ON PRINCIPAL TORUS BUNDLES OVER A HOMOGENEOUS CONTACT MANIFOLD

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Introduction. W. M. Boothby and H. C. Wang [3] have studied a compact manifold with a regular contact structure, and showed that it is a principal fiber bundle over a symplectic manifold with a structural group T^1($=$1-dimensional torus group). Especially it has been shown that a compact, simply connected homogeneous contact manifold is a principal T^1-bundle over a homogeneous Kählerian manifold.

If we take a contact manifold M, then we can provide it a contact metric structure choosing a suitable metric on M. Then for any principal T^1-bundle P over M, we can give an almost complex structure using the contact metric structure on M and an infinitesimal connection on P. The integrability condition of this almost complex structure is a question. In this note, we shall study the case when M is moreover compact, simply connected and homogeneous contact manifold, and we shall show that any principal T^1-bundle P over M has a homogeneous complex structure using the above fibering of Boothby-Wang. Since a compact, simply connected homogeneous contact manifold M has a normal contact metric structure [8], if P is a trivial principal bundle, then the proposition is easily shown taking a flat connection. Therefore it is essential when P is not trivial.

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1. Preliminaries. Let M be a compact, simply connected homogeneous manifold. It is known that the Lie group G which acts transitively on M has a compact semi-simple subgroup which acts transitively on M. Therefore we can take the first G as a compact semi-simple Lie group, and M is a coset space of G by a closed subgroup L. Moreover, we can suppose without loss of generality that the compact, semi-simple Lie group G is simply connected. In fact it is sufficient to take a universal covering group of G. Therefore we assume that in the following G is compact and simply connected. In this situation, L is also compact, and G/L is a reductive homogeneous space in the sense of K. Nomizu [1]. If we represent the Lie algebra of the Lie group G and L by $\widehat{G}$ and $\widehat{L}$, then there exists a subspace $\widehat{M}$ of $\widehat{G}$ such that

$$\widehat{G} = \widehat{L} + \widehat{M}, \qquad \widehat{L} \cap \widehat{M} = (0), \qquad [\widehat{L}, \widehat{M}] \subset \widehat{M}.$$

We consider a natural principal fiber bundle $G(G/L, \pi, L)$ with a canonical projection $\pi: G \to G/L$. Then we have a natural G-invariant connection ω on G in accordance with the above decomposition. We consider in the following that the Lie algebra $\widehat{G}$ of G (composed of all left invariant vector fields on G) is identified with the tangent space of G at the unit element e. In this identification, the differential of the right translation of G by the element g of G can be considered as the adjoint operation of $T_e(G)$ by g. We denote it by A_g. The differentiable connection ω is defined for any element $X \in \widehat{G}$, and $g \in G$,

$$\omega(X) = X_{\hat{L}}, \qquad L_g{}^* \omega = \omega$$

where $X_{\hat{L}}$ means the $\widehat{L}$-component of X. It can be shown that ω satisfies the two conditions of connection on G:

$$\omega(\bar{g} l) = A_l^{-1} \omega(\bar{g}), \qquad \bar{g} \in T_g(G),\ l \in L,$$
$$\omega(g \bar{l}) = l^{-1} \bar{l}, \qquad g \in G,\ \bar{l} \in T_l(L).$$

Next we define a 1-form Ψ on G using a homomorphism μ of L onto T^1 by

$$\Psi = \mu \cdot \omega.$$

Then Ψ is G-invariant and takes value in $\widehat{T^1}$. If we restrict Ψ on G, then it satisfies the following equations:

$$\Psi A_l = \Psi, \qquad l \in L,$$
$$\Psi(\bar{l}) = \mu(\bar{l}), \qquad \bar{l} \in \widehat{L}.$$

