INTEGRAL FORMULAS FOR HYPERSURFACES IN A RIEMANNIAN MANIFOLD AND THEIR APPLICATIONS

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Introduction. The following Liebmann-Süss theorem[1]:

A convex hypersurface M with constant mean curvature in Euclidean space E is a sphere of $\dim M$,

has been generalized by Y. Katsurada[2] (1964) [2] to the case in which E is replaced with an Einstein space admitting a suitable conformal vector field. Her tools of the verification of the theorem are some integral equalities and the first one of the Newton inequalities on symmetric square matrices.

On the other hand, S. S. Chern (1959), [1] has proved uniqueness theorems for closed hypersurfaces in Euclidean spaces, making use of some integral formulas which are obtained by a remarkable method by virtue of moving frames due to E. Cartan.

The object of this note is to prove theorems more generalized than Katsurada's theorem, making use of Chern's methods.

1. Preliminaries. Let M and $\bar{M}$ be oriented differentiable Riemannian manifolds of dimensions n and $n+1$ respectively, and let $x: M \rightarrow \bar{M}$ be an isometric immersion. Let $F(M)$ and $F(\bar{M})$ be the bundles of orthonormal frames of M and $\bar{M}$ such that their orientations are coherent with those of M and $\bar{M}$. Let $\xi(p)$, $p \in M$, be the unit normal vector at $x(p)$ such that for any orthonormal frame $b = \{p, e_1, \cdots, e_n\} \in F(M)$, $\bar{b} = \{x(p), dx(e_1), \cdots, dx(e_n), \xi(p)\} \in F(\bar{M})$. We denote this mapping of $F(M)$ into $F(\bar{M})$ by $\tilde{x}$. Let $\bar{\omega}_\lambda$ be the basic forms for the frame bundle $F(\bar{M})$[3] and $\omega_{\lambda\mu} = -\omega_{\mu\lambda}$ be the connection forms for the Levi-Civita's connection of $\bar{M}$, then we have

$$(1.1) \qquad d\bar{\omega}_\lambda = \sum \bar{\omega}_\mu \wedge \bar{\omega}_{\mu\lambda}, \qquad d\bar{\omega}_{\lambda\mu} = \sum \bar{\omega}_{\lambda\rho} \wedge \bar{\omega}_{\rho\mu} + \bar{\Omega}_{\lambda\mu},$$

1) This theorem dues to H. Liebmann (1901) [3] in case $\dim E=3$ and W. Süss (1929) [4] in case $\dim E>3$.

2) K. Yano [5] has recently generalized Katsurada's theorem to the case in which E is replaced with suitable Riemannian manifolds.

3) In this note, Greek indices run from 1 to $n+1$ and Latin indices from 1 to n.

(1.2) $$\bar{\Omega}_{\lambda\mu} = \frac{1}{2}\sum \bar{R}_{\lambda\mu\nu\rho}\,\bar{\omega}_\nu \wedge \bar{\omega}_\rho\,, \qquad \bar{R}_{\lambda\mu\nu\rho} = -\bar{R}_{\lambda\mu\rho\nu}\,.$$

Putting $\omega_\lambda = \tilde{x}^*\bar{\omega}_\lambda$, $\omega_{\lambda\mu} = \tilde{x}^*\bar{\omega}_{\lambda\mu}$, as is well known, ω_i and ω_{ij} are the basic forms and the connection forms on $F(M)$ for the Levi-Civita's connection of M, and so we have

(1.3) $$d\omega_i = \sum \omega_j \wedge \omega_{ji}\,, \qquad d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}\,,$$

(1.4) $$\Omega_{ij} = \frac{1}{2}\sum R_{ijhk}\,\omega_h \wedge \omega_k\,, \qquad R_{ijhk} = -R_{ijkh}\,.$$

Furthermore we have

$$\omega_{n+1} = 0\,, \qquad \sum \omega_i \wedge \omega_{i\ n+1} = 0$$

and

$$\Omega_{ij} = \tilde{x}^*\bar{\Omega}_{ij} - \omega_{i\ n+1} \wedge \omega_{j\ n+1}\,.$$

Hence, putting

$$\omega_{i\ n+1} = \sum h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,,$$

we have

(1.5) $$R_{ijkl} = \bar{R}_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}\,,$$

where h_{ij} are the components of the second fundamental tensor of the immersion of M into $\bar{M}$.

2. Integral formulas. In this section, assume that a vector field $\bar{\xi}$ is given on $\bar{M}$. According to Chern [1], we introduce the differential $(n-1)$-form on M

(2.1) $$A = (\bar{\xi}, \xi, \underbrace{dx, \cdots, dx}_{n-1})\,,$$

where $dx = \sum \omega_i e_i$, $b = \{p, e_1, \cdots, e_n\} \in F(M)$, and the expression is a determinant of order $n+1$ in the following sense, whose columns are the components of the respective vectors or vector valued differential forms with respect to frames b or $\tilde{x}(b)$, with the convention that in the expansion of the determinant the multiplication of differential forms is in the sense of exterior multiplication and $(e_1, \cdots, e_n, e_{n+1}) = 1$, e_i identifying $dx(e_i)$ and $e_{n+1} = \xi(p)$.

