ON CONFORMAL AND PROJECTIVE TRANSFORMATIONS IN KÄHLERIAN MANIFOLDS

YOSHIHIRO TASHIRO

(Received March 15, 1962)

(Revised October 10, 1962)

In this paper, we shall briefly say that a vector field v is isometric, conformal or projective if it generates a one-parameter group of isometric, conformal or projective transformations respectively.

A few years ago, A.Lichnerowicz [2] proved that, in a compact Kählerian manifold M^{2n} of complex dimension $n \geqq 2$, a conformal vector is always isometric. Recently, the theorem was generalized for almost-Kählerian manifolds by S.Tachibana [3] and S.I.Goldberg [1] in different ways. Either of them used some integral formulas.

The purpose of the present paper is to give an alternative proof of Lichnerowicz' Theorem, which is free from integral formula in some sense. Moreover we shall prove a theorem on a projective vector in the same method.

Our notations follow those of K.Yano's book [4]. Latin indices, running from 1 to $2n$, indicate that equations are referred to a general coordinate system (x^h), and Greek indices, running from 1 to n, indicate that equations are referred to a complex coordinate system (z^α, z^{α^*}). The dimension of manifold will always mean complex dimension.

1. Conformal vector. A conformal vector $v = (v^h)$ in a Riemannian manifold M^{2n} with metric tensor g_{ji} is characterized by the equation

$$\pounds_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \tag{1. 1}$$

$\pounds_v$ indicating the Lie differentiation with respect to v and ρ being a real-valued scalar, called the associated scalar with v. If the associated scalar ρ is zero or a non-zero constant, the vector field v is reduced to an isometric or homothetic one respectively.

Concerning a conformal vector v, we know the following formulas [4,p.160]:

$$\pounds_v \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} = \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji}\rho^h, \tag{1. 2}$$

$$\pounds_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - g_{ji}\nabla_k \rho^h + g_{ki}\nabla_j \rho^h, \tag{1. 3}$$

$$\pounds_v K = -2\rho K - 2(2n-1)\nabla_i \rho^i, \tag{1. 4}$$

where we have put

$$\rho_i = \partial_i \rho, \ \rho^h = \rho_i g^{ih} \tag{1. 5}$$

and K is the contracted scalar curvature.

First we shall prove the following

LEMMA 1. *In a Kählerian manifold M^{2n} of complex dimension n, the associated scalar ρ of a conformal vector is the real part of an analytic function for $n > 2$, and we have*

$$\Delta\rho \equiv g^{ji}\nabla_j\nabla_i\rho = 0 \tag{1. 6}$$

for $n \geqq 2$.

PROOF. By (1. 3) and the well-known formula of Lie derivative of a tensor, we have

$$\begin{aligned} v^l\nabla_l K_{kjih} + K_{ljih}\nabla_k v^l + K_{klih}\nabla_j v^l + K_{kjlh}\nabla_i v^l - K_{hji}{}^l\nabla_l v_h \\ = - g_{kh}\nabla_j\rho_i + g_{jh}\nabla_k\rho_i - g_{ji}\nabla_k\rho_h + g_{ki}\nabla_j\rho_h. \end{aligned} \tag{1. 7}$$

We take a complex coordinate system (z^α, z^{α^*}) in M^{2n}, and put the indices $h = \alpha$, $i = \beta$, $j = \gamma^*$ and $k = \delta^*$ in (1. 7). Then by the hybridism of the metric tensor g_{ji} and the covariant curvature tensor K_{kjih} of a Kählerian manifold, we obtain the eqaution

$$g_{\delta^*\alpha}\nabla_{\gamma^*}\rho_\beta - g_{\gamma^*\alpha}\nabla_{\delta^*}\rho_\beta + g_{\gamma^*\beta}\nabla_{\delta^*}\rho_\alpha - g_{\delta^*\beta}\nabla_{\gamma^*}\rho_\alpha = 0.$$

By contracting this equation with $g^{\delta^*\alpha}$,we have

$$(n-2)\nabla_{\gamma^*}\rho_\beta + g_{\gamma^*\beta}g^{\delta^*\alpha}\nabla_{\delta^*}\rho_\alpha = 0, \tag{1. 8}$$

and, by contracting again with $g^{\gamma^*\beta}$,

$$2(n-1)g^{\gamma^*\beta}\nabla_{\gamma^*}\rho_\beta = 0,$$

from which we obtain (1. 6) for $n \geqq 2$. Further it follows from (1. 8) that

$$\nabla_{\gamma^*}\rho_\beta = 0 \tag{1. 9}$$

for $n > 2$. Since ρ is real valued, the equation (1. 9) implies that the associated scalar ρ is the real part of an analytic function. Q. E. D.

