SOME REMARKS ON ANTISYMMETRIC DECOMPOSITIONS OF FUNCITON ALGEBRAS

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1. Let X be a compact Hausdorff space and A a function algebra on X. Recently Bishop [3] and Glicksberg [4] succeded to express the algebra A in terms of antisymmetric algebras in the following form:

THEOREM A. *Every antisymmetric set of A is contained in a maximal antisymmetric set. The collection of all maximal antisymmetric sets forms a pairwise disjoint, closed covering of X satisfying*:

(1) $f \in C(X)$ *and* $f|K \in A|K$ *for every maximal antisymmetric sets imply* $f \in A$,

(2) $A|K$ *is closed in* $C(K)$.

In succeeding sections, we shall be concerned with the meaning of this decomposition for general function algebras. We shall show (Theorem 2) that the decomposition of the algebra A in the representing space is always deduced from that of A in the maximal ideal space and the collection of all maximal antisymmetric sets consisting of a single point plays a special rôle. Next, the relationship between the decomposition and the essential set of A is made explicit in Theorem 3, from which preceding results relating for essential sets such as Bear [2] and Hoffman and Singer [5] are easily derived, adding some more precise formulations.

2. By a function algebra A on a compact Hausdorff space X we mean a closed subalgebra of $C(X)$, the usual algebra of continuous complex functions on X, containing constants and separating points of X. In this case we call X the representing space of A. $M(A)$ always denotes the space of all maximal ideals of A (with Gelfand topology) and ∂_A the Šilov boundary of A. We notice that $M(A)$ is the largest representing space of A and ∂_A the smallest one. A closed set F (in X) is called the peak set of A (in X) if $F = \{p \in X \mid f(p)=1\}$ for some $f \in A$ where $|f(p)| < 1$ for $p \notin F$. A representing measure μ_p associated to a point $p \in M(A)$ is the positive Borel measure on ∂_A such as $f(p) = \int f d\mu_p$ for every $f \in A$.

The following theorem is the slight modification of Theorem 5.2 in [5].

THEOREM 1. *Let F be an intersection of peak sets of A in $M(A)$, then $A|F$ is closed and*

1. $M(A|F)=F$,
2. *if $p\in F$, then any measure μ_p on ∂_A which represents p is supported in $\partial_A\cap F$,*
3. $\partial_{A|F}\subset\partial_A\cap F$.

PROOF. The closedness of the restriction of A to F, $A|F$, was pointed out in [4] and $M(A|F)=F$ is almost known, because F is hull-kernel closed in $M(A)$. Let $p\in F$ and μ_p be a representing measure of p. Take an arbitrary point $q\notin F$, then there exist disjoint open sets O_1, O_2 in $M(A)$ such as $q\in O_1$ and $F\subset O_2$. By compactness of $M(A)$ some finite intersection K of the peak sets containing F is contained in O_2 and K is again a peak set of A. Let f be a function of A which peaks on K. Then, as pointed out in [5], the relation

$$1=f(p)=\int_{\partial_A} f d\mu_p \quad\text{and}\quad |f|\leqq 1$$

imply that we must have $f=1$ on the support of μ_p, hence the support of $\mu_p\subset K$. Therefore the support of μ_p is contained in $\partial_A\cap F$ which proves 2.

The assertion 3 is an easy consequence of the assertion 2 because, for a point $p\in F$ and a function f in A we have $f(p)=\int_{\partial_A} f d\mu_p=\int_{\partial_A\cap F} f d\mu_p$.

A subset K (of X) is called an antisymmetric set of A if, for f in A, f real valued on K implies f is constant on K. Let P be the collection of all maximal antisymmetric sets of A in $M(A)$ which consist of a single point.