Since $\bar{\zeta} = \bar{\zeta}^\lambda e_\lambda$, $D\bar{\zeta} = (d\bar{\zeta}^\lambda + \bar{\omega}_\mu{}^\lambda \bar{\zeta}^\mu)\, e_\lambda = \bar{\zeta}^\lambda{}_{,\mu} \bar{\omega}^\mu e_\lambda$, putting $\bar{\omega}_\mu{}^\lambda = \bar{\omega}_{\mu\lambda}$ etc., on $\bar{M}$ and $D(dx) = (d\omega^i + \omega_j{}^i \wedge \omega^j) e_i = 0$ on M, we have

$$\begin{aligned} dA &= (D\bar{\zeta}, \xi, dx, \cdots, dx) + (\bar{\zeta}, D\xi, dx, \cdots, dx) \\ &= (-1)^{n-1}(\bar{\zeta}^j{}_{,i}\omega^i e_j, dx, \cdots, dx, e_{n+1}) \\ &\quad + (-1)^n <\bar{\zeta}, e_{n+1}> (\omega_{n+1}{}^i e_i, dx, \cdots, dx, e_{n+1}), \end{aligned}$$

that is

$$dA = (-1)^{n-1} n! \left\{ <\bar{\zeta}, \xi> P_1(H) + \frac{1}{n} \bar{\zeta}^j{}_{,j} \right\} dV, \tag{2.2}$$

where

$$P_1(H) = \frac{1}{n} \sum h_{ii}, \qquad H = ((h_{ij})), \tag{2.3}$$

$$dV = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n. \tag{2.4}$$

Now, for any two vector fields $\bar{\eta}$, $\bar{\rho}$ of $\bar{M}$ along $x: M \to \bar{M}$, we define a vector field of M by

$$D\bar{\eta}(\bar{\rho}) = \sum \bar{\eta}^\lambda{}_{,i} \bar{\rho}_\lambda e_i, \quad D\bar{\eta} = \bar{\eta}^\lambda{}_{,i} \omega^i e_\lambda, \tag{2.5}$$

especially

$$D\xi(\bar{\zeta}) = (\omega_{n+1}{}^i e_i)(\bar{\zeta}) = (-h_j{}^i \omega^j e_i)(\bar{\zeta}) = -h_j{}^i \bar{\zeta}^j e_i.$$

Let ζ be the orthogonal projection of $\bar{\zeta}$ onto $T(M)$, that is

$$\zeta = \bar{\zeta} - <\bar{\zeta}, \xi> \xi, \tag{2.6}$$

then the above equation can be written as[4)]

$$D\xi(\bar{\zeta}) = \omega_{n+1}{}^i(\zeta) e_i. \tag{2.7}$$

Then, we introduce the second differential $(n-1)$-form on M

$$B = (D\xi(\bar{\zeta}), \xi, \underbrace{dx, \cdots, dx}_{n-1}) \tag{2.8}$$

4) ζ is here identified with its horizontal lift on $F(M)$ with respect to the Levi-Civita's connection of M.

from which we have

$$dB = (D(D\xi(\bar{\zeta})), \xi, dx, \cdots, dx).$$

By (2.7), we have

$$\begin{aligned} D(D\xi(\bar{\zeta})) &= D(\omega_{n+1}{}^i(\zeta)\, e_i) \\ &= \{d(\omega_{n+1}{}^i(\zeta)) + \omega_{n+1}{}^j(\zeta)\,\omega_j^i\}\, e_i + \omega_{n+1}{}^j(\zeta)\,\omega_j^{n+1} e_{n+1}\,. \end{aligned}$$

Making use of the same notations for the basic horizontal tangent vector field corresponding to e_i and ζ on $F(M)$ and using (1.1), we get

$$\begin{aligned} \{d(\omega_{n+1}{}^i(\zeta)) + \omega_{n+1}{}^j(\zeta)\,\omega_j^i\}(e_i) &= e_i(\omega_{n+1}{}^i(\zeta)) + \omega_{n+1}{}^j(\zeta)\,\omega_j^i(e_i) \\ &= 2(d\omega_{n+1}{}^i - \omega_{n+1}{}^j \wedge \omega_j^i)(e_i, \zeta) + \omega_{n+1}{}^j(e_i)\,\omega_j^i(\zeta) \\ &\quad + \zeta(\omega_{n+1}{}^i(e_i)) + \omega_{n+1}{}^i([e_i, \zeta]) \\ &= (\bar{R}_{n+1}{}^i{}_{hk}\omega^h \wedge \omega^k)(e_i, \zeta) - \zeta(\mathrm{tr}\,(H)) + \zeta^j{}_{,i}\,\omega_{n+1}{}^i(e_j) \\ &= \bar{R}_{n+1}{}^i{}_{ik}\zeta^k - \zeta(\mathrm{tr}\,(H)) - h_j{}^i\zeta^j{}_{,i}\,. \end{aligned}$$

