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INEQUALITIES FOR WEIGHTED ENTIRE FUNCTIONS

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1. Suppose $h(x)$ is a given function continuous on the whole real axis. Let $W_\sigma^{p,h}(p\geqq 1)$ denote the class of all entire functions $f(z)$ of exponential type σ such that

$$\|f\|_{p,h}^p=\int_{-\infty}^{\infty}|h(x)f(x)|^p dx<\infty.$$

Set

$$U_\sigma^{p,h}=\{f\in W_\sigma^{p,h}\,|\,f(-z)=f(z)\}$$

and

$$V_\sigma^{p,h}=\{f\in W_\sigma^{p,h}\,|\,f(-z)=-f(z)\}$$

Different types of weight functions $f(x)$ have been considered previously. Ibragimov and Mamedhanov [3] have considered the case when $h(x)=\dfrac{1}{\phi(x)}$ where $\phi(x)\geqq 1$ and obtained some interesting inequalities connecting the weighted norms on lines parallel to the real axis for functions belonging to the class $Б_\sigma^{p,h}$ where

$$Б_\sigma^{p,h}=\{f\in W_\sigma^{p,h}\,|\ \ |f(x+iy)|\leqq|f(x-iy)|, y>0\}$$

In a previous paper [7], the author has discussed the case when $h(x)=x^\alpha(\alpha\geqq 0)$ and obtained some results connected with the mean values. It was also proved there that if $\|f\|_{p,h}<\infty$ then $\|f'\|_{p,h}<\infty$ and

$$\|f(x+iy)\|_{p,h}^p=\int_{-\infty}^{\infty}|h(x)f(x+iy)|^p dx$$
$$\leqq e^{p\sigma|y|}\|f\|_{p,h}^p \tag{1}$$

The proofs of these results are given in [8], where other inequalities are also proved.

In what follows $U_\sigma^{p,\alpha}$ will denote the class $U_\sigma^{p,h}$ when $h(x)=x^\alpha$ and $\text{Б}_\sigma^{p,\phi}$ will denote the class $\text{Б}_\sigma^{p,h}$ when $h(x)=\dfrac{1}{\phi(x)}$.

In this paper first we obtain an inequality between the norms $\|f(x+iy)\|_{p,\alpha}$ and $\|f\|_{p,\alpha}$ where $1\leqq p<p'\leqq\infty$ for functions $f\in U_\sigma^{p,\alpha}$.

Further we estimate $\|f'\|_{p,\phi}$ in terms of $\|f\|_{p,\phi}$ for functions $f\in \text{Б}_\sigma^{p,\phi}$.

In the former case we use the representation of functions belonging to the class $U_\sigma^{p,\alpha}$ by means of Hankel transforms obtained by the author [6].

The following theorem of Titchmarsh [5] will be needed.

If $f(x)\in L^p(0,\infty)$, $1<p\leqq 2$, then the integral

$$\int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)f(t)\,dt$$

converges in the mean to a function $F(x)$ such that $F(x)\in L^q(0,\infty)$, $\left(\dfrac{1}{p}+\dfrac{1}{q}=1\right)$ and

$$\left(\int_0^\infty |F(x)|^q dx\right)^{\frac{1}{q}}\leqq A\left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}$$

where $\nu\geqq-\dfrac{1}{2}$, $J_\nu(x)$ is the Bessel function of first kind of order ν, and A is a constant depending on p only with equality for $p=2$ with $A=1$.

A will be referred to as Titchmarsh constant.

2. For functions belonging to the class $U_\sigma^{p,\alpha}$, we have the following

THEOREM 1. *Let* $f(z)\in U_\sigma^{p,\alpha}$, $1\leqq p<\infty$. *If* $p'>p$, *then*

$$\|f\|_{p',\alpha}\leqq\begin{cases}(JA)^{\frac{p'-p}{p'}}\left(\dfrac{\sigma}{2}\right)^{\frac{1}{p}-\frac{1}{p'}}\|f\|_{p,\alpha} & 1\leqq p\leqq 2\\[2ex] \left(\dfrac{(JA)^r\sigma p}{2}\right)^{\frac{1}{p}-\frac{1}{p'}}\|f\|_{p,\alpha} & 2^k<p\leqq 2^{k+1}\end{cases}\tag{1}$$

$$\|f(x+iy)\|_{p',\alpha}\leqq\begin{cases}e^{\sigma|y|}(JA)^{\frac{p'-p}{p'}}\left(\dfrac{\sigma}{2}\right)^{\frac{1}{p}-\frac{1}{p'}}\|f\|_{p,\alpha} & 1\leqq p\leqq 2\\[2ex] e^{\sigma|y|}\left(\dfrac{(JA)^r\sigma p}{2}\right)^{\frac{1}{p}-\frac{1}{p'}}\|f\|_{p,\alpha} & 2^k<p\leqq 2^{k+1}\end{cases}\tag{2}$$

where $r=\dfrac{p}{2^k}$, $J=\max\limits_{-\infty<x<\infty}|x^{\frac{1}{2}}J_\nu(x)|$, $\alpha=\nu+\dfrac{1}{2}$ *and* A *is Titchmarsh constant.*

