THE HOMOLOGY DECOMPOSITION FOR A COFIBRATION

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1. Introduction. As a homology analogue of the Postnikov decomposition, the homology decomposition of a 1-connected polyhedron was introduced by B. Eckmann and P. J. Hilton ([3], [5]). Moreover, as a generalization of this notion, B. Eckmann and P. J. Hilton ([4], [5]) and J. C. Moore ([6]) introduced the notion of the homology decomposition of a map. However, the homology decomposition of a map seems to be inconvenient for the actual applications.

Now as an intermediate notion of the above two decompositions, we introduce a notion of the homology decomposition for a cofibration $B \xrightarrow{q} X \xrightarrow{p} F$. This notion corresponds with a homology analogue of the Moore-Postnikov decomposition ([1]) and seems to have many applications in the algebraic topology.

In §3, we shall give a definition of the homology decomposition for a cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ and its actual construction. If B reduces to a point, then such decomposition reduces to the usual homology decomposition for X. The decomposition for the cofibre F in such decomposition gives the usual one for F.

In §4 we introduce the notion of weak H'-cofibration as a generalization of the induced cofibration. The weak H'-cofibration weakens the notion of H'-cofibration defined in [7]. In §4, we explain the relations between the weak H'-cofibration and the homology decomposition for a cofibration.

2. Preliminaries. All spaces have base points denoted by $*$ and respected by maps $f, g, \cdots$ and their homotopies $f, g, \cdots$. Let $\pi(X, Y)$ denote the set of all homotopy classes of maps $X \to Y$. The homotopy class of a map $f: X \to Y$ is denoted by $[f]$. Let $K'(G, n)$ be a polyhedron with abelian fundamental group such that $H_r(K'(G, n))=0$ for $r \neq n$ and $H_n(K'(G, n))=G$. The homotopy type of the polyhedron $K'(G, n)$ is uniquely determined for $n \geqslant 2$. $K'(G, n)$ $(n \geqslant 2)$ has an H'-structure and we define the n-th homotopy group of X with coefficients in G by $\pi_n(G, X) = \pi(K'(G, n), X)$ and the n-th homotopy group of a map $f: X \to Y$ with coefficients in G by $\pi_n(G, f) = \pi_1(K'(G, n-1), f)$ ([2], [5]). Let $B \xrightarrow{q} X \xrightarrow{p} F$ be a cofibration and let $f: Y \to X$ be a map. Let C_f (resp. C_{pf}) denotes the space obtained by attaching the reduced cone over Y to X (resp. F) by means of f (resp. pf), i.e. $C_f = CY \cup_f X$ (resp. $C_{pf} = CY \cup_{pf} F$). Then $F \xrightarrow{s} C_{pf} \to \Sigma Y$ is an inclusion cofibration and the following diagram is commutative:

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \xrightarrow{p} & F \\ \iota\downarrow & & \downarrow i & & \downarrow s \\ CY & \xrightarrow{k} & C_f & \xrightarrow{\bar{p}} & C_{pf} \end{array}$$

where ι k, i and s are inclusion maps and $\bar{p}$ is defined by

$$\bar{p}(y,t) = (y,t) \quad (y,t) \in CY \text{ and } \bar{p}(x) = px \quad x \in X$$

Since $\bar{p}(y,1) = (y,1) = pf(y)$ and $\bar{p}(fy) = p(fy)$, $\bar{p}$ is well defined. Then the following lemma is an obvious consequence of these considerations.

LEMMA 2.1 *$B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a cofibration.*

3. The homology decomposition for an (inclusion) cofibration.

In this section we only consider 1-connected polyhedra.

DEFINITION 3.1 The homology decomposition for an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ consists of a sequence of spaces and maps $(X_n, F_n, i_n, j_n, q_n, p_n)_{n=1,2,\dots}$ subject to the following conditions;

(I) $X_1 = B \quad j_1 = q \quad q_1 = id$.

(II) $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is an inclusion cofibration.

(III) $q = j_n \cdot q_n \quad : \quad B \xrightarrow{q_n} X_n \xrightarrow{j_n} X$

(IV) $X_{n-1} \xrightarrow{i_n} X_n \longrightarrow K'(H_n(F), n)$ is an inclusion cofibration $(n \geqslant 2)$ (where $i_2 = q_2$).

(V) maps q_n, j_n induce the following;

(1) $j_{n*} : H_r(X_n) \xrightarrow{\approx} H_r(X)$ for $r < n$,

(2) in the sequence $H_n(B) \xrightarrow{q_{n*}} H_n(X_n) \xrightarrow{j_{n*}} H_n(X)$, q_{n*} is a monomorphism, j_{n*} is an epimorphism and $\mathrm{Im}.\, q_{n*} \supset \mathrm{Ker}.\, j_{n*}$.

(3) $q_{n*} : H_r(B) \xrightarrow{\approx} H_r(X_n)$ for $r > n$.