On the other hand, we have

$$D\bar{\zeta}^j = d\bar{\zeta}^j + \bar{\zeta}^l\omega_l^j + \bar{\zeta}^{n+1}\omega_{n+1}{}^j = d\zeta^j + \zeta^l\omega_l{}^j + \bar{\zeta}^{n+1}\omega_{n+1}{}^j\,,$$

$$\bar{\zeta}^j{}_{,i} = \zeta^j{}_{,i} - \langle\bar{\zeta}, \xi\rangle\, h_i{}^j\,. \tag{2.9}$$

Let $\bar{R}_{\lambda\mu} = \bar{R}_\lambda{}^\rho{}_{\rho\mu}$ be the components of the Ricci tensor of $\bar{M}$. Then we have finally

$$\{d(\omega_{n+1}{}^i(\zeta)) + \omega_{n+1}{}^j(\zeta)\,\omega_j^i\}(e_i) = \bar{R}_{\lambda\mu}\xi^\lambda\zeta^\mu - \zeta(\mathrm{tr}\,(H)) - h_j{}^i\bar{\zeta}^j{}_{,i} - \langle\bar{\zeta}, \xi\rangle\, h_j{}^i h_i{}^j\,.$$

Now we define $P_r(H)$ by means of the equation

$$\det\,(I + Hy) = \sum_{r=0}^{n} \binom{n}{r} P_r(H)\, y^r\,. \tag{2.10}$$

We have easily

$$h_j{}^i h_i{}^j = \mathrm{tr}\, H^2 = n^2(P_1(H))^2 - n(n-1)\, P_2(H)\,.$$

Using the above equations, we get

$$
(2.11)\qquad dB = (-1)^{n-1}\, n!\, [-\langle\bar{\xi}, \xi\rangle \{n(P_1(H))^2 - (n-1)\, P_2(H)\} - \zeta(P_1(H)) - \frac{1}{n} h_j{}^i \bar{\xi}^j{}_{,i} + \frac{1}{n} \bar{R}_{\lambda\mu} \xi^\lambda \zeta^\mu]\, dV .
$$

From (2.2) and (2.11), we get a formula

$$
(2.12)\qquad dB + P_1(H)\, dA = (-1)^{n-1}\, n!\, [(n-1)\langle\bar{\xi}, \xi\rangle \{P_2(H) - (P_1(H))^2\} - \zeta(P_1(H)) + \frac{1}{n} \{P_1(H)\bar{\xi}^j{}_{,j} - h_j{}^i \bar{\xi}^j{}_{,i}\} + \frac{1}{n} \bar{R}_{\lambda\mu} \xi^\lambda \zeta^\mu]\, dV .
$$

3. Liebmann-Süss Theorem in general cases. According to Katsurada [2], assume furthermore that M is compact and $\bar{\xi}$ is conformal. Then

$$
(3.1)\qquad \bar{\xi}_{\lambda,\mu} + \bar{\xi}_{\mu,\lambda} = 2\bar{\phi}\,\delta_{\lambda\mu}
$$

in $\bar{M}$ and along $x: M \to \bar{M}$

$$
\mathrm{tr}\,(\bar{\xi}^j{}_{,i}) = n\bar{\phi}\,, \quad h_j{}^i \bar{\xi}^j{}_{,i} = \frac{1}{2} h^{ij}(\bar{\xi}_{j,i} + \bar{\xi}_{i,j}) = \bar{\phi}\,\mathrm{tr}\,(H) = n\bar{\phi} P_1(H)\,.
$$

Hence (2.12) becomes

$$
(3.2)\qquad dB + P_1(H)\, dA = (-1)^{n-1}\, n!\, [(n-1)\langle\bar{\xi}, \xi\rangle \{P_2(H) - (P_1(H))^2\} - \zeta(P_1(H)) + \frac{1}{n} \bar{R}_{\lambda\mu} \xi^\lambda \zeta^\mu]\, dV .
$$

