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ON KITAGAWA'S FUNCTIONAL INTEGRAL

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The purposes of this note are to show that the measure underlying T. Kitagawa's functional integral is the measure induced by a Gaussian process, and that furthermore this process is an extension of the Brownian process into 2-dimensional parameter space.

T. Kitagawa defined functional integration [1] in the space C_2 of real valued continuous functions $x(t,\tau)$ on the unit square $0 \leqq t,\ \tau \leqq 1$ satisfying $x(0,\tau) = x(t,0) = 0$, and for real valued functionals of the type $H[x(t_1,\tau_1), \cdots, x(t_r,\tau_s)]$ where $\{t_h\}$, $\{\tau_k\}$ are preassigned division points satisfying $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_r \leqq t_{r+1} = 1$, $0 = \tau_0 \leqq \tau_1 \leqq \cdots \leqq \tau_s \leqq \tau_{s+1} = 1$ to be

$$(1) \qquad \int_{C_2}^{w} H[x(t_1,\tau_1), \cdots, x(t_r,\tau_s)]\, d_w x = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H[\eta_{11}, \cdots, \eta_{rs}] \prod_{h=1}^{r} \prod_{k=1}^{s} P(\Delta_{hk})\, d\eta_{11} \cdots d\eta_{rs}$$

where

$$(2) \qquad P(\Delta_{hk}) = [\pi(t_h - t_{h-1})(\tau_k - \tau_{k-1})]^{-\frac{1}{2}} \exp\left\{ -\frac{(\eta_{hk} - \eta_{h,k-1} - \eta_{h-1,k} + \eta_{h-1,k-1})^2}{(t_h - t_{h-1})(\tau_k - \tau_{k-1})} \right\}.$$

J. Yeh proved [3] that the family of distributions

$$(3) \qquad F_{(t_1,\tau_1)\cdots(t_r,\tau_s)}(\alpha_{11}, \cdots, \alpha_{rs}) = \int_{-\infty}^{\alpha_{rs}} \cdots \int_{-\infty}^{\alpha_{11}} \prod_{h=1}^{r} \prod_{k=1}^{s} P(\Delta_{hk})\, d\eta_{11} \cdots d\eta_{rs}$$

obtained from the Kitagawa functional integral can be extended to a measure w that is defined on the algebra of Borel cylinders on the space C_2.

In Theorem 1, we shall show that the distribution (3) is the joint distribution of the random vector $\{\xi_{(t_1,\tau_1)} \cdots \xi_{(t_r,\tau_s)}\}$ obtained from the Gaussian process $\{\xi_{t,\tau} : 0 \leqq t \leqq 1, 0 \leqq \tau \leqq 1\}$ with mean values $E(\xi_{t,\tau}) = 0$ and covariance $E(\xi_{t_h,\tau_k} \xi_{t_p,\tau_q}) = \dfrac{1}{2} \min(t_h, t_p) \min(\tau_k, \tau_q)$. Theorem 2 gives further properties of this process.

THEOREM 1. *Let* $\{\xi_{t,\tau}: 0 \leqq t \leqq 1,\ 0 \leqq \tau \leqq 1\}$ *be the Gaussian process with mean values* $E(\xi_{t,\tau}) = 0$ *and covariance* $E(\xi_{t_h,\tau_k}\xi_{t_p,\tau_q}) = \left(\frac{1}{2}\right)\min(t_h, t_p)$ $\min(\tau_k, \tau_q)$. *Let* $\{t_h\}$ *and* $\{\tau_k\}$ *be points satisfying* $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_r \leqq t_{r+1} = 1$, $0 = \tau_0 \leqq \tau_1 \leqq \cdots \leqq \tau_s \leqq \tau_{s+1} = 1$. *Then*

$$(4)\quad P\{\xi_{t_1,\tau_1} < \alpha_{11}, \cdots, \xi_{t_r,\tau_s} < \alpha_{rs}\} = \int_{-\infty}^{\alpha_{rs}} \cdots \int_{-\infty}^{\alpha_{11}} \prod_{h=1}^{r}\prod_{k=1}^{s} P(\Delta_{hk})\, d\eta_{11} \cdots d\eta_{rs}$$

where

$$(5)\quad P(\Delta_{hk}) = [\pi(t_h - t_{h-1})(\tau_k - \tau_{k-1})]^{-\frac{1}{2}} \exp\left\{-\frac{(\eta_{hk} - \eta_{h,k-1} - \eta_{h-1,k} + \eta_{h-1,k-1})^2}{(t_h - t_{h-1})(\tau_k - \tau_{k-1})}\right\}$$

