(M^n)$.*

Of course, the n-plane of fD_1 at $f(z)$ is spanned by $\bar{e}_a$. We denote the vectors of fD_1 at z by $\partial_i + \underset{f}{\Gamma}_{ij}\partial^j$.

If a homogeneous contact transformation f satisfies the relation

$$\underset{f}{\Gamma} = \Gamma, \tag{6.23}$$

then we say that the contact distribution D_1 determined by Γ is invariant under f. We shall study homogeneous contact transformations which leave Γ invariant.

As an example we shall prove the following

THEOREM 6.6. *Suppose D_1 is the contact distribution of the first kind associated with a symmetric affine connection Γ of M^n.* (i) *The extention of every affine transformation of M^n leaves D_1 invariant.* (ii) *If the extended group of a Lie group G of diffeomorphisms of M^n leaves D_1 invariant, G is a group of affine transformations.*

PROOF. Let us take the local expression of f as in (6.10). In order that the contact distribution of the first kind D_1 is invariant under f it is necessary and sufficient that

$$\Gamma_{ab}(\bar{x},\bar{p})(\partial_i \bar{x}^b + \partial^j \bar{x}^b \Gamma_{ji}) = \partial_i \bar{p}_a + \partial^j \bar{p}_a \Gamma_{ji}. \tag{6.24}$$

(i) Suppose f be an extension of an affine transformation of M^n with a symmetric affine connection Γ. Then, we have

$$\bar{x}^a = \bar{x}^a(x),\ \bar{p}_a = p_i \bar{\partial}_a x^i \tag{6.25}$$

and

$$\Gamma^c_{ab}(\bar{x}) = \partial_k \bar{x}^c (\bar{\partial}_a \bar{\partial}_b x^k + \Gamma^k_{ij} \bar{\partial}_a x^i \bar{\partial}_b x^j). \tag{6.26}$$

Noticing (6. 5) and making use of $(6.25)_2$ and (6.26) we get

$$\Gamma_{ab}(\bar{x},\bar{p}) = \bar{\partial}_a \bar{\partial}_b x^k p_k + \Gamma_{ij}(x,p) \bar{\partial}_a x^i \bar{\partial}_b x^j. \tag{6.27}$$

Comparing the last equation with (6.23) we see that D_1 is invariant under f.

(ii) We consider U_μ coincides with U_λ and take an infinitesimal homogeneous contact transformation defined by

$$\bar{x}^i = x^i + \partial^i U\delta t,\ \bar{p}_i = p_i - \partial_i U\delta t, \tag{6.28}$$

where U is a coray function of degree 1 over ${}^cT(M^n)$. From (6.27), we can easily see that the transformation (6.28) leaves D_1 invariant if and only if the equation

$$\partial_k\Gamma_{ia}\partial^k U - \partial^k\Gamma_{ia}\partial_k U + \Gamma_{ij}e_a\partial^j U + e_a\partial_i U = 0. \tag{6.29}$$

Now, we consider (6. 28) which is the extension of an infinitesimal diffeomorphism defined by a vector field X^i of M^n.

Putting (6. 5) into (6.29), we have

$$X^h{}_{ia} + R^h_{iak}X^k = 0, \text{ i.e. } \mathfrak{L}(X)\Gamma^h_{ia} = 0.$$

So, X^i is an infinitesimal affine transformation. If we take vector fields which generate the given Lie group G instead of X^i, we can see that G is a Lie group of affine transformations of M^n.

7. Contact distribution of the second kind. We consider another n-dimensional distribution D_2 such that the n-space of D_2 is disjoint and complementary to the n-space of the contact distribution of the first kind D_1 at every point of ${}^cT(M^n)$. As the bases of n-spaces of D_2, we may take n vectors of the form $\lambda^j{}_ie_j + \mu_{ij}\partial^j$. As the n-space of D_2 is disjoint and complementary to the n-space of D_1 at every point of ${}^cT(M^n)$, we have $|\mu^i_j| \neq 0$. So we may assume that the bases are given by n vectors of the form

$$e^i = \partial^i - \Pi^{ij}(\partial_j + \Gamma_{jk}\partial^k). \tag{7. 1}$$

Here, we assume that

$$\Pi^{ij} = \Pi^{ji} \tag{7. 2}$$

The assumption (7. 1) is independent upon the choice of local coordinates. For, if (6. 4) is a coordinate transformation and it extension, then we have