THEOREM 3.1. *Let $\bar{M}$ be an orientable $(n+1)$-dimensional Riemannian manifold admitting a conformal vector field $\bar{\xi}$. M be a compact orientable n-dimensional Riemannian manifold and $x: M \to \bar{M}$ be an isometric immersion. Let ζ be the vector field of M which is the naturally defined orthogonal projection of $\bar{\xi}$ onto $T(M)$. If the mean curvature of M in $\bar{M}$ is constant, $\bar{\xi}$ is not tangent to $x(M)$ everywhere, and ζ and the normal vector field ξ of M is conjugate with respect of the Ricci tensor of $\bar{M}$, then M is umbilical at every point.*

PROOF. By the assumption we have

$$
P_1(H) = \text{const.}, \qquad \zeta(P_1(H)) = 0\,,
$$

$$
(3.3)\qquad \bar{R}_{\lambda\mu} \xi^\lambda \zeta^\mu = 0\,.
$$

From (3. 2), we get

$$0 = \int_M (dB + P_1(H)\, dA) = (-1)^{n-1}(n-1)n! \int_M <\bar{\xi}, \xi> \{P_2(H) - (P_1(H))^2\}\, dV .$$

By the Newton inequality, we have

$$P_2(H) \leqq (P_1(H))^2 . \tag{3.4}$$

By the assumption. $<\bar{\xi}, \xi> \neq 0$ at every point, we must have

$$P_2(H) = (P_1(H))^2 ,$$

which implies that $H = P_1(H)I$. Hence, M is umbilical at every point.

REMARK. In this theorem, if $\bar{M}$ is an Einstein space, the condition (3.3) is satisfied automatically. In this case, Theorem 3.1 becomes Katsurada's one.

4. Tensors derived from the second fundamental tensor *H*. For the second fundamental tensor H of the immersion $x: M \to \bar{M}$, we introduce some tensors of type (1,1) and (2,2) as follows.

Let $h_i^j = h_{ij}$ be the components of H with respect to an orthonormal frame $b = (p, e_1, \cdots, e_n) \in F(M)$. Let

$$P(y, H) = \det(I + Hy) = \sum_{r=0}^{n} \binom{n}{r} P_r(H) y^r , \tag{4.1}$$

where y is a parameter.

Now, we denote the cofactors of h_i^k and the minor $h_i^k h_j^l - h_i^l h_j^k$ $(i<j, k<l)$ of H by $H_{(i)}^{(k)}$ and $H_{(ij)}^{(kl)}$ respectively and use the analogous notations for the identity matrix $I = (\delta_i^k)$. Let

$$P_i{}^k(y, H) = \det(I_{(i)}^{(k)} + H_{(i)}^{(k)} y) = \sum_{r=0}^{n-1} \binom{n-1}{r} P_{i(r)}^k(H) y^r \tag{4.2}$$

and

$$P_{ij}^{kl}(y, H) = \det(I_{(ij)}^{(kl)} + H_{(ij)}^{(kl)} y) = \sum_{r=0}^{n-2} \binom{n-2}{r} P_{ij(r)}^{kl}(H) y^r . \tag{4.3}$$

LEMMA 4.1. *$(-1)^{i+k} P_{i(r)}^k(H)$ are components of the tensor $H_{(r)}$ defined by*

$$\binom{n-1}{r}\underset{(r)}{H} = \binom{n}{r}P_r(H)\,I - \binom{n}{r-1}P_{r-1}(H)\,H + \cdots \tag{4.4}$$

$$+ (-1)^s\binom{n}{r-s}P_{r-s}(H)H^s + \cdots + (-1)^r H^r\,,$$

$$r = 0, 1, 2, \cdots, n-1\,.$$

PROOF. With respect to a suitable frame $b \in F(M)$, H is written as

$$H = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix}.$$

Then, we get easily $P_i^k(y, H) = 0$ and so $P_{i(r)}^k(H)$ for $i \neq k$. For $i = k = 1$, we have

$$\det(I_{(1)}^{(1)} + H_{(1)}^{(1)}y) = (1+k_2 y)\cdots(1+k_n y) = \frac{\det(I+Hy)}{1+k_1 y}$$

$$= \sum_{r=0}^{n}\binom{n}{r}P_r(H)\,y^r\cdot(1 - k_1 y + k_1{}^2 y^2 - \cdots)\,,$$

hence

$$\binom{n-1}{r}P_{1(r)}^1(H) = \binom{n}{r}P_r(H) - \binom{n}{r-1}P_{r-1}(H)\,k_1 + \cdots$$

$$+ (-1)^s\binom{n}{r-s}P_{r-s}(H)\,k_1{}^s + \cdots + (-1)^r k_1{}^r\,.$$

The right hand side is equal to the corresponding component of $\binom{n-1}{r}\underset{(r)}{H}$.