We take a differentiable T^1-bundle P over M. S. Murakami [2] showed that if the Lie group G is compact and simply connected, then any differentiable principal bundle P over a homogeneous space G/L with an abelian structural group T^1 corresponds to a differentiable homomorphism $\mu: L \to T^1$. P is obtained from μ and the principal fiber bundle $G(G/L, \pi, L)$ as follows: Let $G \times T^1$ be the product space of G and T^1, and we introduce an equivalence relation $(\sim)$ in $G \times T^1$ by the rule

$$(g, a) \sim (g l, \mu(l)^{-1} a), \quad l \in L.$$

Then the manifold $P = G \times T^1/(\sim)$ is a differentiable principal bundle space over G/L with the structural group T^1. We take a 1-form Ψ for this homomorphism μ. If we write the kernel of μ by K, then K is clearly a closed subgroup of L and of course that of G. We suppose that the homogeneous space G/L admits a non-trivial principal T^1-bundle P over it. Then by virtue of the theorem of Murakami cited above, there exists a non-trivial homomorphism μ of L onto T^1. This fact is also equivalent to the condition that the compact Lie group L is not semi-simple.

LEMMA 1. *The principal fiber bundle $P(G/L, p, T^1)$ is equivalent to the principal fiber bundle $G/K(G/L, p, L/K)$ in a natural fashion.*

PROOF. Let $\lambda: G \to G/K$ be a canonical map which coincides with μ on L. We correspond any element $\lambda(g)$ of G/K to the class $[(g, e)]$ of P. This correspondence is well defined, and is clearly onto, one-to-one. We take an element $\lambda(g)$ of G/K and an element $\mu(l)$ of $L/K = T^1$. Since K is an invariant subgroup of L, we have

$$R_{\mu(l)} \lambda(g) = \lambda(g l),$$

where $R_{\mu(l)}$ denotes the right translation on G/K by the structural group L/K. Therefore the above correspondence between G/K and P commutes with the right translations of the structural groups, that is,

$$R_{\mu(l)} [(g, e)] = [(gl, e)].$$

This shows that the correspondence $\lambda(g) \to [(g, e)]$ is a bundle map. Therefore P and G/K have equivalent bundle structures. Q. E. D.

By virtue of this lemma, we can identify the principal fiber bundle $P(G/L, p, T^1)$ with $G/K(G/L, p, L/K)$. In the following we always take this identification. Since the Lie algebra $\widehat{T^1}$ of the 1-dimensional Lie group $T^1 = L/K$ is the 1-dimensional real line, we can consider every $\widehat{T^1}$-valued mapping as a real valued mapping. Therefore Ψ is a real valued left invariant 1-form on G. The center of $\widehat{L}$ is mapped onto $\widehat{T^1}$ by $\mu: \widehat{L} \to \widehat{T^1}$ induced from the homomorphism $L \to T^1$. Therefore we can choose an element α of the center of $\widehat{L}$ such as $\mu(\alpha)=1$. Then it holds

$$\Psi(\alpha) = \mu \cdot \omega(\alpha) = \mu(\alpha) = 1.$$

$\widehat{L}$ can be decomposed to the direct sum of the Lie algebra $\widehat{K}$ of K and the space $\{\alpha\}$ generated by α.

2. Homogeneous almost contact manifold. Suppose M be a compact, simply connected manifold on which a compact simply connected Lie group G acts transitively. Then there exists an isotropic subgroup L of G and M is diffeomorphic with G/L. As in the previous section, we have a 1-form Ψ. Furthermore we assume that M admits a G-invariant almost contact structure (ϕ, ξ, η). These tensors are of types (1, 1), (1, 0) and (0, 1) and satisfy the following relations:

$$\eta(\xi) = 1,$$
$$\phi(\xi) = 0,$$
$$\phi\phi = -\delta + \eta \otimes \xi.$$

From the projection $\pi: G \to G/L$ we have the inverse images ϕ^*, η^* on G

$$\phi^* = \pi^* \phi, \qquad \eta^* = \pi^* \eta.$$

ϕ^* is a $T(M)$-valued G-invariant mapping on $T(G)$ and η^* is a G-invariant tensor fields of type (0, 1) on G. The restrictions of ϕ^* and η^* on $\widehat{G}$ satisfy the following relations;

$$\phi^* A_l = \phi^*, \qquad \eta^* A_l = \eta^*, \qquad l \in L,$$
$$\phi^*(\widehat{L}) = \eta^*(\widehat{L}) = 0.$$

