A CHARACTERIZATION OF CONTACT TRANSFORMATIONS

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1. Introduction. Let M^{2n+1} be a contact manifold with a contact form η [2], [3]. By definition, η satisfies the relation

$$(1) \qquad \eta \wedge \overbrace{d\eta \wedge \cdots \wedge d\eta}^{n} \neq 0,$$

where d means the exterior differentiation and $\wedge$ means the exterior multiplication. The Pfaffian equation

$$(2) \qquad \eta = 0$$

determines in M^{2n+1} a $2n$-dimensional distribution D which we shall call the *contact distribution*. We say that a tangent vector X of M^{2n+1} belongs to the distribution D if and only if

$$(3) \qquad \eta(X) = 0$$

is satisfied.

An r-dimensional submanifold F in M^{2n+1} is said to be an *integral submanifold* (of the contact distribution D) if and only if every tangent vector of F at every point p of F belongs to D. An integral submanifold of dimension r in M^{2n+1} is said to be a *maximal intergral submanifold* if it is not a pure subset of any other integral submanifold of dimension r.

A diffeomorphism of M^{2n+1} is said to be a *contact transformation* [3] of M^{2n+1} if and only if

$$(4) \qquad f_* \eta = \sigma \eta$$

holds good, where f_* is the dual map of f which acts on the vector space of 1-forms of M^{2n+1} and σ is a function over M^{2n+1} which does not vanish at any point of M^{2n+1}.

The purpose of this paper is to prove Theorem B which characterizes contact transformations. Theorem A on the highest dimension of integral submanifold of D is given as a preliminary of Theorem B.

THEOREM A. *Let M^{2n+1} be a contact manifold. Then the highest dimension of integral submanifolds of the contact distribution D is equal to n.*

THEOREM B. *A diffeomorphism f of a contact manifold M^{2n+1} transforms every integral submanifold of D of the highest dimension to another such if and only if f is a contact transformation of M^{2n+1}.*

2. Proof of Theorem A. First we shall show the existence of n-dimensional integral submanifolds of D. By virtue of a theorem of E. Cartan [1] every point p of M^{2n+1} has a neighborhood U with local coordinates (x^α, y^α, z) $(\alpha=1,2,\cdots,n)$ such that the contact form η can be written as

$$\eta = dz - \sum y^\alpha\, dx^\alpha \tag{5}$$

in U. We call such local coordinates as adapted local coordinates for brevity. In U, we can give as an example a piece of n-dimensional integral submanifold F of D containing p defined by $x^\alpha=x_0^\alpha$, $z=z_0$ where $(x_0^\alpha, y_0^\alpha, z_0)$ are local coordinates of p.

A maximal integral submanifold of dimension n which contains F as a pure subset gives an example of a global integral submanifold of dimension n.

Secondly, we shall show that there exists no integral submanifold of D whose dimension is higher than n. To prove it, we introduce a contact metric structure (ϕ, ξ, η, g) associated to the contact form [3]. They satisfy the relations

$$\begin{cases} \phi^i_j\xi^j = 0, \qquad \phi^i_j\eta_i = 0, \\ \xi^i\eta_i = 1, \qquad \phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_k \\ \qquad \eta_i = g_{ij}\xi^j \\ \qquad g_{ij}\phi^i_h\phi^j_k = g_{hk} - \eta_h\eta_k\,. \end{cases} \tag{6}$$

Suppose F were an integral submanifold of dimension r and

$$x^i = x^i(u^1, u^2, \cdots, u^r) \qquad (i,j=1,2,\cdots,2n+1) \tag{7}$$

were a local parametric representation of F. Then, as every tangent vector of F belongs to D, we have

$$\eta_i X^i_\lambda = 0 \quad \left(X^i_\lambda = \frac{\partial x^i}{\partial u^\lambda}\right) \qquad (\lambda, \mu=1,\cdots,r), \tag{8}$$

where $\eta = \eta_i dx^i$. Differentiating the last equation with respect to u^μ, and subtracting the equation obtained by interchanging λ and μ from the last equation, we have

$$\phi_{ij} X^i_\lambda X^j_\mu = 0\,, \tag{9}$$

where ϕ_{ij} is given by

$$\phi_{ij} = g_{ih}\phi^h_j = \frac{\partial \eta_j}{\partial x^i} - \frac{\partial \eta_i}{\partial x^j}\,. \tag{10}$$

As (9) can be written as

$$g_{ih}(\phi^h_j X^j_\mu)\, X^i_\lambda = 0\,,$$

the vector X^i_λ and $\phi^i_j X^j_\mu$ are othogonal. On the other hand we can easily see from $(6)_5$ and $(6)_1$ that ξ^i are orthogonal to X^i_λ and $\phi^i_j X^j_\mu$. Hence $(2r+1)$ vectors ξ^i, X^i_λ, $\phi^i_j X^j_\mu$ are linearly independent. Therefore we see that $r \leqq n$.