LEMMA 4.2. *$(-1)^{i+j+k+l}P_{ij(r)}^{kl}(H)$ are components of the tensor $H\underset{(r)}{\wedge}H$ defined by*

$$\binom{n-2}{r}H\underset{(r)}{\wedge}H = \binom{n}{r}P_r(H)I\wedge I - \binom{n}{r-1}P_{r-1}(H)(H\wedge I + I\wedge H) + \tag{4.5}$$

$$\cdots + (-1)^s\binom{n}{r-s}P_{r-s}(H)(H^s\wedge I + H^{s-1}\wedge H + \cdots$$

$$+ H\wedge H^{s-1} + I\wedge H^s) + \cdots\,, \qquad r = 0, 1, 2, \cdots, n-2\,.$$

PROOF. Analogously to the proof of Lemma 4.1, we have $P^{kl}_{ij}(y, H) = 0$ and so $P^{kl}_{ij(r)}(H)=0$ for $(i,j) \neq (k,l)$. For $i=k=1$, $j=l=2$, we have

$$\det(I^{(12)}_{(12)} + H^{(12)}_{(12)} y) = (1+k_3 y)\cdots(1+k_n y) = \frac{\det(I+Hy)}{(1+k_1 y)(1+k_2 y)}$$

$$= \sum_{r=0}^{n} \binom{n}{r} P_r(H) y^r \cdot (1-k_1 y + k_1{}^2 y^2 - \cdots)(1-k_2 y + k_2{}^2 y^2 - \cdots),$$

hence

$$\binom{n-2}{r} P^{12}_{12(r)}(H) = \binom{n}{r} P_r(H) - \binom{n}{r-1} P_{r-1}(H)(k_1+k_2)$$
$$+ \binom{n}{r-2} P_{r-2}(H)(k_1{}^2 + k_1 k_2 + k_2{}^2) + \cdots + (-1)^s \binom{n}{r-s} P_{r-s}(H)(k_1{}^s$$
$$+ k_1{}^{s-1} k_2 + \cdots + k_1 k_2{}^{s-1} + k_2{}^s) + \cdots + (-1)^r (k_1{}^r + k_1{}^{r-1} k_2$$
$$+ \cdots + k_1 k_2{}^{r-1} + k_2{}^r).$$

The right hand side is equal to the corresponding component of the tensor $\binom{n-2}{r} H \underset{(r)}{\wedge} H$.

REMARK. From (4.4) and (4.5) we have especially

$$\underset{(0)}{H} = I, \quad \underset{(1)}{H} = \frac{n}{n-1} P_1(H) I - \frac{1}{n-1} H,$$

$$\underset{(2)}{H} = \frac{n}{n-2} P_2(H) I - \frac{2n}{(n-1)(n-2)} P_1(H) H + \frac{2}{(n-1)(n-2)} H^2$$

and

$$H \underset{(0)}{\wedge} H = I \wedge I, \quad H \underset{(1)}{\wedge} H = \frac{n}{n-2} P_1(H) I \wedge I - \frac{1}{n-2} (H \wedge I + I \wedge H).$$

5. Main theorem. As in §3, we shall assume that M is compact and $\bar{\xi}$ is conformal. Putting

(5.1) $$\bar{\xi}_{\lambda,\mu} - \bar{\xi}_{\mu,\lambda} = 2\bar{S}_{\lambda\mu}$$

by (3.1) we have along $x: M \to \bar{M}$

$$D\bar{\xi} = \sum \bar{\xi}_{\lambda j}\omega^j e_\lambda = \sum (\bar{\phi}\delta_{\lambda j}+\bar{S}_{\lambda j})\,\omega^j e_\lambda\,,$$

that is

$$D\bar{\xi} = \bar{\phi}\,dx + \sum \bar{S}_{\lambda j}\,\omega^j e_\lambda\,. \tag{5.2}$$

Now, we introduce the differential forms on M

$$A_{rs} = \rho^s(\bar{\xi}, \xi, \underbrace{D\xi, \cdots, D\xi}_{n-r-1}, \underbrace{dx, \cdots, dx}_{r}) \qquad 0 \leqq s \leqq n-r-1\,, \tag{5.3}$$

$$D_{rs} = \rho^s(\xi, \underbrace{D\xi, D\xi, \cdots, D\xi}_{n-r}, \underbrace{dx, \cdots, dx}_{r})\,, \qquad 0 \leqq s \leqq n-r \tag{5.4}$$

where ρ is an arbitrary function on M. For simplicity, put

$$<\bar{\xi}, \xi> = <\bar{\xi}, e_{n+1}> = z\,.$$

We have

$$dA_{rs} = \rho^s(\bar{\xi}, D\xi, \cdots, D\xi, dx, \cdots, dx) - \rho^s(\xi, D\xi, \cdots, D\xi, D\bar{\xi}, dx, \cdots, dx)$$
$$+(n-r-1)\,\rho^s(\bar{\xi}, \xi, D^2\xi, D\xi, \cdots, D\xi, dx, \cdots, dx) + s\,d\log\rho \wedge A_{rs}\,.$$