We define an endomorphism $I: \widehat{G} \to \widehat{G}$ by

$$\pi(IX) = \phi^*(X) + \Psi(X)\xi,$$
$$\omega(IX) = -\eta^*(X)\alpha.$$

Then we have

$$\Psi(IX) = -\eta^*(X).$$

THEOREM. 1. *The endomorphism I satisfies the following properties*:

(i) $IX = 0$ *if and only if* $X \in \widehat{K}$,

(ii) $I^2X \equiv -X \pmod{\widehat{K}}, \quad X \in \widehat{G}$,

(iii) $IA_kX \equiv A_k IX \pmod{\widehat{K}}, \quad k \in K, \quad X \in \widehat{G}$,

(iv) $I\alpha \in \widehat{M}, \quad A_l I\alpha = I\alpha, \quad l \in L$.

Therefore I defines a homogeneous almost complex structure on G/K.

PROOF. (i). If $X \in \hat{K}$, then $\pi(IX) = 0$, $\omega(IX) = 0$. Therefore $IX = 0$. Conversely if $IX = 0$ for $X \in \hat{G}$, then we have $\pi X = 0$, and so $X \in \hat{L}$. We denote X_α the $\{\alpha\}$-component of X, with respect to the decomposition $\hat{L} = \hat{K} + \{\alpha\}$. Then $\pi(IX) = \Psi(X_\alpha)\xi = 0$, therefore $X_\alpha = 0$. This shows that X belongs to $\hat{K}$.

(ii). For any $X \in \hat{G}$, we have

$$\pi(I^2 X) = \phi(\pi IX) + \Psi(IX)\xi = -\pi(X),$$

$$\Psi(I^2 X) = -\eta(\pi IX) = -\Psi(X).$$

On the other hand X satisfies $\pi(X) = \Psi(X) = 0$ if and only if $X \in \hat{K}$. Therefore from the above two equalities, we have

$$I^2 X \equiv -X \pmod{\hat{K}}.$$

(iii). Since ϕ, ξ and η are G-invariant, we have for $k \in K$, $X \in \hat{G}$,

$$\begin{aligned}\pi(IA_k X) &= \phi(L_k^{-1}\pi X) + \Psi(X) L_k^{-1}\xi \\ &= L_k^{-1}\{\phi(\pi X) + \Psi(X)\xi\} \\ &= L_k^{-1}\pi(IX) = \pi(A_k IX),\end{aligned}$$

$$\Psi(IA_k X) = -\eta(\pi A_k X) = -\eta(\pi X),$$

$$\Psi(A_k IX) = -\eta(\pi X).$$

Therefore it holds $IA_k X \equiv A_k IX \pmod{\hat{K}}$ for all $X \in \hat{G}$. We see more precisely that in this equation the element of K can be replaced with the element of L, since the above three equations hold when we take $l \in L$ in place of $k \in K$.

(iv). If we put $\beta = I\alpha$, then

$$\pi(\beta) = \phi(\pi\alpha) + \Psi(\alpha)\xi = \xi \neq 0,$$

$$\omega(\beta) = -\eta(\pi\alpha) = 0.$$

Therefore β is non-zero and belongs to $\hat{M}$. As the Lie algebra $\hat{G}$ is reductive we have $A_l\beta \in \hat{M}$ for $l \in L$. On the other hand, we have seen that $IA_l\alpha \equiv A_l I\alpha \pmod{\hat{K}}$ for $l \in L$. As α is an element of the center of $\hat{L}$, it holds that $A_l\alpha=\alpha$ for $l\in L$. Therefore we have $A_l\beta\equiv\beta$ (mod $\widehat{K}$). Therefore $A_l\beta-\beta$ belongs to both $\widehat{M}$ and $\widehat{K}$, which means

$$A_l\beta=\beta, \qquad l\in L. \qquad \text{Q.E.D.}$$

The conditions (i)~(iii) of this theorem show that any T^1-bundle P over $M=G/L$ has a homogeneous almost complex structure.