$$\begin{aligned}\partial^i - \Pi^{ij}(\partial_j + \Gamma_{jk}\partial^k)\\ = \partial^i\bar{p}_a\,\bar{\partial}_a - \Pi^{ij}\partial_j\bar{x}^b(\bar{\partial}_b + \bar{\Gamma}_{bc}\bar{\partial}^c)\end{aligned}$$

by virtue of an analogous argument for Γ_{ij} in §6. The last equation is easily transformed to $\bar{\partial}_a x^i\{\bar{\partial}^a - \bar{\Pi}^{ab}(\bar{\partial}_b + \bar{\Gamma}_{bc}\bar{\partial}^c)\}$, where we have put

$$\bar{\Pi}^{ab} = \Pi^{ij}\partial_i\bar{x}^a\partial_j\bar{x}^b. \tag{7. 3}$$

Hence, D_2 is spanned by $e^a = \bar{\partial}^a - \bar{\Pi}^{ab}(\bar{\partial}_b + \bar{\Gamma}_{bc}\bar{\partial}^c)$ and $\bar{\Pi}^{ab}$'s are symmetric.

We call such distribution D_2 as a *contact distribution of the second kind* assocciated to the contact distribution of the first kind D_1. The set of n vectors e^i is called a *contact frame of the second kind.* We say that Π^{ji}'s are parameters of the contact frame.

EXAMPLE 1. The distribution determined by the set of all vertical n-spaces of ${}^cT(M^n)$.

EXAMPLE 2. We endow a positive definite Riemannian metric g to M^n and define

$$\Gamma_{ij} = \left\{ {k \atop ij} \right\} p_k, \quad \Pi^{ij} = g^{ij}/\sqrt{g^{hk}p_h p_k} \tag{7. 4}$$

at every coordinate neighborhood $\pi^{-1}(U)$ of ${}^cT(M^n)$, where U is a coordinate neighborhood of M^n and g^{ij}, $\left\{ {k \atop ij} \right\}$ are the fundamental tensors and the Christoffel's symbols of the Riemannian manifold M^n. Then, Γ_{ij} and Π^{ij} determine contact distributions of the first and second kind.

THEOREM 7.1. *Suppose the contact distribution D_1 of the first kind is invariant under the transformation T_c. Then, the contact distribution D_2 of the second kind is invariant under T_c, if and only if $\Pi^{ij}(x,p)$'s are coray functions of degree -1.*

PROOF. As the n-space of D_2 at a point $z = (x,p)$ is spanned by n vectors e^a with components $(-\Pi^{ai}, \delta_i^a - \Pi^{ab}\,\Gamma_{bi})$, we can easily verify that it is defined also by n equations $\omega_i = 0$, where we have put

$$\omega_i = dx^i + \Pi^{ij}(x,p)\ \{dp_j - \Gamma_{jk}(x,p)dx^k\}. \tag{7. 5}$$

The equations of the type $\omega_i = 0$ at $T_c z$ are satisfied by vectors of the n-space which is the image of the n-space of D_1 at z under T_c if and only if Π^{ij}'s are coray functions of degree -1 as Γ_{ij}'s are coray functions of degere $+1$ by assumption. Hence, our theorem is proved.

Now, we consider the vectors e^i as operators in the sense

$$e^i f = \partial^i f - \Pi^{ij}(\partial_j f + \Gamma_{jk}\partial^k f)$$

and put

$$R^{ijk} = e^k\Pi^{ij} - e^j\Pi^{ik} + \Pi^{ih}(\Pi^{aj}e^k\Gamma_{ah} - \Pi^{ak}e^j\Gamma_{ah}). \tag{7. 6}$$

Then, we get the following

THEOREM 7.2. *The contact distribution of the second kind D_2 is completely integrable if and only if there exist the relations*

$$R^{ijk} = 0 \tag{7. 7}$$

PROOF. n-spaces of D_2 are spanned by n vectors e^i defined by (7. 1). As e^a for fixed a has components $(-\Pi^{ia}, \delta_i^a - \Pi^{ab}\Gamma_{bi})$ with respect to natural frames, we can easily see that

$$[e^j,e^k]^i = e^k\Pi^{ij} - e^j\Pi^{ik},$$
$$[e^j,e^k]^{n+i} = e^k(\Gamma_{ia}\Pi^{aj}) - e^j(\Gamma_{ia}\Pi^{ak}).$$