THEOREM 2. *Let $M(A)=P\cup K_\alpha\cup K_\beta\cup\cdots$ be the decomposition of $M(A)$ into antisymmetric parts for A, then for any representing space X of A,*

$$X=P\cup(K_\alpha\cap X)\cup(K_\beta\cap X)\cup\cdots$$

forms the decomposition of X into antisymmetric parts for A, and

1. *the set P is invariant, that is, the collection of all maximal antisymmetric sets in X consisting of a single point coincides with P,*
2. *each $K_\alpha\cap X$ contains a perfect set,*
3. *each K_α is connected.*

The fact that every maximal antisymmetric set containing more than one point (hence containing infinitely many points) contains a perfect set was pointed out in Glicksberg [4] using Rudin's result in [8], but we shall show that the assertion 2 is a direct consequence of the boundary behavior of $A|K_\alpha$, hence we get Rudin's results [8; Theorem 3 and 4], as a corollary of the above theorem. The assertion 3 also appears in [4].

PROOF OF THEOREM 2. Let K be a maximal antisymmetric set of A in $M(A)$. Since, by Theorem 1 and Lemma 2.3 in [4], we have

$$\partial_{A|K} \subset \partial_A \cap K \subset X \cap K,$$

$X \cap K$ is always non-empty. Hence we see that $P \subset X$. Suppose $f \in A$ is a real function on $X \cap K$, then f is real valued on $\partial_{A|K}$, the Šilov boundary of $A|K$, hence f is a real function on K and reduces to a constant function on K. Thus $X \cap K$ is an antisymmetric set of A in X, and moreover a maximal antisymmetric set in X. Therefore

$$X = P \cup (X \cap K_\alpha) \cup (X \cap K_\beta) \cup \cdots$$

is the decomposition of X into antisymmetric parts for A. Notice that, for each index α, $X \cap K_\alpha$ does not reduce to a single point because $X \cap K_\alpha \supset \partial_{A|K_\alpha}$. Hence the assertion 1 holds.

For the assertion 2 we proceed as follows; since K_α is connected by a well known theorem of Šilov [7; p. 168] it suffices to prove 2 when $\partial_{A|K_\alpha} \neq K_\alpha$, but in this case it is known that $\partial_{A|K_\alpha}$ contains a perfect set (for example cf. [7; Theorem 3.3.21]).

It is to be noticed that by Theorem 2 we have

$$M(A) \sim X = \{(M(A) \sim X) \cap K_\alpha\} \cup \{M(A) \sim X) \cap K_\beta\} \cup \cdots$$

Therefore the behavior of a real function f of A on $M(A) \sim X$ is parallel to that of f on $X \sim P$. The decomposition of the representing space of A into its antisymmetric parts is deduced from that of A in its maximal ideal space $M(A)$ and the type of this decomposition is unaltered in X. Hence we give the following

DEFINITION 1. *If there exists only one maximal antisymmetric set for A containing more than one point, the decomposition is called the first type and we call A an almost antisymmetric algebra. Otherwise, the decomposition is called the second type.*

3. The essential set E of A in X is the set which is a hull of the largest ideal of $C(X)$ contained in A (Bear [1]). Thus, the essential set is the minimal closed set E in X such that for any continuous function f, if $f=0$ on E, then $f \in A$. If $E=X$, A is called an essential algebra.

Let P^i be the interior of P in $M(A)$. We note that $P \subset \partial_A$.

LEMMA 1. *Let P_1 be the interior of P in X, then $P_1 = P^i$, that is, P^i is unaltered in X.*

PROOF. Let f be an arbitrary function of $C(X)$ vanishing on $X \sim P_1$. Then by (1) of Theorem A f belongs to A. Take a point $p_0 \in P_1$. There exists a real function f in A such as $f(p_0) \neq 0$ and $f(X \sim P_1) = 0$. By Theorem 2, $f(K_\alpha) = 0$ for all index α and then $f(M(A) \sim P) = 0$. Therefore the open set $U = \{p \in M(A) | f(p) \neq 0\}$ is clearly contained in P. Thus $P_1 \subset P^i$ which completes the proof since the other inclusion is clear.