(VI) (1) a map $\bar{j}_n : F_n \to F$ induced by j induces

$$\bar{j}_{n*} : H_r(F_n) \xrightarrow{\approx} H(F) \text{ for } r \leqslant n ,$$

(2) $$H_r(F) = 0 \text{ for } r > n .$$

CONSTRUCTION. We will construct the homology decomposition for an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ inductively. From the homology exact sequence for a map q (cf. [2], [5])

$$\longrightarrow H_r(B) \xrightarrow{q_*} H_r(X) \xrightarrow{J} H_r(q) \xrightarrow{\partial} H_{r-1}(B) \longrightarrow$$

and the homology exact sequence for an inclusion cofibration q

$$\longrightarrow H_r(B) \xrightarrow{q_*} H_r(X) \xrightarrow{p_*} H_r(F) \xrightarrow{\partial} H_{r-1}(B) \longrightarrow ,$$

we have $H_r(q) \approx H_r(F)$ for all r. (1)
We first describe the case $n=2$. By the universal coefficient theorem for the homotopy group of a map ([5]), we have an exact sequence

$$\pi_2(H_2(F), q) \xrightarrow{\eta} \mathrm{Hom}\,(H_2(F), \pi_2(q)) \longrightarrow 0 .$$

F is 1-connected and so $H_1(q) = 0$. Hence by the generalized Hurewicz theorem ([5]), $\pi_2(q) \approx H_2(q)$. Thus we have an isomorphism $\theta_1 : \pi_2(q) \approx H_2(F)$ and $[(u_1, v_1)] \in \pi_2(H_2(F), q)$ such that $\eta[(u_1, v_1)] = \theta_1^{-1}$. Hence we have the following commutative diagram :

$$\begin{array}{ccc} K'(H_2(F),1) & \xrightarrow{u_1} & B \\ \iota\downarrow & & \downarrow q \\ CK'(H_2(F),1) & \xrightarrow{v_1} & X \end{array}$$

Let $X_2 = CK'(H_2(F),1) \cup_{u_1} B$ and $F_2 = K'(H_2(F),2)$; then, $B \xrightarrow{q} X_2 \xrightarrow{p} F_2$ is an inclusion cofibration, where q is an inclusion and p a projection. Next we define $j_2 : X_2 \to X$ by $j_2|B = q$ and $j_2|CK'(H_2(F),1) = v_1$. Evidently j_2 is well defined and $j_2 q_2 = q$, if we denotes the injection $B \to X_2$ by q_2.
Now we consider the commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(B) & \xrightarrow{q_{2*}} & H_2(X_2) & \xrightarrow{p_{2*}} & H_2(F_2) & \longrightarrow & 0 \\ \downarrow & & \downarrow id & & \downarrow j_{2*} & & \downarrow \bar{j}_{2*} & & \downarrow \\ H_3(F) & \xrightarrow{\partial} & H_2(B) & \xrightarrow{q_*} & H_2(X) & \xrightarrow{p_*} & H_2(F) & \longrightarrow & 0 \end{array}$$

where the upper sequence is a part of the homology exact sequence for an inclusion cofibration q_2 and the lower sequence is that of an (inclusion) cofibration q. Then it is evident that q_{2*} is a monomorphism, j_{2*} is an epimorphism and $\mathrm{Im}.\, q_{2*} \supset \mathrm{Ker}.\, j_{2*}$. Also obviously we have $j_{2*}: H_r(X_2) \approx H_r(X)$ for $r < 2$ and $H_r(B) \approx H_r(X_2)$ for $r > 2$.

Thus a sequence of spaces and maps $(X_2, F_2, j_2, p_2, q_2)$ was constructed so as to satisfy the conditions in Definition 3. 1.

Now we assume that spaces and maps $(X_m, F_m, i_m, j_m, p_m, q_m)$ for $m < n$ were constructed so as to satisfy the conditions in Definition 3.1.

From the homology exact sequence for a map j_{n-1} and the condition (V), (1), (2) in 3.1.

$$H_r(j_{n-1}) = 0 \quad \text{for} \quad r \leqslant n-1\,. \tag{2}$$

Since $q = j_{n-1} q_{n-1}$ (the condition (III) for $n-1$), the following homology sequence is exact (cf. [5]).

$$\longrightarrow H_r(q_{n-1}) \longrightarrow H_r(q) \longrightarrow H_r(j_{n-1}) \longrightarrow H_{r-1}(q_{n-1}) \longrightarrow .$$