Conversely, we take a T^1-bundle P over a compact simply connected homogeneous manifold $M=G/L$ and suppose that $P=G/K$ admits a homogeneous almost complex structure. Then there exists an endomorphism $I\colon \widehat{G}\to\widehat{G}$ which satisfies the conditions (i)~(iii) of Theorem 1. If there exists an element $\alpha\in\widehat{L}$ such that

$$\text{(iv)} \qquad \beta=I\alpha\in\widehat{M}, \qquad A_l\beta=\beta \qquad \text{for} \quad l\in L,$$

then the base manifold G/L admits a homogeneous almost contact structure. In fact, by virtue of (iv), the non-zero vector $\pi\beta\in T_{x_0}(G/L)$ can be extended to a G-invariant vector field ξ over G/L which takes value $\pi\beta$ at $x_0=\pi(e)\in G/L$. For this purpose it is sufficient to define $\xi_x=L_g\xi_{x_0}$ for $x=L_gx_0$. Other tensors ϕ, η over G/L can be defined as follows: for any vector $\bar{x}\in T_{x_0}(G/L)$, taking a lift X of $\bar{x}$ with respect to the connection ω on $G(G/L,\pi,L)$ (see § 1), we define at x_0

$$\phi(\bar{x})=\pi(IX),$$

$$\eta(\bar{x})=-\Psi(IX).$$

For any $x=L_gx_0\in G/L$, we define $\phi_x=L_g\phi_{x_0}L_g^{-1}$, $\eta_x=\eta_{x_0}\cdot L_g^{-1}$. Then for a lift $X\in\widehat{G}$ of a vector $\bar{x}\in T_{x_0}(G/L)$, X belongs to $\widehat{M}$ and so the lift of $L_l\bar{x}\in T_{x_0}(G/L)$ is $A_lX\in\widehat{M}$ for $l\in L$. Therefore we have by virtue of (i) and (iii)

$$L_l^{-1}\phi(L_l\bar{x})=L_l^{-1}\pi(IA_lX)=L_l^{-1}\pi(A_lIX)=\pi(IX)=\phi(\bar{x}),$$

$$\eta(L_l\bar{x})=-\Psi(IA_lX)=-\Psi(A_lIX)=\eta(\bar{x}).$$

The tensors ϕ, η are well-defined, and clearly are G-invariant. Next for any element X of $\widehat{G}$, we have $\pi(IX_{\hat{L}})=\Psi(X)\xi$. Since the element $X_{\hat{M}}$ is a lift of a vector in $T_{x_0}(G/L)$, we see that

$$\phi(\pi X_{\hat{M}})=\pi I(X-X_{\hat{L}})=\pi(IX)-\Psi(X)\xi.$$

Therefore

$$\pi(IX) = \phi(\pi X) + \Psi(X)\xi .$$

In a similar way

$$\Psi(IX) = -\eta(\pi X)$$

for any $X \in \widehat{G}$. Since the lift of a vector $\phi\bar{x}$ is $IX - \Psi(X)\beta$ when X is a lift of $\bar{x} \in T_{x_0}(G/L)$, using these equations and the condition (ii), we see

$$\eta(\xi) = 1 , \qquad \eta(\phi\bar{x}) = 0 , \qquad \phi(\xi) = 0 ,$$
$$\phi^2(\bar{x}) = -\bar{x} + \eta(\bar{x})\xi .$$

Therefore (ϕ, ξ, η) admits a homogeneous almost contact structure on M. This proves the following theorem.

THEOREM 2. *Let M be a compact simply connected homogeneous manifold. If a principal T^1-bundle P over M admits a homogeneous almost complex structure which can be defined by an endomorphism I: $\widehat{G} \to \widehat{G}$ satisfying the condition* (iv), *then M admits a homogeneous almost contact structure.*

