GENERALIZED HIRZEBRUCH POLYNOMIALS

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Introduction. We have generalized the Hirzebruch polynomials in [8] and proved their integrality in [9]. In this paper we shall generalize the Hirzebruch polynomials in the most general way without loss of integrality. Moreover we shall utilize these polynomials for study of the cobordism coefficients or the cobordism ring. The cobordism ring of modulo 2 has been completely studied by Dold ([3]). Recently Milnor and Wall have completely made clear the torsion of the cobordism ring Ω ([6], [11]). However, there are still many unsolved problems concerning the free part of the Ω. We shall introduce a finite group which seems to be the centre of these problems.

1. Let X^{4k} be a compact orientable differentiable manifold whose dimension is $4k$. Let

$$X^{4k} \sim \sum_{i_1+\cdots+i_t=k} A^k_{i_1\cdots i_t} P_{2i_1}(c)\cdots P_{2i_t}(c) \qquad \text{mod torsion} \tag{1. 1}$$

be the cobordism decomposition of X^{4k} based on the complex projective spaces $P_{2i}(c)$'s. It is known that ([8])

$$\text{(1. 2)} \quad \begin{cases} \text{(a)} & 9A^4_4 = (-4p_4 + 4p_3p_1 + 2p_2^2 - 4p_2p_1^2 + p_1^4)[X^{16}], \\ \text{(b)} & 21A^4_{31} = (36p_4 - 33p_3p_1 - 18p_2^2 + 33p_2p_1^2 - 8p_1^4)[X^{16}], \\ \text{(c)} & 25A^4_{22} = (18p_4 - 18p_3p_1 - 7p_2^2 + 16p_2p_1^2 - 4p_1^4)[X^{16}], \\ \text{(d)} & 45A^4_{211} = (-180p_4 + 159p_3p_1 + 80p_2^2 - 150p_2p_1^2 + 36p_1^4)[X^{16}], \\ \text{(e)} & 81A^4_{1111} = (165p_4 - 137p_3p_1 - 70p_2^2 + 127p_2p_1^2 - 30p_1^4)[X^{16}], \\ \text{(f)} & 3^3\cdot 5^2\cdot 7\tau = 3^3\cdot 5^2\cdot 7(A^4_4 + \cdots + A^4_{1111}) \\ & \qquad = (381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4)[X^{16}], \end{cases}$$

where τ denotes the index of X^{16} and p_i denotes the Pontryagin class of the dimension $4i$. In [8] we introduced a multiplicative series such that

$$\prod_i \frac{\sqrt{r_i}}{\mathrm{tgh}\sqrt{r_i}}(1 + y\,\mathrm{tgh}^2\sqrt{r_i}) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} y^j \Gamma_{ij}(p_1,\cdots,p_i), \tag{1. 3}$$

where

$$\prod_i (1 + r_i) = \sum_{i=0}^{\infty} p_i. \tag{1. 4}$$

If we put $y = 0$ in (1. 3) we obtain the Hirzebruch polynomials. For brevity we put as follows:

(1. 5) $$\Gamma_{ij}(p_1,\cdots,p_i)[X^{4i}]=\Gamma_{ij}.$$

We have proved in [9] that the Γ_{ij}'s are integers. Some of them are as follows:

(1. 6) $$\begin{cases} \text{(g)} & \Gamma_{44}=p_4[X^{16}],\\ \text{(h)} & 3\Gamma_{43}=(8p_4-p_3p_1)[X^{16}],\\ \text{(i)} & 45\Gamma_{42}=(108p_4-27p_3p_1-7p_2^2+6p_2p_1^2)[X^{16}],\\ \text{(j)} & 3^3\cdot5\cdot41\Gamma_{41}=(744p_4-325p_3p_1-176p_2^2+248p_2p_1^2-51p_1^4)[x^{16}]. \end{cases}$$

The parametric multiplicative series (1. 3) admits the following generalization:

(1. 7) $$\prod_i\frac{\sqrt{r_i}}{\mathrm{tgh}\sqrt{r_i}}(1+y_1\mathrm{tgh}^2\sqrt{r_i}+y_2\mathrm{tgh}^4\sqrt{r_i}+\cdots+y_m\mathrm{tgh}^{2m}\sqrt{r_i})^{*}$$
$$=\sum_{i=0}^{\infty}\sum_{\alpha_1\cdots\alpha_p}y_{\alpha_1}\cdots y_{\alpha_p}L^i_{\alpha_1\cdots\alpha_p}(p_1,\cdots,p_i),$$

where $L^i_{\alpha_1\cdots\alpha_p}$' s denote certain polynomials of $p_1,\cdots,p_i$ whose weight is i. The integrality of