PROOF. $1<p\leqq 2$. If $f(z)\in U_\alpha^{p,\beta}$, by the representation theorem for entire functions [6], we have

$$f(z)=z^{-\nu}\int_0^\sigma t^{-\nu}J_\nu(zt)\phi(t)dt \tag{2}$$

with

$$\begin{aligned}\left(\int_0^\sigma |t^{-\alpha}\phi(t)|^q dt\right)^{\frac{1}{q}} &\leqq A\left(\int_0^\infty |x^\alpha f(x)|^p dx\right)^{\frac{1}{p}}\\ &=A.2^{-\frac{1}{p}}\|f\|_{p,\alpha}\end{aligned} \tag{3}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Then

$$x^\alpha f(x)=\int_0^\sigma (xt)^{\frac{1}{2}}J_\nu(xt)t^{-\alpha}\phi(t)dt$$

so that, by Hölder's inequality,

$$|x^\alpha f(x)|\leqq\left(\int_0^\sigma |(xt)^{\frac{1}{2}}J_\nu(xt)|^p dx\right)^{\frac{1}{p}}\left(\int_0^\sigma |t^{-\alpha}\phi(t)|^q dt\right)^{\frac{1}{q}}$$

from which it follows that, using (3),

$$\max_{-\infty<x<\infty}|x^\alpha f(x)|\leqq AJ\left(\frac{\sigma}{2}\right)^{\frac{1}{p}}\|f\|_{p,\alpha} \tag{4}$$

Now, when $p'>p$, we have

$$\begin{aligned}\|f\|_{p',\alpha}^{p'}&=\int_{-\infty}^\infty |x^\alpha f(x)|^{p'-p}|x^\alpha f(x)|^p dx\\ &\leqq\left(\max_{-\infty<x<\infty}|x^\alpha f(x)|\right)^{p'-p}\|f\|_{p,\alpha}^p\\ &\leqq(AJ)^{p'-p}\left(\frac{\sigma}{2}\right)^{\frac{p'-p}{p}}\|f\|_{p,\alpha}^{p'}\quad\text{by (4)}\end{aligned}$$

Hence

$$\|f\|_{p',\alpha} \leqq (AJ)^{\frac{p'-p}{p}} \left(\frac{\sigma}{2}\right)^{\frac{1}{p}-\frac{1}{p'}} \|f\|_{p,\alpha}, \tag{5}$$

$2^k < p \leqq 2^{k+1}$. If $r = \frac{p}{2^k}$, the function

$$g(z) = f(z)^{2^k}$$

belongs to the class $U^{r,\alpha 2^k}_{\sigma 2^k}$ and $\|g\|^r_{r,\alpha.2^k} = \|f\|^p_{p,\alpha}$. Since $1 < r \leqq 2$, we apply (5) to $g(z)$ to obtain

$$\max_{-\infty<x<\infty} |x^{\alpha.2^k} g(x)|^r \leqq (JA)^r \frac{\sigma.2^k}{2} \|g\|^r_{r,\alpha.2^k}$$

or

$$\max_{-\infty<x<\infty} |x^\alpha f(x)|^p \leqq (JA)^r \frac{\sigma p}{2} \|f\|^p_{p,\alpha}$$

so that

$$\max_{-\infty<x<\infty} |x^\alpha f(x)| \leqq (JA)^{r/p} \left(\frac{\sigma p}{2}\right)^{\frac{1}{p}} \|f\|_{p,\alpha} \tag{6}$$

When $p' > p$, we have

$$\|f\|^{p'}_{p',\alpha} \leqq \left(\max_{-\infty<x<\infty} |x^\alpha f(x)|\right)^{p-p} \|f\|^p_{p,\alpha}$$

from which we obtain, applying (6),

$$\|f\|_{p',\alpha} \leqq \left(\frac{(AJ)^r \sigma p}{2}\right)^{\frac{1}{p}-\frac{1}{p'}} \|f\|_{p,\alpha} \tag{7}$$

We have yet to consider the case when $p = 1$. We choose p_n such that $1 < p_n \leqq 2$, $p_n > p_{n+1}$, $p_n \to 1$. Then

$$f(z) \in U^{p_n,\alpha}_{r}$$

so that if $p' > p$, then $p' > p_n$ for $n > m$ and

$$\|f\|_{p',\alpha} \leqq (AJ)^{\frac{p'-p_n}{p}} \left(\frac{\sigma}{2}\right)^{\frac{1}{p'}-\frac{1}{p_n}} \|f\|_{p_n,\alpha}$$

from which the result follows by letting $n \to \infty$.