Using the condition (II) in 3.1. and the observation done in the beginning of the construction, $H_r(q_{n-1}) \approx H(E_{n-1})$ for all r. (3)
Combining these facts and the condition (VI) in 3.1, we have

$$H_r(j_{n-1}) \approx H_r(q) \quad \text{for} \quad r \geqslant n\,. \tag{4}$$

Let M be the mapping cylinder of j_{n-1}. Then j_{n-1} can be factorized into the composite map $X_{n-1} \xrightarrow{l_{n-1}} M \xrightarrow{\alpha} X$, where l_{n-1} is an inclusion cofibration and α a homotopy equivalence. Then it is clear that

$$H_r(j_{n-1}) \approx H_r(l_{n-1}) \quad \text{for all } r\,. \tag{5}$$

Now by the universal coefficient theorem for the homotopy group of a map, we see that

$$\pi_n(H_n(F), l_{n-1}) \xrightarrow{\eta} \mathrm{Hom}(H_n(F), \pi_n(l_{n-1})) \longrightarrow 0 \text{ is exact.}$$

By (2) and (5), $H_r(l_{n-1}) = 0$ for $r \leqslant n-1$. Hence by the generalized Hurewicz theorem, $\pi_n(l_{n-1}) \approx H_n(l_{n-1})$. Combining (1), (4) and (5), we have an isomorphism $\pi_n(l_{n-1}) \approx H_n(l_{n-1}) \approx H_n(j_{n-1}) \approx H_n(q) \approx H_n(F)$. Let θ_{n-1} be such an isomorphism. Then there exists $[(u_{n-1}, v_{n-1}) \in \pi_n(H_n(F), l_{n-1})$ such that $\eta[(u_{n-1}, v_{n-1})] = \theta_{n-1}^{-1}$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc} K'(H_n(F), n-1) & \xrightarrow{u_{n-1}} & X_{n-1} & \xrightarrow{id} & X_{n-1} \\ \downarrow \iota & & \downarrow l_{n-1} & & \downarrow j_{n-1} \\ CK'(H_n(F), n-1) & \xrightarrow{v_{n-1}} & M & \xrightarrow{\alpha} & X . \end{array}$$

We set $X_n = CK'(H_n(F), n-1) \cup_{u_{n-1}} X_{n-1}$ and define $j_n : X_n \to X$ by $j_n | CK'(H_n(F), n-1) = \alpha v_{n-1}$ and $j_n | X_{n-1} = j_{n-1}$. Obviously j_n is well defined.

Let $i_n : X_{n-1} \to X_n$ be an inclusion map. Then we see immediately that i_n is an inclusion cofibration with cofibre $K'(H_n(F), n)$ and

$$H_r(i_n) = 0 \quad \text{for} \quad r \neq n \quad \text{and} \quad H_n(i_n) \approx H_n(F) . \tag{6}$$

Next we define $q_n : B \to X_n$ to be a composite map $q_n = i_n \cdot q_{n-1}$. Then q_n is an inclusion cofibration. We denote its cofibre F_n. From the definition of j_n, it is evident that $j_n q_n = q$ and $j_{n-1} = j_n i_n$. From the homology exact sequence for the composite map $j_{n-1} = j_n i_n$,

$$\to H_r(i_n) \to H_r(j_{n-1}) \to H_r(j_n) \to H_{r-1}(i_n) \to \text{ is exact.}$$

Hence by (2), (4), and (6),

$$H_r(j_n) = 0 \quad \text{for} \quad r \leqslant n \quad \text{and} \quad H_r(j_n) \approx H_r(q) \quad \text{for} \quad r > n . \tag{7}$$

Moreover $\to H_{r+1}(j_n) \xrightarrow{\partial} H_r(X_n) \xrightarrow{j_{n*}} H_r(X) \xrightarrow{J} H_r(j_n) \to$ is exact, and hence by (7), $j_{n*} : H_r(X_n) \approx H_r(X)$ for $r < n$.

On the other hand, from the homology exact sequence for the composite map $q_n = i_n \cdot q_{n-1}$,

$$\to H_{r+1}(i_n) \to H_r(q_{n-1}) \to H_r(q_n) \to H_r(i_n) \to \text{ is exact.} \tag{8}$$

But $H_r(q_{n-1}) \approx H_r(F_{n-1})$ for all r ($H_r(q_n) \approx H_r(F_n)$ for all r).
Hence by (VI) in 3.1. and (6),

$$H_r(F_n) = 0 \quad \text{for} \quad r > n . \tag{9}$$

Since $\to H_{r+1}(F_n) \to H_r(B) \xrightarrow{q_{n*}} H_r(X_n) \xrightarrow{p_{n*}} H_r(F_n) \to$ is exact, it follows from (9) that $q_{n*} : H_r(B) \approx H_r(X_n)$ for $r > n$.