(1. 8) $$L^i_{\alpha_1\cdots\alpha_p}(p_1,\cdots,p_i)[X^{4i}]$$

can be proved in the same way as in the case of (1. 5) (Appendix, [9]). In the case of X^{16} we need only the following one:

(1. 9) $$\prod_i\frac{\sqrt{r_i}}{\mathrm{tgh}\sqrt{r_i}}(1+u\,\mathrm{tgh}^2\sqrt{r_i}+v\,\mathrm{tgh}^4\sqrt{r_i})=\sum_i\sum_{\alpha,\beta}u^\alpha v^\beta K^i_{\alpha\beta}(p_1,\cdots,p_i).$$

In particular we need the following term:

(1.10) $$45K^4_{01}(p_1,\cdots,p_4)=(-216p_4+156p_3p_1+94p_2^2-147p_2p_1^2+33p_1^4).$$

We put as follows:

(1.11) $$K^4_{01}(p_1,\cdots,p_4)[X^{16}]=K^4_{01}.$$

Of course the K^4_{01} is an integer.

2. We see from 2(f) + (c) that $5A^4_{22}$ is an integer. It follows from 2(f) + (e) that $9A^4_{1111}$ is an integer. The integrality of $3A^4_{31}$ follows from 2(f) + (b). We see from 111(j) + 126(f) − (d) + 70(i) that A^4_{211} is an integer. From (a) it is clear that $9A^4_4$ is an integer. From above facts and the integrality of $A^4_{1111}+A^4_4+A^4_{31}+A^4_{22}+A^4_{211}+A^4_{1111}$ we see that A^4_{22} and $3A^4_4+3A^4_{1111}$ are integers. From 2(f) + (e) + 9(j) + 9(h) + 9(d) + 9(b) we see that $3A^4_{1111}+A^4_{31}$ is an integer. Since A^4_{211}, K^4_{01} and $3A^4_{31}$ are integers we see from − 2(d) + $45K^4_{01}$ + 3(b) + 4(f) + 4(j) that $7A^4_{31}$ is an integer and hence A^4_{31} is an integer. Therefore $3A^4_{1111}$ and $3A^4_4$ are also integers. Since $A^4_4+\cdots+A^4_{1111}$ is an integer we see that $A^4_4+A^4_{1111}$ is

*) $\mathrm{tgh}^{2m}\sqrt{r_i}\equiv(\mathrm{tgh}\sqrt{r_i})^{2m}$.

an integer. Thus we have the following results:

$$(2.1)\quad \begin{cases} \text{(i)} & 3A^4_4,\ A^4_{31},\ A^4_{22},\ A^4_{211},\ 3A^4_{1111}\ \textit{are integers}, \\ \text{(ii)} & A^4_4 + A^4_{1111}\ \textit{is an integer}. \end{cases}$$

In the case of the manifold $W = F_4/\mathrm{Spin}(9)$ ([2], p.534) we have

$$(2.2)\qquad A^4_4 = -\frac{28}{3},\ A^4_{31} = 36,\ A^4_{22} = 18,\ A^4_{211} = -92,\ A^4_{1111} = \frac{145}{3}.$$

This is an example of non-integral A^4_4. Another example is found in the 16-dimensional submanifolds of the $P_9(c)$. Let X^{16} be a submanifold of X^{18}, i.e. $X^{16} \xrightarrow{j} X^{18}$. Let X^{16} correspond to a cohomology class $v \in H^2(X^{18}, Z)$. Then the cobordism coefficients of X^{16} are given by

$$(2.3)\quad \begin{cases} \text{(a)}\ A^4_4 = \frac{1}{9}\{-v(4p_4 - 4p_3p_1 - 2p_2^2 + 4p_2p_1^2 - p_1^4) - v^9\}[X^{18}], \\ \text{(b)}\ A^4_{31} = \frac{1}{21}\{v(36p_4 - 33p_3p_1 - 18p_2^2 + 33p_2p_1^2 - 8p_1^4) + v^3(-3p_3 \\ \qquad + 3p_2p_1 - p_1^3) - v^7p_1 + 10v^9\}[X^{18}], \\ \text{(c)}\ A^4_{22} = \frac{1}{25}\{v(18p_4 - 18p_3p_1 - 7p_2^2 + 16p_2p_1^2 - 4p_1^4) + v^5(2p_2 - p_1^2) \\ \qquad + 5v^9\}[X^{18}], \\ \text{(d)}\ A^4_{211} = \frac{1}{45}\{v(-180p_4 + 159p_3p_1 + 80p_2^2 - 150p_2p_1^2 + 36p_1^4) + v^3(21p_3 \\ \qquad - 19p_2p_1 + 6p_1^3) + v^5(-11p_2 + 5p_1^2) + 8v^7p_1 - 55v^9\}[X^{18}], \\ \text{(e)}\ A^4_{1111} = \frac{1}{81}\{v(165p_4 - 137p_3p_1 - 70p_2^2 + 127p_2p_1^2 - 30p_1^4) + v^3\cdot \\ \qquad (-28p_3 + 23p_2p_1 - 7p_1^3) + 3v^5(5p_2 - 2p_1^2) - 12v^7p_1 + 55v^9\}[X^{18}]. \end{cases}$$