We can decompose $\widehat{G} = \widehat{L} + \widehat{M}$ as $\widehat{L} = \widehat{K} + \{\alpha\}$, $\widehat{M} = \{\beta\} + \widehat{M}'$,

$$\widehat{G} = \widehat{K} + \{\alpha\} + \{\beta\} + \widehat{M}' , \qquad \text{(direct sum)}$$

where $\{\beta\}$ denotes the 1-dimensional subspace generated by β. By virtue of Theorems 1 and 2, a compact simply connected homogeneous space G/L admits a homogeneous almost contact structure when and only when there exists an endomorphism of $\widehat{G}$ whose component is

$$\begin{array}{cc} \begin{array}{ccc} \overbrace{\qquad}^{r} & \overbrace{\qquad\quad}^{2} & \overbrace{\qquad\quad}^{2n} \end{array} & \\ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & E \\ -E & 0 \end{matrix} \end{array}\right) & \begin{array}{l} \}\, r \\ \}\, 2 \\ \}\, 2n \end{array} \end{array} \qquad \begin{array}{l} r = \dim K , \\ 2n + 1 = \dim M , \end{array}$$

for an adequate basis of $\widehat{G}$.

3. Homogeneous contact manifold. Let M be a compact, simply connected homogeneous contact manifold. Then M is a coset space of a compact and simply connected Lie group G by an isotropy subgroup L. Let η be the contact form on M such that $\eta \wedge (d\eta)^n \neq 0$ on M. Then we shall prove the following

THEOREM 3. *There exists on M a homogeneous almost contact metric structure which associates to the homogeneous contact form η.*

PROOF. As M is a compact homogeneous manifold G/L, M admits a Riemannian metric which is invariant by the left operation of G. We take such a G-invariant metric h. Since the 2-form $d\eta$ is skew-symmetric and has rank $2n$, we can adopt the method of Y. Hatakeyama [7]. Consider two G-invariant distributions D_1, D_2 at each point of M:

$$D_1(x) = \{\bar{x} \in T_x(M)\,;\ \eta(\bar{x}) = 0\}\,,$$

$$D_2(x) = \{\bar{x} \in T_x(M)\,;\ d\eta(\bar{x}, T_x(M)) = 0\}\,.$$

Then D_1 and D_2 span the whole tangent space $T_x(M)$ and are complement to each other. Therefore we can take at each point x of M a coordinate neighbourhood U and orthonormal frames $e_1, e_2, \cdots, e_{2n}, e_\Delta$ with respect to the metric h, such that $e_1, \cdots, e_{2n} \in D_1$ and $e_\Delta \in D_2$. Now for any $x \in M$, $g \in G$, we put $y = L_g x \in M$. Then we can take open neighbourhoods U of x and V of y such that $L_g U \subset V$ and they have the above adapted frames. Since for the element g of G, L_g is an isometric transformation, it has with respect to the adapted frame the component

$$L_g = \begin{pmatrix} A_V^U & 0 \\ 0 & 1 \end{pmatrix},$$

where A_V^U is an orthogonal matrix of degree $2n$. On the other hand, the components of 2-form $d\eta$ on U are

$$\psi_U = \begin{pmatrix} \psi'_U & 0 \\ 0 & 0 \end{pmatrix}$$

where $\psi'_U \in GL(2n)$. We can decompose the regular matrix ψ'_U as the product of $\phi'_U \in O(2n)$ and $g'_U \in H(2n)$ where $O(2n)$ and $H(2n)$ denote the orthogonal group of degree $2n$ and the set of all positive definite symmetric matrices of degree $2n$. Since $d\eta$ is G-invariant, we have $L_g \psi_U = \psi_V L_g$, therefore

$$A_V^U \psi'_U = \psi'_V A_V^U .$$

And so

$$A_V^U \phi'_U {}^t A_V^U A_V^U g'_U {}^t A_V^U = \phi'_V g'_V .$$

From the uniqueness of the decomposition $GL(2n) \to O(2n) \times H(2n)$ and from the fact that A_V^U is an orthogonal matrix, we have

$$\phi'_V = A_V^U \phi'_U {}^t A_V^U ,$$
$$g'_V = A_V^U g'_U {}^t A_V^U .$$

Therefore, for the tensors g and ϕ whose components are

$$g_U = \begin{pmatrix} g'_U & 0 \\ 0 & 1 \end{pmatrix}, \qquad \phi_U = \begin{pmatrix} \phi'_U & 0 \\ 0 & 0 \end{pmatrix},$$

it holds that $L_g g_U = g_V L_g$, $L_g \phi_U = \phi_V L_g$. This shows that the almost contact metric structure (ϕ, ξ, η, g) is G-invariant. Q. E. D.