Applying the five lemma to the commutative diagram:

$$\begin{array}{ccccccccccc} H_{r+1}(F_n) & \xrightarrow{\partial} & H_r(B) & \xrightarrow{q_{n*}} & H_r(X_n) & \xrightarrow{p_{n*}} & H_r(F_n) & \xrightarrow{\partial} & H_{r-1}(B) & \to & H_{r-1}(X_n) \\ \downarrow \bar{j}_{n*} & & \downarrow id & & \downarrow j_{n*} & & \downarrow \bar{j}_{n*} & & \downarrow id & & \downarrow j_{n*} \\ H_{r+1}(F) & \xrightarrow{\partial} & H_r(B) & \xrightarrow{q_*} & H_r(X) & \xrightarrow{p_*} & H_r(F) & \xrightarrow{\partial} & H_{r-1}(B) & \to & H_{r-1}(X) , \end{array} \tag{10}$$

where the upper sequence is the homology exact sequence for an inclusion cofibration q_n and the lower is that of an inclusion cofibration q, we obtain

$$H_r(F_n) \approx H_r(F) \quad \text{for} \quad r < n .$$

If we apply condition (6), (9) and (11) to the sequence (8) with $r=n$, we have

$$H_n(F_n) \approx H_n(F) . \tag{11}$$

Finally we consider again the above commutative diagram (10) with $r=n$. Then, by (9) and (11), we easily see that q_{n*} is a monomorphism and j_{n*} is an epimorphism and $\mathrm{Im}.\, q_{n*} \supset \mathrm{Ker}.\, j_{n*}$.

REMARK 1. If $H_m(F)=0$ for some m, then $X_m = X_{m-1}$ and $F_m = F_{m-1}$.

REMARK 2. If F is $(q-1)$-connected $(q \geqslant 2)$, $F_m = *$ and $X_m = B$ for $m \leqslant q-1$ and F_m $(m \geqslant q)$ is $(q-1)$-connected. In addition for $p \leqslant q$, if B is $(p-1)$-connected, then each X_m is $(p-1)$-connected.

REMARK 3. If $H_r(F)=0$ for $r > m$, then sequences $\{X_i\}$ and $\{F_i\}$ terminate with X_m and F_m respectively. Then maps $j_m : X_m \to X$ and $\bar{j}_m : F_m \to F$ are homotopy equivalences.

As the assertion on $\bar{j}_m$ is obvious and we prove only about j_m. By (V) in 3.1, $j_{m*} : H_r(X_m) \approx H_r(X)$ for $r < m$. As for $r \geqslant m$, we consider the preceding commutative diagram (10) and apply the five lemma to obtain the isomorphism $j_{m*} : H_r(X_m) \approx H_r(X)$ for $r \geqslant m$. Thus j_m induces the singular homotopy equivalence. In the construction of each X_i, we may choose u_{i-1} to be cellular and we may arrange so that X_i is itself a polyhedron. Hence j_m is an actual homotopy equivalence.

REMARK 4. Generally we may form $X_\infty = \cup X_n$ and $F_\infty = \cup F_n$, and give them the weak topology. We define $j_\infty : X_\infty \to X$ by $j_\infty | X_n = j_n$, and $\bar{j}_\infty : F_\infty \to F$ by $\bar{j}_\infty | F_n = \bar{j}_n$. Then j_∞ and $\bar{j}_\infty$ are homotopy equivalences. Also two cofibration $B \to X_\infty \to F_\infty$ and $B \to X \to F$ are equivalent in the sense of [7; Definition 2.5]. The assertion on $\bar{j}_\infty$ is obvious (cf. [4]) and the assertion on j_∞ follows from the similar argument as in Remark 3.

REMARK 5. If an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is obtained by applying the suspension functor Σ to an (inclusion) cofibration $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$ with all spaces 1-connected polyhedra, then the homology decomposition $(X_n, F_n,$

$i_n, j_n, p_n, q_n)_{n=2,3,\dots}$ for $B \xrightarrow{q} X \xrightarrow{p} F$ may be obtained by applying the suspension functor to the homology decomposition $(X'_{n-1}, F'_{n-1}, i'_{n-1}, j'_{n-1}, p'_{n-1}, q'_{n-1})_{n=2,3,\dots}$ for $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$; i.e.

$$X_n = \Sigma X'_{n-1},\ F_n = \Sigma F'_{n-1},\ i_n = \Sigma i'_{n-1},\ j_n = \Sigma j'_{n-1},\ p_n = \Sigma p'_{n-1} \text{ and } q_n = \Sigma q'_{n-1} \quad (n = 2, 3, \cdots).$$

REMARK 6. In the preceding construction, each F_n was defined by $F_n = X_n/B$. However we may also construct F_n in the usual way (cf. [3]). We consider the composite map $p_2u_2 : K'(H_3(F), 2) \xrightarrow{u_2} X_2 \xrightarrow{p_2} F_2$ where $F_2 = K'(H_2(F), 2)$ and maps u_2, p_2 are those defined in the peeceding construction.