On the other hand, by the theory of distributions the distribution D_2 is completely integrable if and only if there exist functions $\lambda^{jk}{}_h$ such that

$$[e^j, e^k] = \lambda^{jk}{}_h e^h,$$

i.e.

$$e^k\Pi^{ij} - e^j\Pi^{ik} = -\lambda^{jk}{}_h\Pi^{hi},$$
$$e^k(\Gamma_{ia}\Pi^{aj}) - e^j(\Gamma_{ia}\Pi^{ak}) = \lambda^{jk}{}_h(\delta^h_i - \Pi^{hl}\Gamma_{li}).$$

Eliminating $\lambda^{jk}{}_h$ from the last two equations we see that our assertion is true.

COROLLARY. *If the distribution of the second kind D_2 can be transformed to the distribution determined by vertical n-spaces by a homogeneous contact transformation f, then* (7. 7) *holds good.*

PROOF. The distribution determined by the vertical n-spaces are completely integrable. So, its inverse image D_2 by f^{-1} is also completely integrable. Hence, by virtue of Theorem 7.3, we see that our assertion is true.

Let f be a homogeneous contact transformation which is regular with respect to the contact distribution D_1. We consider now the equations

$$\underset{f}{\overline{\Pi}}({}^t\overline{A} + \underset{f}{\overline{\Gamma}}\,{}^t\overline{B}) = (A + B\Gamma)\Pi - B.^{2)} \tag{7. 8}$$

Then, $\underset{f}{\overline{\Pi}}{}^{ab}$'s are defined in $f\circ\pi^{-1}(U_\lambda) \cap \pi^{-1}(U_\mu)$ uniquely, as

$$|\overline{A} + \overline{B}\ {}^t\underset{f}{\overline{\Gamma}}| \neq 0 \tag{7. 9}$$

by virtue of (6.19).

LEMMA 7.1. *$\underset{f}{\overline{\Pi}}{}^{ab}$'s defined by* (7. 8) *are symmetric with respect to a and b.*

PROOF. We multiply $A + B\Gamma$ to both sides of (7. 8) and contract with respect to i. Then the left hand side reduces to $\underset{f}{\overline{\Pi}}$ by virtue of (6.19), the right hand side reduces to

$$(A + B\Gamma)\Pi({}^tA + \Gamma\,{}^tB) - B\,{}^tA - B\Gamma\,{}^tB.$$

It is evident that the first term and the third term of the last formula are symmetric. The second term is also symmetric by virtue of $(6.14)_1$. So the right hand side is symmetric. Hence we can see that

$$\underset{f}{\overline{\Pi}}{}^{ab} = \underset{f}{\overline{\Pi}}{}^{ba}. \tag{7.10}$$

2) Considering the homogeneous contact transformation (2. 13) as a coordinate transformation, Y. Muto [8] and T. C. Doyle [3] independently introduced the object Π which is transformed by an equation of the form (7. 8).

The geometric meaning of $\underset{f}{\Pi}^{ab}$ is given by the following

THEOREM 7.3. *If a homogeneous contact transformation f of M^n is regular with respect to the contact distribution D_1 determined by Γ, then the image of the n-space spanned by e^i under f is the n-space spanned by* $\bar{\partial}^a - \underset{f}{\Pi}^{ab}(\bar{\partial}_b + \underset{f}{\Gamma}_{bc}\bar{\partial}^c)$.

PROOF. As the vector e^j for a fixed j has components $(-\Pi^{ij}, \delta^i_j - \Gamma_{ik}\Pi^{kj})$ with respect to the natural frame (∂_i, ∂^i), the image of the vector under the homogeneous contact transformation f has as its components the j-th columns of the set of matrices

$$\{B - (A + B\Gamma)\Pi,\ D - (C + D\Gamma)\Pi\}$$

with respect to the natural frame $(\bar{\partial}_a, \bar{\partial}^a)$. By virtue of (7.18), this is transformed to

$$\{-\underset{f}{\bar{\Pi}}({}^t\bar{A} + \underset{f}{\bar{\Gamma}}\,{}^t\bar{B}),\ D - (C + D\Gamma)\Pi\}.$$

Now, we take linear combinations of these n vectors by multiplying ${}^t(A + B\Gamma)$ and summing up for j, we get by (6.17)

$$\{-\underset{f}{\bar{\Pi}}, D\,{}^t(A + B\Gamma) - (C + D\Gamma)\Pi\,{}^t(A + B\Gamma)\}. \tag{7.11}$$