This proves the first part of the theorem.

Now, if $f \in U_\sigma^{p,\alpha}$ then $f \in U_\sigma^{p',\alpha}$ for every $p' > p$. Hence the second part follows from (1) applying the first part of the theorem.

Now, using the inequality for the Bessel function,

$$|J_\nu(z)| \leqq \frac{|(z/2)^\nu|}{\Gamma(\nu+1)} \exp(|\mathrm{Im}\, z|) \tag{8}$$

we obtain from (2), by Hölder's inequality,

$$\begin{aligned} |f(x)| &\leqq \int_0^\sigma \frac{1}{2^\nu \Gamma(\nu+1)} |\phi(t)|\, dt \\ &\leqq \frac{1}{2^\nu \Gamma(\nu+1)} \left(\int_0^\sigma t^{p\alpha} dt\right)^{\frac{1}{p}} \left(\int^\sigma |t^{-\alpha}\phi(t)|^q dt\right)^{\frac{1}{q}} \\ &= \frac{1}{2^\nu \Gamma(\nu+1)} \left(\frac{\sigma^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}} \left(\int_0^\sigma |t^{-\alpha}\phi(t)|^q dt\right)^{\frac{1}{q}} \end{aligned}$$

then instead of (4) and (6), the following theorem can be easily proved.

THEOREM 2. *If $f(z) \in U_\sigma^{p,\alpha}$ $(1 \leqq p < \infty)$, then*

$$\max_{-\infty<x<\infty} |f(x)| \leqq \begin{cases} \left(\dfrac{A}{(2^\nu \Gamma(\nu+1)}\right) \left(\dfrac{\sigma^{p\alpha+1}}{2(p\alpha+1)}\right)^{\frac{1}{p}} \|f\|_{p,\alpha} & 1 \leqq p \leqq 2 \\ \left(\dfrac{A}{2^\nu \Gamma(\nu+1)}\right)^{\frac{r}{p}} \left(\dfrac{\sigma^{p\alpha+1}}{2(p\alpha+1)}\right)^{\frac{1}{p}} \cdot \|f\|_{p,\alpha} & 2^k < p \leqq 2^{k+1} \end{cases}$$

where $r = \frac{p}{2^k}$ and A is Titchmarsh constant.

Analogous results for functions belonging to the Class $W_\sigma^{p,h}$ when $h(x) = 1$ are given by Ibragimov [2].

3. Along with Ibragimav and Mamedhanov we impose the following condition on $\phi(x)$:

$$\alpha_\phi(t)=\sup_{\substack{-\infty<x<\infty\\|y|\leqq t}}\frac{\phi(x+y)}{\phi(x)}\leqq P_m(t)=\sum_{k=0}^{m}A_k t^k$$

where $A_k\geqq 0$ $(k=0,1,2,\cdots,m)$ are the coefficients of the polynomial $P_m(t)$.

Generalizing a result of Boas and Rahman [1], it was then proved for a function $f(z)\in B_\sigma^{p,\phi}$, we have the inequality

$$\|f(x+iy)\|_{p,\phi}\leqq MD_p[(\sigma+m)y]\ \cosh(\sigma+m)y\left(\frac{y}{\sinh y}\right)^m\|f\|_{p,\phi}$$

where

$$D_p(u)=\left\{\frac{1}{2B\left(\frac{p+1}{2},\frac{1}{2}\right)}\int_0^{2\pi}(1-\sin^2\omega\,\mathrm{sech}^2 u)^{p/2}d\omega\right\}^{\frac{1}{p}} \tag{9}$$

and $B(\lambda,\mu)$ is Euler's beta function.
For $p'>p$, it was also shown that

$$\|f(x+iy)\|_{p,\phi}$$

$$\leqq M\left(\frac{s\mu}{\pi}\right)^{\frac{1}{p}-\frac{1}{p'}}\left(\frac{y}{\sinh y}\right)^m[D_{p/s}(s\mu y)\ \cosh(s\mu y)]^{\frac{p'-p}{sp'}}\ [D_p(\mu y)\cosh\mu y]^{p/p}\,\|f\|_{p,\phi}$$

with $\mu=\sigma+m$ and s is given by

$$s=\arg\{\cos(w+i\mu y)\}$$

w being a real parameter.