By Lemma 2.1, $B \to C_{u_2} \to C_{p_2u_2}$ is a cofibration. But $C_{u_2} = X_3$. Hence we have $H_r(C_{p_2u_2}) \approx H_r(F_3)$ for all r. Consider the homology exact sequence of the cofibration $F_2 \to C_{p_2u_2} \to K'(H_3(F), 3)$, then $H_2(F_2) \approx H_2(C_{p_2u_2})$ and $H_3(F) \approx H_3(C_{p_2u_2})$. It follows from [3: Proposition 4′] that p_2u_2 is homologically trivial. Thus $C_{p_2u_2} = CK'(H_3(F), 2) \cup_{p_2u_2} F$ obtained by attaching the cone $CK'(H_3(F), 2)$ to F by a homologically trivial map p_2u_2 has the homotopy type of F_3. The same considerations are done for F_n $(n > 3)$.

Thus we may also built up the homotopy type F_∞ of F by an usual process of successively attaching cones $CK'(H_n(F), n-1)$ by homologically trivial maps.

DEFINITION 3.2. The 1-connected polyhedron X is said to be normal if it admits a filtration into 1-connected subcomplexes

$$X_2 \subset X_3 \subset \cdots \subset X_n \subset \cdots; \ \cup X_n = X$$

with $H_r(X_n) = 0$ for $r > n$ and $i_*: H_r(X_n) \approx H(X)$ for $r \leqslant n$.

REMARK 7. $F_\infty = \cup F_n$ in Remark 3 is a normal polyhedron. Now we consider an inclusion cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ with a normal polyhedron F. Let $\{F_2 \subset F_3 \subset \cdots \subset F_n \subset \cdots; \cup F_n = F\}$ be a normalization of F and we set $X_n = p^{-1}(F_n)$. Then $X_1 = B$ and $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is an inclusion cofibration where q_n is an inclusion map and $p_n = p|X$.

Since $\to H_{r+1}(F_n) \to H_r(B) \to H_r(X_n) \to H_r(F_n) \to$ is exact and $H_r(F_n) = 0$ for $r > n$, it follows that $q_{n*} : H_r(B) \approx H_r(X_n)$ for $r > n$.

Next we consider the commutative diagram (10). Then by the values of the homology groups of F_n and five lemma, we have $H_r(X_n) \approx H(X)$ for $r < n$. Moreover, for $r = n$, we easily see that q_{n*} is a monomorphism, j_{n*} is an epimorphism and Im. $q_{n*} \supset$ Ker. j_{n*}.

4. Weak H′-cofibration and the homology decomposition for a cofibration. In this section we assume that the cofibrations whose homology decompositions are considered constitute 1-connected polyhedra.

DEFINITION 4.1. ([7]) A cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is called weak H'-cofibration if there exists a map $\phi: X \to F \vee X$ and a homotopy $H_t: X \to F \times X$ such that

(a)
$$\begin{tikzcd} B \arrow[r, "i_2"] \arrow[d, "q"'] & F \vee B \arrow[d, "1 \vee q"] \\ X \arrow[r, "\phi"] & F \vee X \end{tikzcd}$$
is homotopy-commutative

where i_2 is the injection into the second factor.

(b) $H_0 = j\phi$ (where j: $F \vee X \to F \times X$ is the injection) and $H_1 = (p \times 1)\Delta_X$.

Let Y be an H'-space with comultiplication μ and let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak H'-cofibration.

DEFINITION 4.2. ([7]) A map $f: Y \to X$ is said to be coprimitive if the diagram:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "f"'] & Y \vee Y \arrow[d, "pf \vee f"] \\ X \arrow[r, "\phi"] & F \vee X \end{tikzcd}$$

is homotopy-commutative.

EXAMPLE. Let $f: A \to B$ be a map. Then the induced cofibration $B \to C_f \to \Sigma A$ via f is a weak H'-cofibration. In fact, following to [7], we define a map ϕ: $C_f \to \Sigma A \vee C_f$ by

$$\phi(b) = (*, b) \qquad b \in B \subset C_f$$

$$\phi(a,t) = \begin{cases} (\langle a, 2t \rangle, *) & 0 \leqslant t \leqslant 1/2 \\ (*, (a, 2t-1)) & 1/2 \leqslant t \leqslant 1 \end{cases} \qquad (a,t) \in CA \subset C_f\,.$$

Then the condition (a) in 4.1 holds evidently. Now we define a homotopy H_s: $C_f \to \Sigma A \times C_f$ by

$$H_s(b) = (*, b) \qquad b \in B \subset C_f\,,$$

$$H_s(a,t) = \begin{cases} \left(<a, \frac{2t}{1+s}>, \left(a, \frac{2st}{1+s}\right)\right) & 0 \leqslant t \leqslant \frac{1+s}{2}, \\ (*, (a, 2t-1)) & \frac{1+s}{2} \leqslant t \leqslant 1 \end{cases} \qquad (a,t) \in CA \subset C_j.$$

Then H_s is well-defined and satisfies the condition (b) in 4.1.