We have

$$(\bar{\xi}, D\xi, \cdots, D\xi, dx, \cdots, dx) = z(\xi, D\xi, \cdots, D\xi, dx, \cdots, dx)\,,$$

$$(\xi, D\xi, \cdots, D\xi, D\bar{\xi}, \underbrace{dx, \cdots, dx}_{r}) = \bar{\phi}(\xi, D\xi, \cdots, D\xi, \underbrace{dx, \cdots, dx}_{r+1})$$
$$+ (\xi, D\xi, \cdots, D\xi, \sum \bar{S}_{\lambda j}\,\omega^j e_\lambda, \underbrace{dx, \cdots, dx}_{r})$$

and

$$(\xi, D\xi, \cdots, D\xi, \sum \bar{S}_{ij}\,\omega^j e_i, dx, \cdots, dx) = 0\,,$$

because $D\xi = -\sum h_{ij}\,\omega^j e_j$, $h_{ij} = h_{ji}$, $\bar{S}_{ij} = -\bar{S}_{ji}$. Hence we have

$$dA_{rs} = zD_{rs} - \bar{\phi}D_{r+1,s} + (n-r-1)\,\rho^s(\zeta, \xi, D^2\xi, D\xi, \cdots, D\xi, dx, \cdots, dx)$$
$$+ s\,d\log\rho \wedge A_{rs}\,.$$

Now, we prepare some equalities for the above equation. Firstly

$$\sum_{r=0}^{n}\binom{n}{r}D_{r0}y^{n-r}=(-1)^n\sum_{r=0}^{n}\binom{n}{r}(D\xi,\cdots,D\xi,dx,\cdots,dx,\xi)\,y^{n-r}$$

$$=(-1)^n\sum_{i_1,\cdots,i_n}\varepsilon_{i_1,\cdots,i_n}(\omega^{i_1}+\omega^{i_1}_{n+1}y)\wedge\cdots\wedge(\omega^{i_n}+\omega^{i_n}_{n+1}y)$$

$$=(-1)^n n!(\omega^1+\omega^1_{n+1}y)\wedge\cdots\wedge(\omega^n+\omega^n_{n+1}y)$$

$$=(-1)^n n!\det(I-Hy)\,\omega^1\wedge\cdots\wedge\omega^n$$

$$=(-1)^n n!\sum_{r=0}^{n}\binom{n}{r}P_{n-r}(H)(-y)^{n-r}\,dV\,,$$

hence

$$D_{rs}=(-1)^r n!\,\rho^s P_{n-r}(H)\,dV\,. \tag{5.5}$$

Now, put

$$F_r=(\zeta,\xi,D^2\xi,\underbrace{D\xi,\cdots,D\xi}_{n-r-2},\underbrace{dx,\cdots,dx}_{r})\,.$$

Since $D\xi=\omega^i_{n+1}e_i$, $D^2\xi=(d\omega^i_{n+1}+\omega^i_j\wedge\omega^j_{n+1})e_i$, we get

$$\sum_{r=0}^{n-2}\binom{n-2}{r}F_r y^{n-2-r}=(-1)^{n-1}\sum_{r=0}^{n-2}\binom{n-2}{r}(\zeta,D^2\xi,D\xi,\cdots,D\xi,dx,\cdots,dx,\xi)\,y^{n-2-r}$$

$$=(-1)^{n-1}\sum_{i_1,\cdots,i_n}\varepsilon_{i_1\cdots i_n}\zeta^{i_1}\bar{\Omega}^{i_2}_{n+1}\wedge(\omega^{i_3}+\omega^{i_3}_{n+1}y)\wedge\cdots\wedge(\omega^{i_n}+\omega^{i_n}_{n+1}y)$$

$$=(-1)^{n-1}(n-2)!\sum_{i<j}(-1)^{i+j-1}(\zeta^i\bar{\Omega}^j_{n+1}-\zeta^j\bar{\Omega}^i_{n+1})\wedge(\omega^1+\omega^1_{n+1}y)\wedge$$

$$\cdots\wedge\widehat{(\omega^i+\omega^i_{n+1}y)}\wedge\cdots\wedge\widehat{(\omega^j+\omega^j_{n+1}y)}\wedge\cdots\wedge(\omega^n+\omega^n_{n+1}y)\ ^{5)}$$

$$=(-1)^{n-1}(n-2)!\sum_{i<j,k<l}(-1)^{i+j+k+l}(\zeta^i\bar{R}_{n+1}{}^j{}_{kl}-\zeta^j\bar{R}_{n+1}{}^i{}_{kl})\times$$

$$\times\sum_{r=0}^{n-2}\binom{n-2}{r}P^{kl}_{ij(n-2-r)}(H)(-y)^{n-2-r}\,dV\,.$$

Using Lemma 4.2, we get

5) The notation " $\frown$ " means the omission of the symbols under it.