By virtue of this Theorem 3, we can consider G/L as a homogeoeous almost contact manifold. We take a non-trivial T^1-bundle over a compact simply connected homogeneous contact manifold $M = G/L$. Then by virtue of § 1, this principal bundle space can be identified with the homogeneous space G/K where K is the kernel of the homomorphism $\mu : L \to T^1$. Let K_0 be the identity component of K. As G is simply connected, $P_0 = G/K_0$ is a universal covering space of $G/K = P$. Moreover we know owing to Boothby-Wang [3] that a compact simply connected homogeneous contact manifold G/L is a principal T^1-bundle over a homogeneous Kähler manifold G/H. The fiber H/L is a 1-parameter torus group T^1 which is generated at each point of M by an associated direction field ξ of the contact form η.

LEMMA 2. *The Lie algebra $\hat{H}$ of the Lie subgroup H is spanned by $\hat{L}$ and β.*

PROOF. Since $\dim(G/H) = 2n$, we have $\dim \hat{H} = \dim \hat{L} + 1$. We take an element $\gamma \in \hat{G}$ which spans $\hat{H}$ in company with $\hat{L}$. The Lie algebra of $T^1 = H/L$ which can be considered as the image of $\hat{H}$ by the natural projection $\pi : G \to G/L$ is isomorphic with the vertical subspace of a tangent space to G/L at any point of G/L (with respect to the connection η). The latter is spanned by ξ. Therefore we have $\{\pi\gamma\}=\{\pi\beta\}$, and γ is equal to β modulo $\hat{L}$. This proves our Lemma.

LEMMA 3. *The homogeneous space H/K_0 is a complex 1-dimensional torus.*

PROOF. By virtue of the condition (iv) of Theorem 1, we know $[\hat{L},\beta]=0$. Therefore $\hat{H}=\hat{L}+\{\beta\}$ is contained in the centralizer of β in $\hat{G}$. Since $[\hat{H},\hat{H}]=[\hat{L},\hat{L}]\subset\hat{K}$, the Lie algebra $\hat{K}$ is an ideal of $\hat{H}$, and the quotient subalgebra $\hat{H}/\hat{K}$ is abelian. Consequently the compact connected abelian Lie group H/K_0 associating to the Lie algebra $\hat{H}/\hat{K}$ is isomorphic with the 2-dimensional torus T^2. However we can verify easily that the endomorphism $I: \hat{G}\to\hat{G}$ gives a homogeneous almost complex structure on H/K_0. Since the generators α,β of the Lie algebra $\hat{H}/\hat{K}$ satisfy $I\alpha=\beta$, $I\beta=-\alpha$, we have

$$N(\alpha,\beta)=[\alpha,\beta]+I[I\alpha,\beta]+I[\alpha,I\beta]-[I\alpha,I\beta]=0\,.$$

Therefore this almost complex structure on H/K_0 is integrable, and so H/K_0 is a complex 1-dimensional torus.

From this Lemma, we can give the structure of a complex Lie algebra of H/K_0 by the relation

$$\beta=\sqrt{-1}\,\alpha\,.$$

Next we consider the integrability condition of the almost complex structure defined in Theorem 1. As the base space G/H of the fibering of Boothby-Wang is a homogeneous Hodge manifold, we can take a normal contact metric structure (ϕ,ξ,η,g) on G/L associated to the contact form η [8]. In this case, for the G-invariant complex structure J_0 of G/H, it holds

$$q\phi=J_0q$$

where q denotes the natural projection $q: G/L\to G/H$. As η and the metric g is G-invariant, ϕ is also G-invariant. We can give to a principal bundle $P_0=G/K_0(G/L,p,L/K_0)$ an endomorphism I defined as in Theorem 1. A G-invariant almost complex structure I_0 of G/K_0 can be defined by

$$\lambda I=I_0\lambda\,,$$

where λ denotes the natural projection $G\to G/K_0$.