PROPOSITION 4.1. *Let an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ be obtained by applying the suspension functor Σ to an (inclusion) cofibration $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$. Then there is the homology decomposition $\{X_n, F_n, i_n, j_n, p_n, q_n\}$ for $B \xrightarrow{q} X \xrightarrow{p} F$ such that $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is a weak H'-cofibration for each n.*

PROOF. From §3 Remark 5, the homology decomposition for $B \xrightarrow{q} X \xrightarrow{p} F$ may be obtained by applying the suspension functor Σ to the homology decomposition for $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$, i.e. $X_n = \Sigma X'_{n-1}$, $F_n = \Sigma F'_{n-1}$, $q_n = \Sigma q'_{n-1}$ and $p_n = \Sigma p'_{n-1}$.

Now we define a map $\phi\colon X_n \to F_n \vee X_n$ to be the composite

$$\Sigma X'_{n-1} \xrightarrow{\mu} \Sigma X'_{n-1} \vee \Sigma X'_{n-1} \xrightarrow{\Sigma p'_{n-1} \vee 1} \Sigma F'_{n-1} \vee \Sigma X'_{n-1},$$

where μ is a comultiplication in $\Sigma X'_{n-1}$.

Then we can show that the conditions (a) and (b) in 4.1 are satisfied for a cofibration $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$.

First we consider the following diagram:

$$\begin{array}{ccc} \Sigma B' & \xrightarrow{i_2} & \Sigma F'_{n-1} \vee \Sigma B' \\ \Big\downarrow{\scriptstyle \Sigma q'_{n-1}} & & \Big\downarrow{\scriptstyle 1 \vee \Sigma q'_{n-1}} \\ \Sigma X'_{n-1} & \xrightarrow{\phi} & \Sigma F'_{n-1} \vee \Sigma X'_{n-1}. \end{array}$$

From the definition of ϕ, for $<x,t> \in \Sigma B' = B$

$$(\phi \cdot \Sigma q'_{n-1}) <x,t> = \begin{cases} (*,*) & \text{for } \ 0 \leqslant t \leqslant 1/2, \\ (*, \ <q'_{n-1}x, 2t-1>) & \text{for } \ 1/2 \leqslant t \leqslant 1. \end{cases}$$

On the other hand,

$$(1 \vee \Sigma q'_{n-1}) \cdot i_2 <x,t> = (*, <q'_{n-1}\, x, t>) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

Thus the above diagram is homotopy-commutative and condition (a) in 4.1 is satisfied.

Next we consider the diagram :

$$\begin{tikzcd}
\Sigma X'_{n-1} \arrow[r, "\mu"] \arrow[dr, "\Delta"'] & \Sigma X'_{n-1} \vee \Sigma X'_{n-1} \arrow[r, "\Sigma p'_{n-1} \vee 1"] \arrow[d, "j"] & \Sigma F'_{n-1} \vee \Sigma X'_{n-1} \arrow[d, "j"] \\
& \Sigma X'_{n-1} \times \Sigma X'_{n-1} \arrow[r, "\Sigma p'_{n-1} \times 1"] & \Sigma F'_{n-1} \times \Sigma X'_{n-1} .
\end{tikzcd}$$

Since $j \cdot \mu \simeq \Delta$, we have $j \cdot \phi = j \cdot (\Sigma p'_{n-1} \vee 1) \mu \simeq (\Sigma p'_{n-1} \times 1) \cdot \Delta = (p_{n-1} \times 1) \cdot \Delta$. Thus condition (b) in 4.1 is satisfied. Q.E.D.

THEOREM 4.2. *Let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak H'-cofibration, Y an H'-space with comultiplication μ, $f: Y \to X$ coprimitive, and $X \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma Y$ an induced cofibration via f. Then $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a weak H'-cofibration.*

PROOF. By Lemma 2.1, $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a cofibration and so it suffices to show that conditions (a) and (b) in 4.1 are satisfied. By the hypothesis $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak H'-cofibration and hence there exists a map $\phi: X \to F \vee X$ satisfying the conditions (a) and (b) in 4.1. First we consider a composite map $(s \vee i) \cdot \phi: X \xrightarrow{\phi} F \vee X \xrightarrow{s \vee i} C_{pf} \vee C_f$ where s and i are inclusion maps. Since

$$\begin{aligned}
(s \vee i) \phi f &\simeq (s \vee i)(pf \vee f) \mu && \text{(by the coprimitivity of } f) \\
&= (spf \vee if) \mu = (\bar{p} k \iota \vee k \iota) \mu \simeq 0 && \text{(see §2)}
\end{aligned}$$

and $\iota: Y \to CY$ is a cofibration, there exists a homotopy $\omega_t: CY \to C_{pf} \vee C_f$ such that $\omega_1 \iota = (s \vee i) \cdot \phi \cdot f$ and $\omega_0 = *$.

Now we define a map $\lambda: C_f \to C_{pf} \vee C_f$ by

$$\lambda(y, t) = \omega_1(y, t) \quad (y, t) \in CY \quad \lambda(x) = (s \vee i) \cdot \phi(x) \quad x \in X .$$

Since $\lambda(y, 1) = \omega_1(y, 1) = \omega_1 \iota(y) = (s \vee i) \phi \cdot f(y) = \lambda(fy)$, λ is well defined.