$$(5.6) \qquad F_r = (-1)^{r-1}(n-2)! \sum_{\substack{i<j \\ k<l}} (\zeta^i \bar{R}_{n+1}{}^j{}_{kl} - \zeta^j \bar{R}_{n+1}{}^i{}_{kl})(H \underset{(r)}{\wedge} H)^{kl}_{ij}\, dV .$$

Lastly, we have

$$\sum_{r=0}^{n-1} \binom{n-1}{r} A_{r0} y^{n-1-r} = (-1)^{n-1} \sum_{r=0}^{n-1} \binom{n-1}{r} (\zeta, D\xi, \cdots, D\xi, dx, \cdots, dx, \xi)\, y^{n-1-r}$$

$$= (-1)^{n-1}(\zeta, dx + D\xi y, \cdots, dx + D\xi y, \xi)$$

$$= (-1)^{n-1} \sum_{i_1,\cdots,i_n} \varepsilon_{i_1\cdots}\ \zeta^{i_1}(\omega \ + \omega^{i_2}_{n+1} y) \wedge \cdots \wedge (\omega^{i_n} + \omega^{i_n}_{n+1} y)$$

$$= (-1)^{n-1}(n-1)! \sum_i (-1)^{i-1} \zeta^i (\omega^1 + \omega^1_{n+1} y) \wedge \cdots \wedge \widehat{(\omega^i + \omega_{n+1}{}^i y)}$$

$$\wedge \cdots \wedge (\omega^n + \omega^n_{n+1} y)$$

$$= (-1)^{n-1}(n-1)! \sum_{i,k} (-1)^{i-1} \zeta^i \sum_{r=0}^{n-1} \binom{n-1}{r} P^k_{i(n-r-1)}(H)(-y)^{n-r-1} \times$$

$$\times\, \omega^1 \wedge \cdots \wedge \widehat{\omega}^k \wedge \cdots \wedge \omega^n ,$$

hence

$$(5.7) \qquad A_{r0} = (-1)^r (n-1)! \sum_{i,k} (-1)^{i-1} \zeta^i P^k_{i(n-1-r)}(H) \omega^1 \wedge \cdots \wedge \widehat{\omega}^k \wedge \cdots \wedge \omega^n .$$

Putting

$$d \log \rho = \frac{1}{\rho} \sum_i \rho,_i \omega^i ,$$

and using Lemma 4.1, we get

$$(5.8) \qquad d \log \rho \wedge A_{r0} = (-1)^r (n-1)! \sum_{i,k} \zeta^i \frac{\rho,_k}{\rho} \underset{(n-1-r)}{H^k_i}\, dV .$$

Using the above equations, we get finally

$$(5.9) \qquad dA_{rs} = (-1)^r n! \Big[z\rho^s P_{n-r}(H) + \bar{\Phi}\rho^s P_{n-r-1}(H)$$

$$- \frac{(n-r-1)}{n(n-1)} \rho^s \sum_{i<j,k<l} (\zeta^i \bar{R}_{n+1}{}^j{}_{kl} - \zeta^j \bar{R}_{n+1}{}^i{}_{kl})(H \underset{(n-2-r)}{\wedge} H)^{kl}_{ij}$$

$$+ \frac{s}{n} \rho^{s-1} \zeta^i \rho,_k \underset{(n-1-r)}{H^k_i} \Big] dV .$$

Now, we assume that $P_{n-2-r}(H) \neq 0$ at every point of M. Putting

$$\rho = \frac{P_{n-1-r}(H)}{P_{n-2-r}(H)},$$

we have from (5.9) the formula

$$(5.10) \qquad d(A_{r0}+A_{r+1,1}) = (-1)^r n! \Big[z \Big\{ P_{n-r}(H) - \frac{(P_{n-1-r}(H))^2}{P_{n-2-r}(H)} \Big\}$$

$$- \frac{1}{n(n-1)} \sum_{\substack{i<j \\ k<l}} (\zeta^i \bar{R}_{n+1}{}^j{}_{kl} - \zeta^j \bar{R}_{n+1}{}^i{}_{kl}) \times$$

$$\times \Big\{ (n-r-1) (H \underset{(n-2-r)}{\wedge} H)^{kl}_{ij} - (n-r-2) \frac{P_{n-1-r}(H)}{P_{n-2-r}(H)} (H \underset{(n-3-r)}{\wedge} H)^{kl}_{ij} \Big\}$$

$$- \frac{1}{n} \zeta^i \Big(\frac{P_{n-1-r}(H)}{P_{n-2-r}(H)} \Big)_{,k} \underset{(n-2-r)}{H^k_i} \Big] dV .$$