On the other hand we have

$$\underset{f}{\bar{\Pi}} = (A + B\Gamma)\Pi\,{}^t(A + B\Gamma) - B\,{}^t(A + B\Gamma),$$
$$\underset{f}{\bar{\Gamma}}(A + B\Gamma) = C + D\Gamma.$$

So, we have

$$\underset{f}{\bar{\Gamma}}\,\underset{f}{\bar{\Pi}} = (C + D\Gamma)\Pi\,{}^t(A + B\Gamma) - \underset{f}{\bar{\Gamma}}B\,{}^t(A + B\Gamma).$$

By virtue of the last equation (7.11) can be written as

$$\begin{aligned}
&\{-\underset{f}{\bar{\Pi}}, D\,{}^t(A + B\Gamma) - \underset{f}{\bar{\Gamma}}B\,{}^t(A + B\Gamma) - \underset{f}{\bar{\Gamma}}\,\underset{f}{\bar{\Pi}}\}\\
&= \{-\underset{f}{\bar{\Pi}},\ (D - \underset{f}{\bar{\Gamma}}B)\,{}^t(A + B\Gamma) - \underset{f}{\bar{\Gamma}}\,\underset{f}{\bar{\Pi}}\}\\
&= \{-\underset{f}{\bar{\Pi}},\ {}^t(\bar{A} + \bar{B}\underset{f}{\bar{\Gamma}})\,{}^t(A + B\Gamma) - \underset{f}{\bar{\Gamma}}\,\underset{f}{\bar{\Pi}}\}\\
&= \{-\underset{f}{\bar{\Pi}},\ E - \underset{f}{\bar{\Gamma}}\,\underset{f}{\bar{\Pi}}\}
\end{aligned}$$

by virtue of $(6.13)_{1,2}$ and (6.17). Therefore, the image of the n-space spanned by e^i is spanned by

$$\bar{e}^a = \bar{\partial}^a - \underset{f}{\Pi}^{ab}(\bar{\partial}_b + \underset{f}{\Gamma}_{bc}\,\bar{\partial}^c). \tag{7.12}$$

From the proof of Theorem 7.1, we can see that

$$\bar{e}^b = (\partial_j \bar{x}^b + \Gamma_{hj}\partial^h \bar{x}^b) f(e^j), \tag{7.13}$$

where $f(e^j)$ is the vector which is the image of e^j under f.

From Theorems 6.5, 7.3 and Lemma 7.1 we get the following

THEOREM 7.4. *Let D_1 and D_2 be contact distributions of the first and second kind of ${}^cT(M^n)$ determined by Γ and Π and f be a homogeneous contact transformation of M^n. If f is regular at every point of D_1, then f induces new contact distribution of the second kind fD_2 associated to fD_1.*

$\underset{f}{\bar{\Pi}}{}^{ab}$'s are parameters which define fD_2 in $f\circ\pi^{-1}(U_\lambda) \cap \pi^{-1}(U_\mu)$. We denote the parameters which define fD_2 generally by $\underset{f}{\Pi}{}^{ij}$.

If a homogeneous contact transformation f satisfies the relation

$$\underset{f}{\Pi} = \Pi, \tag{7.14}$$

then we say that the contact distribution D_2 determined by Γ and Π is invariant under f.

THEOREM 7.5. *Suppose that we take the distribution determined by vertical spaces as contact distribution of the second kind. Then every homogeneous contact transformation f which leaves this contact distribution invariant is an extension of a diffeomorphism of M^n.*

PROOF. Putting $\Pi = 0$, $\bar{\Pi} = 0$ into (7. 8) we get $\partial^i \bar{x}^a = 0$. Noticing that the 1-form $p_i dx^i$ is invariant under f, we can easily see that our contact transformation f is an extension of a diffeomorphism of M^n.

THEOREM 7.6. *Let M^n be a Riemannian manifold and suppose that we take the contact distributions determined by* (7. 4). *Then the extension of an isometry of M^n leaves both distributions D_1 and D_2 invariant. And if G' is the extended group of a Lie group G of diffeomorphisms of M^n such that every transformation of G' leaves D_1 and D_2 invariant, then G is a group of isometries of M^n.*

The proof is almost evident from that of Theorem 6.6 and the law of transformation (7. 3) of Π^{ij} under an extension of a diffeomorphism of M^n.