Then we have the following inequality between the norms of the function and its derivative.

THEOREM 3. *Let $f(z)\in B_\sigma^{p,\phi}$, $1\leqq p<\infty$. Then*

$$\|f'\|_{p,\phi}\leqq M^2\left(\frac{2^{p+2}(p+2)}{\pi}\right)^{\frac{1}{p}}\frac{\cosh(\sigma+m)\delta}{\delta}D_p(\sigma+m)\delta\|f\|_{p,\phi}$$

and for $p'>p$

$\|f'\|_{p',\phi}$

$$\leqq M^2\left(\frac{s\mu}{\pi}\right)^{\frac{1}{p}-\frac{1}{p}}\left[D_{p/s}(s\mu\delta)\cosh(s\mu\delta)\right]^{\frac{p'-p}{sp'}}\frac{(D_p(\mu\delta)\cosh(\mu\delta))^{p/p'}}{\delta}\left(\frac{2^{p+2}(p'+2)}{\pi}\right)^{\frac{1}{p}}\|f\|_{p,\phi}$$

where δ *is arbitrary and*

$$M=\sup_{\substack{-\infty<x<\infty\\|t|\leqq\delta}}\frac{\phi(x+t)}{\phi(x)}$$

PROOF. Plancherel and Pólya [4, p.127] proved that if $f(z)$ is regular in a square with corners $x+\delta\pm i\delta$, $x-\delta\pm i\delta$, then

$$|f'(x)|^p\leqq P\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}|f(x+u+it)^p dudt$$

where δ is an arbitrary positive number and

$$P=\frac{2^p(p+2)}{\pi\delta^{p+2}}.$$

Then

$$\left|\frac{f'(x)}{\phi(x)}\right|^p\leqq P\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\left|\frac{f(x+u+it)}{\phi(x+u)}\right|^p\left|\frac{\phi(x+u)}{\phi(x)}\right|^p dudt$$

$$\leqq M^pP\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\left|\frac{f(x+u+it)}{\phi(x+u)}\right|^p dudt,$$

so that

$$\|f'\|_{p,\phi}^p\leqq M^pP\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\left\{\int_{-\infty}^{\infty}\left|\frac{f(x+u+it)}{\phi(x+u)}\right|^p dx\right\}dudt \tag{13}$$

$$=M^pP\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\{\|f(x+it)\|_{p,\phi}^p\}\,dudt.$$

Using the fact that $D_p[(\sigma+m)y]\cosh[(\sigma+m)y]\leqq D_p[(\sigma+m)\delta]\cosh[(\sigma+m)\delta]$ for $|y|\leqq\delta$, it follows by applying (8) to the above inequality, that

$$\|f'\|_{p,\phi}^{p}$$

$$\leqq M^{p}P\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\left\{M^{p}D_{p}^{p}[(\sigma+m)t]\cosh^{p}[(\sigma+m)t]\left(\frac{t}{\sin t}\right)^{pm}\|f\|_{p,\phi}^{p}\right\}dudt$$

$$\leqq M^{2p}PD_{p}^{p}[(\sigma+m)\delta]\cosh^{p}[(\sigma+m)\delta]\|f\|_{p,\phi}^{p}\,4\delta^{2}.$$

Hence

$$\|f'\|_{p,\phi}\leqq M^{2}\left(\frac{2^{p+2}(p+2)}{\pi}\right)^{\frac{1}{p}}D_{p}[(\sigma+m)\delta]\cosh[(\sigma+m)\delta]\|f\|_{p,\phi}.$$

(12) is proved analogously from (13) with p' instead of p and applying (10).
When $\phi(x)\equiv 1$, we have $m=0$ and $M=1$. We deduce from Theorem 3.

THEOREM 4. *If* $f(z)\in B_{\sigma}^{p,\phi}$, *then*

$$\|f'\|_{p}\leqq\left(\frac{2^{p+2}(p+2)}{\pi}\right)^{\frac{1}{p}}\frac{D_{p}(\sigma\delta)\cosh(\sigma\delta)}{\delta}\|f\|_{p}$$

and

$$\|f\|_{p'}\leqq\left(\frac{s\sigma}{\pi}\right)^{\frac{1}{p}-\frac{1}{p'}}D_{p/s}(s\sigma\delta)\cosh(s\sigma\delta)^{\frac{p'-p}{sp'}}$$

$$\frac{(D_{p}(\sigma\delta)\cosh(\sigma\delta))^{p/p'}}{\delta}\left(\frac{2^{p'+2}(p+2)}{\pi}\right)^{\frac{1}{p}}\|f\|_{p}.$$

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