3. **Integral element**. We consider adapted local coordinates (x^α, y^α, z) around a point p of M^{2n+1} and suppose $(x^\alpha_0, y^\alpha_0, z_0)$ are coordinates of p. Then, we have the following

LEMMA 1. *In order that r sets of values $(a^\alpha_\lambda, b^\alpha_\lambda, c_\lambda)$ $(\lambda, \mu = 1, 2, \cdots, r)$ are components of linearly independent tangent vectors at p of an integral submanifold of dimension r through p, it is necessary and sufficient that*

$$c_\lambda = \sum y^\alpha_0 a^\alpha_\lambda\,, \tag{11}$$

$$\sum a^\alpha_\lambda b^\alpha_\mu = \sum a^\alpha_\mu b^\alpha_\lambda\,, \tag{12}$$

are satisfied.

PROOF. As

$$\eta_i = (-y^\alpha, 0, 1)\,, \tag{13}$$

$$\phi_{ij} = \begin{cases} 1 & \text{if} \quad j = n+i \quad (1 \leqq i \leqq n)\,, \\ -1 & \text{if} \quad i = n+j \quad (1 \leqq j \leqq n)\,, \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

in adapted local coordinates, the necessity of the lemma is clear from (8) and (9).

To prove the sufficiency, we put

$$
(15)\qquad \begin{cases} x^{\alpha} = x_0^{\alpha} + \sum a_{\lambda}^{\alpha} u^{\lambda}, \\ y^{\alpha} = y_0^{\alpha} + \sum b_{\lambda}^{\alpha} u^{\lambda}, \\ z \ = z_0 + \sum c_{\lambda} u^{\lambda} + \frac{1}{2} \sum c_{\lambda\mu} u^{\lambda} u^{\mu}, \end{cases}
$$

where

$$
(16)\qquad c_{\lambda\mu} = \sum a_{\lambda}^{\alpha} b_{\mu}^{\alpha} = \sum a_{\mu}^{\alpha} b_{\lambda}^{\alpha} .
$$

Then, we can easily see that

$$
(17)\qquad \frac{\partial z}{\partial u^{\lambda}} = \sum y^{\alpha} \frac{\partial x^{\alpha}}{\partial u^{\lambda}}
$$

is satisfied identically is u^{λ}. Hence, (15) gives an integral submanifold of D admitting $(a_{\lambda}^{\alpha}, b_{\lambda}^{\alpha}, c_{\lambda})$ as tangent vectors at $p(u^{\lambda} = 0)$. Q.E.D.

[N. B.] The integral submanifolds which have $(a_{\lambda}^{\alpha}, b_{\lambda}^{\alpha}, c_{\lambda})$ as tangent vectors at p are not unique, because the condition to be an integral submanifold is too weak.

In the proof of Theorem A, we saw that if X_{λ}^{i} $(\lambda=1, 2, \cdots, r)$ are tangent vectors of an integral submanifold of D at a point p, then X_{λ}^{i} satisfy (8) and (9).

Conversely, if X_{λ}^{i} $(\lambda=1, 2, \cdots, r)$ are linearly independent tangent vectors of M^{2n+1} at a point p satisfying (8) and (9), then there exists an integral submanifold F^r which has these vectors as tangent vectors. This can be easily seen by taking adapted local coordinates around p and applying Lemma 1. Hence, we see that the following lemma is true.

LEMMA 2. *In order that r linearly independent vectors X_{λ}^{i} $(\lambda = 1, 2, \cdots, r)$ at a point p of M^{2n+1} are tangent vectors of an integral submanifold of dimension r of the distribution D, it is necessary and sufficient that* (8) *and* (9) *hold at p.*

We say that an r-space determined by r linearly independent vectors X_{λ}^{i} at a point p of M^{2n+1} satisfying (8) and (9) as an r-dimensional *integral element* of D.