THEOREM 7.7. *If we denote the components of an arbitrary vector X with respect to the natural frame by (X^i, P_i), then its components with respect to the contact frame (e_i, e^i) are given by*

$$\left\{\begin{aligned} &\Lambda^i = X^i + \Pi^{ij} M_j = X^i + \Pi^{ij}(P_j - \Gamma_{jk} X^k), \\ &M_i = P_i - \Gamma_{ij} X^j. \end{aligned}\right. \tag{7.15}$$

PROOF. We can easily verify that

$$\Lambda^i e_i + M_i e^i = X^i \partial_i + P_i \partial^i,$$

which shows that our assertion is true.

THEOREM 7.8. *The projection tensors* T_1 *and* T_2 *of an arbitrary vector to the distributions* D_1 *and* D_2 *are given by*

$$\underset{1}{T} = \begin{pmatrix} \delta^i_j - \Pi^{ik}\Gamma_{kj} & \Pi^{ij} \\ \Gamma_{ij} - \Gamma_{ik}\Pi^{kh}\Gamma_{hj} & \Gamma_{ik}\Pi^{kj} \end{pmatrix}, \tag{7.16}$$

$$\underset{2}{T} = \begin{pmatrix} \Pi^{ik}\Gamma_{kj} & -\Pi^{ij} \\ -\Gamma_{ij} + \Gamma_{ik}\Pi^{kh}\Gamma_{hj} & \delta_{ij} - \Gamma_{ik}\Pi^{kj} \end{pmatrix} \tag{7.17}$$

respectively with respect to natural frames.

PROOF. The projection of an arbitrary vector $X = (X^i, P_i)$ on D_1 is given by $\Lambda^i e_i$. The components of the last vector with respect to the natural frame are easily seen to be

$$(\delta^i_j - \Pi^{ik}\Gamma_{kj})X^j + \Pi^{ij}P_j, \ (\Gamma_{ij} - \Gamma_{ik}\Pi^{kh}\Gamma_{hj})X^j + \Gamma_{ik}\Pi^{kj}P_j.$$

The components of $\underset{1}{T}$ are nothing but the coefficients of X^j, P_j of the last vector. So, (7.16) is proved. The proof of (7.17) can be obtained in the same way.

Now we denote the projections of the image of a vector X at z by a homogeneous conact transformation f on fD_1 and fD_2 by $\bar{\Lambda}^a \bar{e}_a$ and $\bar{M}_a \bar{e}^a$, then as $\Lambda^i e_i$ and $M_i e^i$ are transformed by f to $\bar{\Lambda}^a \bar{e}_a$ and $\bar{M}_a \bar{e}^a$ respectively, we can easily see that

$$\bar{\Lambda} = (A + B\Gamma)\Lambda, \ \bar{M}(A + B\Gamma) = M \tag{7.18}$$

hold good by (6.22) and (7.13).

BIBLIOGRAPHY

[1] DAVIES, E. T., On the invariant theory of contact transformations, Math. Zeits., 57 (1953), 415-427.

[2] DAVIES, E. T. and K. YANO, Contact tensor calculus, Ann. di Mat., 37 (1954), 1-36.

[3] DOYLE, T. C. Tensor decomposition with applications to the contact and complex groups, Ann. of Math. 42 (1941), 698-721.

[4] EISENHART, L. P., Contact transformations. Ann. of Math. (2) 30 (1929), 211-249.

[5] EISENHART, L. P., Continuous groups of transformations. Princeton Univ. Press, 1933.

[6] EISENHART, L. P. AND M. S. KNEBELMANN, Invariant theory of homogeneous contact transformations. Ann. of Math. 37 (1936), 747-765

[7] HOSOKAWA, T., Connections in the manifold admitting contact transformations. Journ. Fasc. Sci. Hokkaido Imp. Univ., 2 (1934), 169-176.

[8] MUTÔ, Y., On the connections in the manifold admitting homogeneous contact transformations. Proc. Phys. Math. Soc. Japan, 20 (1938), 451-457.

[9] MUTÔ, Y. AND K. YANO, Sur les transformations de contact et les espaces de Finsler, Tôhoku Math. Journ., 45 (1939), 293-307.

[10] YANO, K., On the theory of linear connections in the manifold admitting contact transformations, Proc. Phys. Math. Soc. Japan, 17 (1935), 39-47.

TÔHOKU UNIVERSITY