Now let us give an alternative proof of Lichnerowicz'

THEOREM 1. *In a compact Kählerian manifold M^{2n} of dimension $n \geqq 2$, a conformal vector v is always isometric.*

PROOF. On a compact manifold M^{2n}, Hopf's maximal principle or Bochner's lemma tell us that the equation (1. 6) implies that ρ is a constant, that is, the vector field v is homothetic. However, it is well known [4,p.222] or can be proved by integration along a trajectory of v [5] that ρ should be equal to zero, i.e., a homothetic vector on a compact manifold is isometric. Q. E. D.

From Lemma 1 and the formula (1. 4), we have immediately

THEOREM 2. *In a Kählerian manifold whose contracted scalar curvature K is a non-zero contant, a conformal vector is isometric.*

Since a Riemannian homogeneous space has a constant scalar curvature, we have

THEOREM 3. *In a Kählerian homogeneous space with non-zero scalar curvature, a conformal vector is isometric.*

2. Projective vector. A projective vector v is characterized by the equation

$$\mathfrak{L}_v \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} = \nabla_j \nabla_i v^h + K_{lji}{}^h v^l = \delta_j^h p_i + \delta_i^h p_j, \tag{2. 1}$$

where p_i is the gradient vector given by

$$p_i = \nabla_i \nabla_h v^h/(2n+1). \tag{2. 2}$$

Putting $p = \nabla_h v^h/(2n+1)$, we call p and p_i the associated scalar and vector with v respectively.

Concerning a projective vector, we know the formula

$$\mathfrak{L}_v K_{kji}{}^h = -\delta_k^h \nabla_j p_i + \delta_j^h \nabla_k p_i \tag{2. 3}$$

or

$$\begin{aligned} v^l \nabla_l K_{kjih} + K_{ljih} \nabla_k v^l + K_{klih} \nabla_j v^l + K_{kjlh} \nabla_i v^l - K_{kji}{}^l \nabla_l v_h \\ = -g_{kh} \nabla_j p_i + g_{jh} \nabla_k p_i. \end{aligned} \tag{2. 4}$$

By the same argument as those in §1, we have easily the following

LEMMA 2. *In a Kählerian manifold M^{2n}, $n \geqq 2$, the associated scalar p of a projective vector is the real part of an analytic function*:

$$\nabla_{\gamma^*} \nabla_\beta p = 0. \tag{2. 5}$$

THEOREM 4. *In a compact Kählerian manifold M^{2n}, $n \geqq 2$, a projective vector is isometric.*

REMARK. By use of the complex structure $F_i{}^h$ and the operations O and O^* in [4], we can give proofs of the above theorems, too, which are essentially same as those given here.

REFERENCES

[1] S. I. GOLDBERG, Groups of transformations of Kähler and almost-Kähler manifolds, Comment. Math. Helv., 35 (1961), 35-46.

[2] A. LICHNEROWICZ, Sur les transformations analytiques des variétés kähleriennes compactes, C. R., 244 (1957), 3011-3013.

[3] S. TACHIBANA, On almost analytic vectors in almost-Kählerian manifolds, Tôhoku Math.

Journ., 11(1959), 247-265.
[4] K. YANO, Theory of Lie Derivatives and its Applications, Amsterdam, 1956.
[5] K. YANO AND T. NAGANO, The de Rham decomposition, isometries and affine transformations in Riemannian space. Jap. Journ. Math., 29(1959), 173-184.

OKAYAMA UNIVERSITY
AND UNIVERSITY OF HONG KONG