These relations are derived from the multiplicative series

$$(2.4)\qquad \prod_i \frac{\sqrt{r_i}}{\mathrm{tgh}\sqrt{r_i}}(1 + y\,\mathrm{tgh}^2\sqrt{r_i})^{-1} = \sum_i \Lambda_i(y, p_1, \cdots, p_i),$$

$$(2.5)\qquad \sum_i \Lambda_i(y, p_1[X^{16}], \cdots, p_i[X^{16}]) = j^*\left[\frac{\mathrm{tgh}v}{v}(1 + y\,\mathrm{tgh}^2 v)\sum_i \Lambda_i(y, p_1[X^{18}], \cdots, p_i[X^{18}])\right]$$

and the relation

$$(2.6)\qquad \Lambda_4(y, p_1, \cdots, p_4)[X^{16}] = \kappa^{18}[\mathrm{tgh}v\,(1 + y\,\mathrm{tgh}^2 v)\sum_i \Lambda_i(y, p_1, \cdots, p_i)][X^{18}].$$

In the case of $P_9(c)$ it is known that

$$(2.7)\qquad p_1 = 10g^2,\ p_2 = 45g^4,\ p_3 = 120g^6,\ p_4 = 210g^8,\ g^9[P_9(c)] = 1.$$

Letting $X^{18} = P_9(c)$ and $v = 2g$ we have from (2. 3)

$$
\begin{cases} A^4_4 = -\dfrac{164}{3}, \quad A^4_{31} = 180, \quad A^4_{22} = 90, \\ A^4_{211} = -400, \quad A^4_{1111} = \dfrac{560}{3}. \end{cases} \tag{2. 8}
$$

We denote above submanifold of $P_9(c)$ by Q_{16}.

We have from (2. 1) and (2. 2) the

THEOREM. *A compact orientable differentiable manifold* X^{16} *admits the following cobordism decomposition*:

$$
X^{16} \sim \bar{A}^4_4 P_8(c) + \bar{A}^4_{31} P_6(c) P_2(c) + \bar{A}^4_{22} P_4(c)^2 + \bar{A}^4_{211} P_4(c) P_2(c)^2 + \bar{A}^4_{1111} P_2(c)^4 + \bar{A}^4_0 Q_{16} \quad \text{mod torsion,} \tag{2. 9}
$$

where the $\bar{A}$*'s denote some integer and the* $\bar{A}^4_0$ *takes the values* 0, 1 *and* 2 *and the* $\bar{A}$*'s are uniquely determined by* X^{16}.

3. We denote by G_k the additive group of the cobordism classes of the compact orientable differentiable manifolds of the dimension $4k$ modulo torsion and let

$$
X^{4k} \sim \sum_{i_1+\cdots+i_t=k} A^k_{i_1\cdots i_t} P_{2i_1}(c) \cdots P_{2i_t}(c) \quad \text{mod torsion} \tag{3. 1}
$$

be the cobordism decomposition of such a manifold. Among them, those cobordism classes whose all coefficients are integers form a sub-group. We denote this group by $\hat{G}_k$.

It is well known that $G_k = \hat{G}_k$, $k = 1,2,3$ ([4]).

Let us consider the factor group $G_k/\hat{G}_k$. We have proved in the last paragraph that

(i) $$G_4/\hat{G}_4 \approx Z_3.$$

Moreover we see that

(ii) $G_k/\hat{G}_k$ *is a finite group.*

PROOF. First of all $G_k/\hat{G}_k$ consists only of torsions because the cobordism coefficients are rational numbers. Moreover the number of these torsions is finite, because all $A^k_{i_1\cdots i_t}$'s $(i_1 + \cdots + i_t = k)$ become integers by multiplying a suitable large integer depending on k ([5] p.p.77 ~ 79).

(iii) *The sequence* $\hat{G}_4/G_4 \to \hat{G}_5/G_5 \to \cdots \to \hat{G}_k/G_k \to \hat{G}_{k+1}/G_{k+1} \to \cdots$ *is decreasing.*

PROOF. If we multiply an element of G_k by the $P_2(c)$, we obtain an element of G_{k+1}. We denote this injection by j:

$$G_k \xrightarrow{j} G_{k+1}.$$

It is clear that the injection j is isomorphic into and induces an injection such that

$$G_k/\widehat{G}_k \xrightarrow{j^*} G_{k+1}/\widehat{G}_{k+1}.$$

It is also clear that j^* is isomorphic into.