4. Proof of Theorem B.

LEMMA 3. *Let X be a tangent vector of a contact manifold M^{2n+1}*

belonging to the contact distribution D at a point p of M^{2n+1}. Then, there exists an r-dimensional integral submanifold F of D through p $(1 \leqq r \leqq n)$ such that X is a tangent vector of F at p.

PROOF. We take a neighborhood containing p and adapted local coordinates (x^α, y^α, z). Suppose $(x_0^\alpha, y_0^\alpha, z_0)$ are coordinates of the point p and $(a_1^\alpha, b_1^\alpha, c_1)$ are components of the vector X with respect to the adapted coordinates. As X belongs to the distribution D there exists a relation

$$c_1 = \sum y_0^\alpha a_1^\alpha . \tag{18}$$

First, suppose $a_1^\alpha \neq 0$ and we will show that we can take $(r-1)$ vectors $(a_\rho^\alpha, b_\rho^\alpha, c_\rho)$ $(\rho = 2, \cdots, r)$ so that they and $(a_1^\alpha, b_1^\alpha, c_1)$ together span an r-dimensional integral element of D. To do so, we take $a_2^\alpha, \cdots, a_r^\alpha$ so that

$$\operatorname{rank}\,(a_\lambda^\alpha) = r \qquad (\lambda = 1, 2, \cdots, r), \tag{19}$$

and define $c_2, \cdots, c_r$ by (11) for $\lambda = 2, \cdots, r$. Then, (11) holds good for $\lambda = 1, 2, \cdots, r$.

We determine $b_2^\alpha, \cdots, b_r^\alpha$ inductively. Assuming that $b_1^\alpha, \cdots, b_\sigma^\alpha$ $(1 \leqq \sigma \leqq r-1)$ is already determined, we take $b_{\sigma+1}^\alpha$ as a set of solutions of

$$\left\{ \begin{array}{l} \sum a_1^\alpha x^\alpha = \sum a_{\sigma+1}^\alpha b_1^\alpha , \\ \cdots\cdots\cdots\cdots\cdots , \\ \sum a_\sigma^\alpha x^\alpha = \sum a_{\sigma+1}^\alpha b_\sigma^\alpha , \end{array} \right. \tag{20}$$

as these equations admit solutions by (19). Then we can see that the constants a_λ^α, b_λ^α, c_λ thus determined satisfy (12) for $\lambda, \mu = 1, 2, \cdots, r$. So $(a_\lambda^\alpha, b_\lambda^\alpha, c_\lambda)$ define an r-dimensional integral element containing X.

Secondly, if $a_1^\alpha = 0$ then we have $c_1 = 0$ by (18). In this case $(0, b_\lambda^\alpha, 0)$ such that rank $(b_\lambda^\alpha) = r$ determine an r-dimensional integral element containing X.

Hence, in any case there exists an integral submanifold of demension r of D which satisfies the required property by § 3.

[N. B.] The case $r = n$ of Lemma 3 is used in the proof of Theorem B.

LEMMA 4. *If a diffeomorphism f of a contact manifold M^{2n+1} with a contact form η transforms every tangent vector X belonging to the contact distribution D again to such one, then f is a contact transformation.*

PROOF. We put $f(p) = q$, $X_p \in D$ and

$$X_q = f' X_p, \tag{21}$$

where f' is the induced map of the tangent space M_p at p onto the tangent space M_q at q by f. Then, as X_q belongs to D by assumption,

$$0 = \eta(X_q) = \eta(f' X_p) = (f_* \eta)(X_p), \tag{22}$$

where f_* is the dual map of f which acts on the vector space of 1-forms at q. As the last equation holds good for every tangent vector X_p of M_p belonging to D we see that the relation

$$f_* \eta = \sigma \eta \tag{23}$$

holds good at p. As p is a general point of M^{2n+1}, (23) shows that f is a contact transformation of M^{2n+1}.

PROOF OF THEOREM B. We put $f(p) = q$ and take an arbitrary tangent vector X_p of M_p belonging to D. By Lemma 3, there exists an n-dimensional integral submanifold F of D such that $p \in F$ and X_p is a tangent vector of F at p. Now, by assumption f transforms F to an n-dimensional integral submanifold fF of D. As $X_q = f' X_p$ is a tangent vector of fF, X_q belongs to D. Hence, by Lemma 4, f is a contact transformation of M^{2n+1}.

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