THEOREM 5.1. *Let $\bar{M}$ be an orientable $(n+1)$-dimensional Riemannian manifold admitting a conformal vector field $\bar{\zeta}$, M be a compact orientable n-dimensional Riemannian manifold and $x: M \to \bar{M}$ be an isometric immersion. Let ζ be the vector field of M which is the naturally defined orthogonal projection of $\bar{\zeta}$ onto $T(M)$. Let ξ and H be a normal unit vector field and the second fundamental tensor with respect to ξ of the immersion. If $<\xi, \bar{\zeta}>$ has the same sign except a subset with measure 0 of M, $P_{n-r-2}(H) \neq 0$ at every point, $\frac{P_{n-1-r}(H)}{P_{n-2-r}(H)}$ is constant, and $\mathrm{tr}\,[(\zeta \wedge \bar{R}(\xi))((n-r-1) P_{n-r-2}(H) H \underset{(n-r-2)}{\wedge} H - (n-r-2) P_{n-r-1}(H) H \underset{(n-r-3)}{\wedge} H) = 0$, then M is umbilical at every point of M.*

Where $\zeta \wedge \bar{R}(\xi)$ denotes the transformation on $T(M) \wedge T(M)$ itself such that $\zeta \wedge \bar{R}(\xi)(X \wedge Y) = \zeta \wedge \bar{R}(\xi, XY)$.

PROOF. By (5.10) and the assumption, we have

$$\int_M \frac{z}{P_{n-r-2}(H)} \{P_{n-r}(H) P_{n-r-2}(H) - (P_{n-r-1}(H))^2\} dV = 0 .$$

Since $\frac{z}{P_{n-r-2}(H)}$ has a fixed sign except a set with measure 0 and by the Newton inequalities

$$P_{n-r}(H)P_{n-r-2}(H) \leqq (P_{n-r-1}(H))^2,$$

it must be

$$P_{n-r}(H)P_{n-r-2}(H) = (P_{n-r-1}(H))^2,$$

which implies $H = P_1(H)I$. Hence M is umbilical at each point of M.

REMARK. Since

$$\sum_{i<j}(\zeta^i \overline{R}_{n+1}{}^j{}_{kl} - \zeta^j \overline{R}_{n+1}{}^i{}_{kl})(H \underset{(0)}{\wedge} H)^{kl}_{ij} = \sum_{i<j,k<l}(\zeta^i \overline{R}_{n+1}{}^j{}_{kl} - \zeta^j \overline{R}_{n+1}{}^i{}_{kl})\,\delta^k_i \delta^l_j$$

$$= \sum \zeta^i \overline{R}_{n+1}{}^j{}_{ij} = -\zeta^i \overline{R}_{n+1\,i} = -\overline{R}_{\lambda\mu}\xi^\lambda \zeta^\mu,$$

for $r=n-2$, (5.10) becomes to

$$d(A_{n-2,0}+A_{n-1,1}) = (-1)^n n!\,[z\{P_2(H)-(P_1(H))^2\} + \frac{1}{n(n-1)}\overline{R}_{\lambda\mu}\xi^\lambda\zeta^\mu - \frac{1}{n}\zeta(P_1(H))]\,dV. \tag{5.11}$$

This formula implies also Theorem 3.1.

On the other hand, by (3.2), (5.8) and $A=A_{n-1,0}$, we get

$$d(B+P_1(H)A) = (-1)^{n-1}n!\,[(n-1)z\{P_2(H)-(P_1(H))^2\} - \zeta(P_1(H)) + \frac{1}{n}\overline{R}_{\lambda\mu}\xi^\lambda\zeta^\mu]\,dV + dP_1(H)\wedge A_{n-1,0}$$

$$= (-1)^{n-1}n!\,[(n-1)z\{P_2(H)-(P_1(H))^2\} - \frac{n-1}{n}\zeta(P_1(H)) + \frac{1}{n}\overline{R}_{\lambda\mu}\xi^\lambda\zeta^\mu]\,dV = -(n-1)\,d(A_{n-2,0}+A_{n-1,1}).$$

Hence Theorem 5.1 is a natural generalization of Theorem 3.1.

REFERENCES

[1] S. S. CHERN, Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems, Journ. Math. and Mech., 8(1959), 947-955.
[2] Y. KATSURADA, On a certain property of closed hypersurfaces in an Einstein space, Comment. Math. Helvetici, 38(1964), 165-171.

[3] H. LIEBMANN, Über die Verbiegung der geschlossenen Flächen positiver Krümmung, Math. Ann., 53(1900), 91–112.
[4] W. SÜSS, Zur relativen Differentialgeometrie V, Tôhoku Math. Journ., 30(1929), 202–209.
[5] K. YANO, Closed hypersurfaces with constant mean curvature in a Riemannian manifold, Journ. of the Math. Soc. of Japan, 17(1965), 333–340.

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