Next we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[dr, "\phi"] \arrow[dd, "l"'] & X\times X \arrow[dr, "p\times 1"] \arrow[dd, "i\times i", crossing over] & \\
 & F\times X \arrow[r, "j"] \arrow[dd, "s\vee i"] & F\times X \arrow[dd, "s\times i"] \\
C_f \arrow[r, "\Delta"] \arrow[dr, "\lambda"'] & C_f\times C_f \arrow[dr, "\bar{p}\times 1"] & \\
 & C_{pf}\times C_f \arrow[r, "j"'] & C_{pf}\times C_f
\end{tikzcd}$$

where the top square is homotopy commutative and all other squares except the bottom square are strict commutative.

Then $\quad j\lambda\, i = j\cdot(s\vee i)\phi = (s\times i)\cdot j\cdot\phi \simeq (s\times i)(p\times 1)\,\Delta_X$ (by (b) in 4.1)
$\qquad = (\bar{p}\times 1)\cdot(i\times i)\,\Delta_X$ (by the definition of $\bar{p}$)
$\qquad = (\bar{p}\times 1)\cdot\Delta\cdot i$.

Since $X\xrightarrow{i} C_f\to\Sigma Y$ is an induced cofibration ([7]), it follows from [7; Lemma 2.2] that there exists a map $w:\Sigma Y\to C_{pf}\times C_f$ such that $(w\nabla j\lambda)\cdot\psi\simeq(\bar{p}\times 1)\Delta$, where $\psi: C_f\to\Sigma Y\vee C_f$ is a cooperation in the induced cofibration $X\xrightarrow{i} C_f\to\Sigma Y$ and ∇ denotes the wedge product of maps (see [7]).

Let $p_1: C_{pf}\times C_f\to C_{pf}$, $p_2: C_{pf}\times C_f\to C_f$ be the projection and $\bar{\mu}:\Sigma Y\to\Sigma Y\vee\Sigma Y$ be the comultiplication for ΣY, then we have

$$j(p_1w\vee p_2w)\,\bar{\mu}\simeq(p_1w\times p_2w)\,\Delta = w\,.$$

If we set $\kappa=(p_1w\vee p_2w)\,\bar{\mu}$ and define a map $\tilde{\phi}: C_f\to C_{pf}\vee C_f$ to be the composite map $\tilde{\phi}=(\kappa\nabla\lambda)\psi$, then

$$j\tilde{\phi}=j(\kappa\nabla\lambda)\psi\simeq(j\kappa\nabla j\lambda)\psi\simeq(w\nabla j\lambda)\psi\simeq(\bar{p}\times 1)\Delta\,.$$

Thus the condition (b) in 4.1 holds.
Also, for $b\in B$,

$$\begin{aligned}
\tilde{\phi}\,i\,q(b) &= (\kappa\nabla\lambda)\cdot\psi\cdot i\cdot q(\mathrm{b})\\
&\simeq(\kappa\nabla\lambda)\cdot(1\vee i)\cdot i_2q(b) \quad \text{(since } i \text{ is an weak } H'\text{-cofibration)}\\
&=(\kappa\nabla\lambda)(1\vee i)(*,q(b))=(\kappa\nabla\lambda)(*,iq(b))\\
&=\lambda q(b) \qquad\qquad \text{(by the definition of } \nabla)
\end{aligned}$$

$$= (s\vee i)\,\phi q(b) \simeq (s\vee i)\cdot(1\vee q)(*, b)$$
$$= (*, i\cdot q(b)) = (1\vee i\cdot q)\cdot i_2(b)\,.$$

Hence the condition (a) in 4.1 holds. Q.E.D.

THEOREM 4.3. *Let $B \xrightarrow{q} X \xrightarrow{p} F$ be an (inclusion) cofibration with B: $(r-1)$-connected and F: $(s-1)$-connected $(2\leqslant r\leqslant s)$. Then there is the homology decomposition $(X_n, F_u, i_n, j_n, p_n, q_n)$ for $B \xrightarrow{q} X \xrightarrow{p} F$ such that $B\to X_n\to F_n$ is a weak H'-cofibration for $n\leqslant r+s-2$.*

PROOF. If $n\leqslant s$, then $F_{n-1}=*$ and $X_{n-1}=B$. Hence $B\xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is an induced cofibration and so a weak H'-cofibration. Thus we may take $n>s$. Inductively we assume that $B\xrightarrow{q_{n-1}} X_{n-1}\xrightarrow{p_{n-1}} F_{n-1}$ is a weak H'-cofibration and we shall show that $B\xrightarrow{q_n} X_n\xrightarrow{p_n} F_n$ is a weak H'-cofibration for $n\leqslant r+s-2$. From Theorem 4.2, it suffices to show that $u_{n-1}\colon K'(H_n(F), n-1)\to X_{n-1}$ is co-primitive.