Now we consider the principal bundle $G/K_0(G/H, r, H/K_0)$. The projection r is the product of the projections p and q. Then we have

$$rI_0 = J_0 r\,.$$

As G/H is a reductive homogeneous space, we can take a natural G-invariant connection $\widetilde{\omega}_0$ on $G(G/H,H)$ with some reduction of G/H. Then on a principal bundle $G/K_0(G/H, r, H/K_0)$ there exists a connection $\widetilde{\omega}$ associated to $\widetilde{\omega}_0$. Then we have

LEMMA 4. *$\widetilde{\omega}$ is of type* (1, 0) *for the almost complex structure I_0 on G/K_0, that is, it holds*

$$\widetilde{\omega}(I_0\bar{u}) = \sqrt{-1}\,\widetilde{\omega}(\bar{u})\,, \quad \bar{u} \in T(G/K_0)\,.$$

PROOF. It is sufficient to prove for the tangent vector at $\lambda(e) \in G/K_0$. We take an element λX of $T_{\lambda(e)}(G/K_0)$, where X belongs to $\widehat{G} = T_e(G)$. Then

$$\widetilde{\omega}(I_0\lambda X) = \widetilde{\omega}_0(IX) = (IX)_{\hat{H}}\,,$$

$X_{\hat{H}}$ denotes the $\widehat{H}$-component of X. As $I\widehat{H} \subset \widehat{H}$, $I\widehat{M}' \subset \widehat{M}'$, we have

$$(IX)_{\hat{H}} = IX_{\hat{H}} = \sqrt{-1}\,\widetilde{\omega}(\lambda X)\,.$$

The principal bundle $G/K_0(G/H, r, H/K_0)$ is therefore almost complex principal bundle. Since the curvature form of the connection $\widetilde{\omega}$ is $d\widetilde{\omega}$, we have the following proposition by virtue of Theorem 2 of [5].

PROPOSITION. *The almost complex structure I_0 is integrable if and only if the* (0, 2)*-component of $d\widetilde{\omega}$ vanishes.*

However, with respect to the homogeneous complex structure of a principal bundle space P over M, we see that P is covered universally by a C-manifold $P_0 = G/K_0$. In fact, since H/K_0 is an abelian group, the commutator subgroup of H is contained in K_0. If we denote the semi-simple parts of groups H and K_0 by H' and K_0', then K_0' coincides with H'. For the homogeneous Kählerian C-manifold G/H, the isotropy subgroup H is a C-subgroup of G. Therefore H' is the semi-simple part of a centralizer of a torus in G, and K_0' is also. This shows that G/K_0 is a C-manifold and $G/K_0(G/H, H/K_0)$ is a complex torus bundle [4]. This proves the following

THEOREM 4. *Let M be a compact, simply connected homogeneous contact manifold. Then any principal T^1-bundle space P over M has a homogeneous complex structure and it is a non-Kählerian complex analytic principal $T^1(C)$-bundle over a Kählerian C-manifold.*

BIBLIOGRAPHY

[1] K. NOMIZU, Invariant affine connections on homogeneous spaces, Amer. Journ. Math., 76(1954), 33-65.

[2] S. MURAKAMI, Sur certains espaces fibrés principaux différentiables et holomorphes, Nagoya Math. Journ., 15(1959), 171-199.

[3] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math., 68(1958), 721-734.

[4] J. HANO AND S. KOBAYASHI, A fibering of a class of homogeneous complex manifolds, Trans. Amer. Math. Soc., 94(1960), 233-243.

[5] J. L. KOSZUL AND B. MALGRANGE, Sur certaines structures fibrées complexes, Arkiv. der Math., 9(1958), 102-109.

[6] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structures II, Tôhoku Math. Journ., 13 (1961), 281-294.

[7] Y. HATAKEYAMA, On the existance of Riemann metrics associated with a 2-form of rank 2r, Tôhoku Math. Journ., 14(1962), 161-166.

[8] ——————, Some notes on differentiable manifolds with almost contact structures, Tôhoku Math. Journ., 15(1963), 176-181.

[9] Y. OGAWA, Some properties on manifolds with almost contact structures, Tôhoku Math. Journ., 15(1963), 148-161.

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