THEOREM 3. *Let E be the essential set of A in X,* then

$$E = X \sim P^i$$

PROOF. Since any function of $C(X)$ vanishing on E is in A it is clear that $X \sim E \subset P$. Hence, by Lemma 1, $X \sim E \subset P^i$. We have $X \sim P^i \subset E$. On the other hand the same argument as in the first part of the proof of Lemma 1 gives the opposite inclusion.

As it is known, every essential maximal algebra are antisymmetric (Helson and Quigley [6], Bear [2] etc.) hence maximal algebras are almost antisymmetric in the sense of Definition 1. As for P, if there is only a finite number of maximal antisymmetric sets containing more than one point P is an open set in X (hence open in $M(A)$ by Lemma 1) and $E = X \sim P$. However this fact does not necessarily hold in general.

Now for the essential set E of a function algebra A, Bear [2] has proved the following results;

1. if $E \neq X$, $X \sim E$ is an (non-zero) open set in $M(A)$,
2. $E \cup (M(A) \sim X)$ is the essential set of A in $M(A)$,
3. for every function $f \in A$, if $f=0$ on E, $f=0$ on $E \cup (M(A) \sim X)$,
4. $M(A|E) = E \cup (M(A) \sim X)$.

The assertions 1 and 2 are easy consequences of Theorem 3. For 3, we get the following more precise form.

THEOREM 4. *Let E_0 be the essential set for A in $M(A)$, then $\partial_{A|E_0} \subset \partial_A \cap E_0$. Hence $A|E$ is isometrically isomorphic to $A|E_0$.*

PROOF. Let p_0 be an arbitrary point in $M(A) \sim E_0$. We can find a continuous function $f \in A$ on $M(A)$ such that $0 \leqq f \leqq 1$, $f | E_0 = 0$ and $f(p_0) = 1$. Put $F(p_0) = \{p \in M(A) | f(p) = 0\}$, then one easily see that $F(p_0)$ is a peak set of A containing E_0. We have $E_0 = \bigcap_{p \in M(A) \sim E_0} F(p)$, and E_0 is an intersection of peak sets, It follows, by 3 of Theorem 1, $\partial_{A|E_0} \subset \partial_A \cap E_0 = \partial_A \sim P^i$.

The assertion 4 follows directly from the above theorem and 1 of Theorem 1.

In [5], Hoffman and Singer considered the following sets in $M(A)$;

$I = M(A) \sim \partial_A$, the interior of $M(A)$;

$L = \bar{I} \sim I$, the accessible set,

S_*; the minimal support set which is obtained by intersecting the closed supports of all measures μ_p on ∂_A representing points p in I,

S^*; the maximal support set which is the closure of the union of the closed supports of all measures μ_p on ∂_A representing points p in I.

Generally these sets are related with the essential set E of A in ∂_A in the following manner:

1. $L \subset E$([5; Theorem 5.3]), 2. $S^* \subset E$ ([5; Theorem 5.4]).

If A is a maximal subalgebra of $C(\partial_A)$, and if $\partial_A \neq M(A)$, $L \subset E = S_* = S^*$ and Hoffman and Singer conjecture that for any algebra A with I non-empty the inclusion $S_* \subset L$ holds.

Now the assertion 1 follows directly from our Theorem 3 because $P^i \cap I = \emptyset$ implies $P^i \cap \bar{I} = \emptyset$. As for 2 we can show more precise result. Let S_α^* be the maximal support set of $A|K_\alpha$ where K_α is an maximal antisymmetric set of A in $M(A)$ containing more than one point, then by 2 of Theorem 1 we have $S_\alpha^* \subset K_\alpha \cap \partial_A$. Hence by Theorem 2 we get the following

THEOREM 5. *$S^* = \overline{\bigcup_\alpha S_\alpha^*}$, the closure of $\bigcup_\alpha S_\alpha^*$, hence $S^* \subset \partial_A \sim P^i$.*

One direct consequence of this theorem is that Hoffman and Singer's conjecture loses its interest unless A is an almost antisymmetric algebra, in fact if it is not the case we have $S_* = \emptyset$.

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