Now we consider the following diagram:

$$\begin{tikzcd}
K'(H_n(F), n-1) \arrow[r, "\mu"] \arrow[d, "u_{n-1}"] & K'(H_n(F), n-1)\vee K'(H_n(F), n-1) \arrow[d, "p_{n-1}u_{n-1}\vee u_{n-1}"] \\
X_{n-1} \arrow[r, "\phi"] \arrow[d, "\Delta"] & F_{n-1}\vee X_{n-1} \arrow[d, "j"] \\
X_{n-1}\times X_{n-1} \arrow[r, "p_{n-1}\times 1"] & F_{n-1}\times X_{n-1}
\end{tikzcd}$$

where μ is a comultiplication for $K'(H_n(F), n-1)$ and ϕ is a map defined for the weak H'-cofibration $B\xrightarrow{q_{n-1}} X_{n-1}\xrightarrow{p_{n-1}} F_{n-1}$.
Then $j\cdot(p_{n-1}u_{n-1}\vee u_{n-1})\cdot\mu\simeq(p_{n-1}u_{n-1}\times u_{n-1})\cdot\Delta_{K'}$,

$$\begin{tikzcd}
K'(H_n(F), n-1) \arrow[r, "\mu"] \arrow[dr, "\Delta_{K'}"'] & K'(H_n(F), n-1)\vee K'(H_n(F), n-1) \arrow[r, "p_{n-1}u_{n-1}\vee u_{n-1}"] \arrow[d, "j"] & F_{n-1}\vee X_{n-1} \arrow[d, "j"] \\
 & K'(H_n(F), n-1)\times K'(H_n(F), n-1) \arrow[r, "p_{n-1}u_{n-1}\times u_{n-1}"] & F_{n-1}\times X_{n-1}.
\end{tikzcd}$$

On the other hand, we have $(p_{n-1}u_{n-1}\times u_{n-1})\Delta_{K'} = (p_{n-1}\times 1)\,\Delta u_{n-1}\,.$

Also, by (b) in 4.1, $(p_{n-1}\times 1)\Delta \simeq j\phi$. Thus we have $j(p_{n-1}u_{n-1}\vee u_{n-1})\mu \simeq j\phi u_{n-1}$. Now we consider the homotopy exact sequence for a map $j: F_{n-1}\vee X_{n-1} \to F_{n-1}\times X_{n-1}$ (cf. [2], [5]).

$$\pi_n(H_n(F), j) \xrightarrow{\partial} \pi_{n-1}(H_n(F), F_{n-1}\vee X_{n-1}) \xrightarrow{j_*} \pi_{n-1}(H_n(F), F_{n-1}\times X_{n-1}).$$

Since B is $(r-1)$-connected and F is $(s-1)$-connected $(r\leqslant s)$, it follows from Remark 2 that each X_m is $(r-1)$-connected and F_m $(m\geqslant s)$ is $(s-1)$-connected. Hence $F_{n-1}\vee X_{n-1}$ is $(r-1)$-connected and $F_{n-1}\# X_{n-1} = F_{n-1}\times X_{n-1}/F_{n-1}\vee X_{n-1}$ is $(r+s-1)$-connected. By using the generalized Blakers-Massey theorem ([5]) for an inclusion cofibration $F_{n-1}\vee X_{n-1}\to F_{n-1}\times X_{n-1}\to F_{n-1}\# X_{n-1}$, we have

$$\pi_i(H_n(F), j) \approx \pi_i(H_n(F), F_{n-1}\# X_{n-1}) \quad \text{for} \quad i<2r+s-2.$$

By the universal coefficient theorem ([5]) for the homotopy group,

$$0\to \mathrm{Ext}(H_n(F), \pi_{n+1}(F_{n-1}\# X_{n-1})) \to \pi_n(H_n(F), F_{n-1}\# X_{n-1})$$
$$\to \mathrm{Hom}\,(H_n(F), \pi_n(F_{n-1}\# X_{n-1})) \to 0 \text{ is exact.}$$

Since $F_{n-1}\# X_{n-1}$ is $(r+s-1)$-connected, then we have

$$\pi_n(H_n(F), F_{n-1}\# X_{n-1}) = 0 \quad \text{for} \quad n\leqslant r+s-2.$$

Hence $j_*: \pi_{n-1}(H_n(F), F_{n-1}\vee X_{n-1}) \to \pi_{n-1}(H_n(F), F_{n-1}\times X_{n-1})$ is a monomorphism for $n\leqslant r+s-2$.

Thus, for $n\leqslant r+s-2$, it can be deduced from $j\cdot(p_{n-1}u_{n-1}\vee u_{n-1})\cdot\mu \simeq j\cdot\phi\cdot u_{n-1}$ that $(p_{n-1}u_{n-1}\vee u_{n-1}) \simeq \phi\cdot u_{n-1}$. Q.E.D.

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