THE INTEGRABILITY OF A STRUCTURE ON A DIFFERENTIABLE MANIFOLD

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1. Let M be an n-dimensional differentiable manifold of class C^ω. Let M_x be the tangent space at each point x of the manifold M and M_x^C the complexification of M_x. A (1, 1) tensor field of class C^ω defines an endomorphism on each tangent space M_x^C. Such an endomorphism is denoted by F. Suppose F satisfies the condition $F^2=\lambda^2 E$ where $\lambda^2=\pm 1$ and E is the identity mapping. F is said to be integrable if at each point of the manifold there exists a coordinate neighborhood in which the field F has numerical components. Let M_x' and M_x'' be the eigen spaces of F associated with the eigen values λ and $-\lambda$ respectively. It is known that F is integrable if and only if the distributions $x \to M_x'$ and $x \to M_x''$ are involutive.

On M we consider a structure $\{F, G\}$ defined by two fields F, G of class C^ω satisfying the following conditions:

(a) $F^2 = \lambda^2 E,\ G^2 = \mu^2 E$ where λ, μ satisfy $\lambda^2 = \pm 1,\ \mu^2 = \pm 1$.

We take λ, μ to be i or 1.

(b) $FG = GF$.

DEFINITION. *The structure* $\{F, G\}$ *satisfying* (a), (b) *is said to be integrable if at each point of the manifold there is a coordinate neighborhood in which the fields* F, G *have simultaneously numerical components.*

If $\lambda^2=-1$ or $\mu^2=-1$ then F or G is an almost complex structure. The dimension n of M turns out to be an even number. So in this case if $\{F, G\}$ is integrable, the coordinate neighborhood is a complex coordinate neighborhood. We are going to investigate a necessary and sufficient condition for such a structure to be integrable. To do this, we define a (1, 2) tensor field $[F, F']$ assciated with two (1, 1) tensor fields F, F' as follows: for any two vector fields X, Y

$$(1)\quad [F,F'](X,Y) = [FX,F'Y] - F[X,F'Y] - F'[FX,Y] + FF'[X,Y] + [F'X,FY] - F'[X,FY] - F[F'X,Y] + F'F[X,Y].$$

Thus $\frac{1}{2}[F, F]$ is the Nijenhuis tensor of F, and $[F, F] = 0$ and $[G, G] = 0$ mean that F, G are integrable respectively. The result is

THEOREM. *The structure* $\{F, G\}$ *satisfying* (a), (b) *is integrable if and only if* $[F, F]=0$, $[G, G]=0$ *and* $[F, G]=0$.

Let $x \to M'_x$ and $x \to M''_x$ be the distributions we mentioned above. Let N'_x and N''_x be the eigen spaces of G associated with the eigen values μ and $-\mu$ respectively. By the above definition, the structure $\{F, G\}$ is integrable if and only if the distributions $x \to M'_x \cap N'_x$, $x \to M'_x \cap N''_x$, $x \to M''_x \cap N'_x$, $x \to M''_x \cap N''_x$ are involutive and the distributions $x \to M'_x \cap N'_x \oplus M''_x \cap N''_x$, $x \to M'_x \cap N''_x \oplus M''_x \cap N'_x$ are involutive.

2. The vector subspaces M'_x, M''_x, N'_x and N''_x are invariant by F and G. In fact, let u be any vector in M'_x then $F(Gu) = G(Fu) = G(\lambda u) = \lambda G(u)$ since $FG = GF$. This proves that M'_x is invariant by G. Similar proofs hold good for the other subspaces.

LEMMA. $M^c_x = M'_x \cap N'_x \oplus M'_x \cap N''_x \oplus M''_x \cap N'_x \oplus M''_x \cap N''_x$. (*Direct sum*)

PROOF. Put

$$P' = (1/2)(E + \lambda F),\ P'' = (1/2)(E - \lambda F),$$
$$Q' = (1/2)(E + \mu G),\ Q'' = (1/2)(E - \mu G).$$

Then P', P'' are projections from M^c_x to the subspaces M'_x, M''_x or M''_x, M'_x respectively, Q', Q'' are projections from M^c_x to the subspaces N'_x, N''_x or N''_x, N'_x respectively. For instance, take the image $P'(M^c_x)$. The following relations are clear:

$$(2) \qquad \begin{aligned} P'F = FP' = \frac{1}{\lambda}P', \qquad P''F = FP'' = -\frac{1}{\lambda}P'', \\ Q'G = GQ' = \frac{1}{\mu}Q', \qquad Q''G = GQ'' = -\frac{1}{\mu}Q''. \end{aligned}$$

Thus if $\lambda^2 = 1$, $F(P'M^c_x) = \lambda P'M^c_x$. Hence $P'M^c_x = M'_x$. If $\lambda^2 = -1$, $F(P'M^c_x) = -\lambda P'M^c_x$. Hence $P'M^c_x = M''_x$. P' is a projection since $P'P' = P'$. We can check that the other mappings P'', Q', Q'' have the above mentioned property by similar processes.