APPENDIX

Integrality of Γ_{ij} ((1. 5)).

It suffices to prove that

(i) the Γ_{ij} does not contain the factor 2 in its denominator when it is written as a quotient of relative prime integers.

(ii) $2^{\alpha}\Gamma_{ij}$ becomes an integer for a suitable integer α.

First of all let us prove (i). It is well known that

$$(1) \qquad \frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}} = 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k z^k \qquad \text{([5] p.13)},$$

$$(2) \qquad \mathrm{tgh}z = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_n z^{2n-1}.$$

Moreover it is known that

(3) a. B_k(Bernoulli number) contains the factor 2 exactly to the first power in its denominator.

b. $(2k)!$ is not divisible by 2^{2k} ([2] II p.341).

The statement (i) easily follows from these facts and (1. 3). Next let us prove (ii). It suffices to show this for the complex algebraic manifold ([6]). In this case we have

$$(4) \qquad \sum_{i=0}^{\infty} p_i = \prod_i (1+\delta_i^2), \quad \sum_i c_i = \prod_i (1+\delta_i)$$

and

$$(5) \qquad \prod_i \frac{\sqrt{r_i}}{\mathrm{tgh}\sqrt{r_i}}(1+y\,\mathrm{tgh}^2\sqrt{r_i}) = \prod_i \frac{\delta_i}{\mathrm{tgh}\delta_i}(1+y\,\mathrm{tgh}^2\delta_i)$$

$$= \prod_i \frac{\delta_i}{1-e^{-\delta_i}}\left(\frac{1}{\mathrm{tgh}\delta_i} + y\,\mathrm{tgh}\delta_i\right)(1-e^{-\delta_i})$$

$$= \prod_i \frac{\delta_i}{1-e^{-\delta_i}} \left\{ \frac{\frac{1}{2}(1+e^{-2\delta_i})}{1-\frac{1-e^{-\delta_i}}{2}} + y \frac{\frac{1}{2}(1-e^{-2\delta_i})(1-e^{-\delta_i})}{1-\frac{1-e^{-2\delta_i}}{2}} \right\}.$$

Hence we have

$$(6) \quad \sum_{j=0}^{n} y^j \Gamma_{nj}(p_1,\cdots,p_n) = \kappa_{4n} \Big[\prod \frac{\delta_i}{1-e^{-\delta_i}} \prod \Big[\frac{1}{2}(1+e^{-2\delta_i})$$

$$\times \left\{ 1 + \frac{1-e^{-\delta_i}}{2} + \left(\frac{1-e^{-\delta_i}}{2}\right)^2 + \cdots + \left(\frac{1-e^{-\delta_i}}{2}\right)^{2n} \right\} + \frac{1}{2} y (1-e^{-2\delta_i})(1-e^{-\delta_i})$$

$$\times \left\{ 1 + \frac{1-e^{-2\delta_i}}{2} + \left(\frac{1-e^{-2\delta_i}}{2}\right)^2 + \cdots + \left(\frac{1-e^{-2\delta_i}}{2}\right)^{2n-2} \right\} \Big] \Big]$$

$$= \kappa_{4n} \left[\left(\sum_t y^t \sum_{a_1,,a_m} A^t_{a_1\cdots a_m} \sum_{i_1 \neq \cdots \neq i_m} e^{a_1\delta_{i_1}+\cdots+a_m\delta_{i_m}} \right) \times \prod_l \frac{\delta_i}{1-e^{-\delta_i}} \right],$$

where A denotes some rational number which becomes integer by multiplying suitable power of 2 and a_i denotes some integer.

Putting

$$(7) \qquad \prod_{i_1 \neq \cdots \neq i_m} (1 + a_1\delta_{i_1} + \cdots + a_m\delta_{i_m}) = \sum_i d_i$$

we see that d_i is an integral cohomology class, i.e. $d_i \in H^{2i}(X^{4n}, Z)$.
Hence we have

$$(8) \quad \Gamma_{nk} = \Gamma_{nk}(p_1,\cdots,p_n)[X^{4n}] = \sum_i b_i T(X^{4n}, W_i) = \sum_i b_i \chi(X^{4n}, W_i),$$

where b_i denotes some rational number which becomes integer by multiplying some power of 2 and $T(X^{4n}, W_i)$ denotes the Todd genus with regard to W_i and W_i denotes a complex analytic vector bundle whose Chern class is Σd_i and $X(X^{4n}, W_i)$ denotes the Riemann-Roch number with regard to W_i([5] p.154). Thus we have proved (ii).

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