The following relations are also easy to verify.

$$
(3)\qquad
\begin{aligned}
&P'+P''=E,\quad Q'+Q''=E.\\
&P'P''=P''P'=0,\quad Q'Q''=Q''Q'=0.\\
&P'Q'=Q'P',\ P'Q''=Q''P',\ P''Q'=Q'P'',\ P''Q''=Q''P''.\\
&E=P'Q'+P'Q''+P''Q'+P''Q''.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
M_x^c &= (P'Q'+P'Q''+P''Q'+P''Q'')M_x^c\\
&= (M_x'\cap N_x')\oplus(M_x'\cap N_x'')\oplus(M_x''\cap N_x')\oplus(M_x''\cap N_x'').
\end{aligned}
$$

Since the products of any two of $P'Q'$, $P'Q''$, $P''Q'$, $P''Q''$ are zero, the sum is a direct sum. This proves our lemma.

3. Now we are going to prove the theorem. If the structure $\{F, G\}$ is integrable, it is clear that F and G are integrable respectively and $[F, G]=0$. Conversely, suppose F and G are both integrable and $[F, G]=0$. It is known that the distributions M' : $x\rightarrow M_x'$ and M'' : $x\rightarrow M_x''$ are involutive if and only if $P''[P'X, P'Y]=0$, $P'[P''X, P''Y]=0$ for any complex vector fields X and Y. Similarly G is integrable if and only if $Q''[Q'X, Q'Y]=0$, $Q'[Q''X, Q''Y]=0$ for any complex fields X and Y. $x\rightarrow M_x'\cap N_x'\oplus M_x''\cap N_x''$ and $x\rightarrow M_x'\cap N_x''\oplus M_x''\cap N_x'$ are involutive if and only if $(P'Q''+P''Q')[(P'Q'+P''Q'')X, (P'Q'+P''Q'')Y]=0$ and $(P'Q'+P''Q'')[(P'Q''+P''Q')X, (P'Q''+P''Q')Y]=0$ for any complex fields X and Y. Assuming that $P''[P'X, P'Y]=0$, $P'[P''X, P''Y]=0$, $Q''[Q'X, Q'Y]=0$ and $Q'[Q''X, Q''Y]=0$, these two conditions are equivalent to

$$
(4)\qquad
\begin{cases}
(P'Q''+P''Q')([P'Q'X, P''Q''Y]+[P''Q''X, P'Q'Y])=0,\\
(P'Q'+P''Q'')([P'Q''X, P''Q'Y]+[P''Q'X, P'Q''Y])=0.
\end{cases}
$$

To prove the distributions

$$
x\rightarrow M_x'\cap N_x',\ x\rightarrow M_x'\cap N_x'',\ x\rightarrow M_x''\cap N_x',\ x\rightarrow M_x''\cap N_x''
$$

are all involutive, we want to show

$$
(P'Q''+P''Q'+P''Q'')[P'Q'X, P'Q'Y]=0
$$

for any complex vector fields X and Y. This result implies that one of the above distributions is involutive.

F and G being integrable, $P''[P'(Q'X), P'(Q'Y)]=0$ and $Q''[Q'(P'X),$

$Q'(P'Y)]=0$ for any vector fields X and Y. These imply $(P'Q''+P''Q'+P''Q'')$ $[P'Q'X, P'Q'Y]=0$ since $P'Q'=Q'P'$ and $P''Q''=Q''P''$.

Similarly we can prove the other distributions are involutive.

Finally, to prove the distributions

$$x \to M'_x \cap N'_x \oplus M''_x \cap N''_x\,, \quad x \to M'_x \cap N''_x \oplus M''_x \cap N'_x$$

are involutive, substituting $P'Q'X$ and $P''Q''Y$ in X and Y of (1) and making use of the relations (2), (3) we can easily show that

$$(P''Q' + P'Q'')[P'Q'X, P''Q''Y] = 0$$

for any complex fields X and Y. Similarly we can prove

$$(P'Q' + P''Q'')[P'Q''X, P''Q'Y] = 0$$

for any complex fields X and Y. Hence the relations (4) follow. Thus we complete the proof of our theorem.

REFERENCES

[1] C. J. HSU, Note on the integrability of a certain structure on differentiable manifold, Tôhoku Math. Journ., 12 (1960), 349-360.

[2] ———, On some structure which are similar to the quaternion structure, Tôhoku Math. Journ., 12 (1960), 403-428.

[3] P. LIBERMANN, Sur le problème d'équivalence de certaines structures infinitésimales, Thèse (1953).

[4] M. OBATA, Affine connections on manifolds with almost complex, quaternion or Hermitian structure, Journ. Math. Soc. Japan, 26 (1956), 43-77.

[5] K. YANO, Affine connections in an almost product space, Kodai Math. Seminar Report, 11 (1959), 1-24.

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