PROOF. Write $\xi_{h,k} = \xi_{t_h,\tau_k}$ and consider the random variables

$$\zeta_{h,k} = \xi_{h,k} - \xi_{h,k-1} - \xi_{h-1,k} + \xi_{h-1,k-1}$$

Observe that the following statements (i), (ii) and (iii) hold.

(i) $E(\zeta_{h,k}) = E(\xi_{h,k} - \xi_{h,k-1} - \xi_{h-1,k} + \xi_{h-1,k-1}) = 0$

$$\begin{aligned}(\text{ii})\quad E(\zeta_{h,k}\zeta_{h,k}) &= E(\xi_{h,k}^2 + \xi_{h,k-1}^2 + \xi_{h-1,k}^2 + \xi_{h-1,k-1}^2 - 2\xi_{h,k}\xi_{h,k-1} - 2\xi_{h,k}\xi_{h-1,k} \\ &\quad + 2\xi_{h,k}\xi_{h-1,k-1} + 2\xi_{h,k-1}\xi_{h-1,k-1} - 2\xi_{h,k-1}\xi_{h-1,k-1} - 2\xi_{h-1,k}\xi_{h-1,k-1}) \\ &= \frac{1}{2}(t_h\tau_k + t_h\tau_{k-1} + t_{h-1}\tau_k + t_{h-1}\tau_{k-1} - 2t_h\tau_{k-1} - 2t_{h-1}\tau_k \\ &\quad + 2t_{h-1}\tau_{k-1} + 2t_{h-1}\tau_{k-1} - 2t_{h-1}\tau_{k-1} - 2t_{h-1}\tau_{k-1}) \\ &= \frac{1}{2}(t_h\tau_k - t_h\tau_{k-1} - t_{h-1}\tau_k + t_{h-1}\tau_{k-1}) \\ &= \frac{1}{2}(t_h - t_{h-1})(\tau_k - \tau_{k-1})\end{aligned}$$

(iii) $E(\zeta_{h,k}\zeta_{p,q}) = 0$ when $(h, k) \neq (p, q)$.

To prove (iii) consider each of the cases $(p = h,\ q < k)$, $(p < h,\ q > k)$, $(p > h,\ q < k)$, $(p < h,\ q = k)$, $(p < h,\ q < k)$, $(p = h,\ q > k)$, $(p > h,\ q = k)$, $(p > h,\ q > k)$, sepearately. The desired answer is a simple consequence of the direct application of the formula $E(\xi_{h,k}\xi_{p,q}) = \frac{1}{2}\min(t_h, t_p)\min(\tau_k, \tau_q)$ to the term on the right of $E(\zeta_{h,k}\zeta_{p,q}) = E\{(\xi_{h,k} - \xi_{h-1,k} - \xi_{h,k-1} + \xi_{h-1,k-1}) \cdot (\xi_{p,q} - \xi_{p-1,q} - \xi_{p,q-1} + \xi_{p-1,q-1})\}$.

Hence the random variables $\zeta_{h,k}$ are independent Gaussian random variables with mean 0 and variance $\sigma_{h,k}^2 = \frac{1}{2}(t_h - t_{h-1})(\tau_k - \tau_{k-1})$. (Because $\zeta_{h,k}$, as linear combinations of Gaussian random variables, are themselves Gaussian and zero covariance implies independence.)

Thus we may write

$$
\begin{aligned}
&P\{\xi_{h,k} < \alpha_{hk}\,;\ h = 1,\cdots r\,; k = 1,\cdots, s\} \\
&= P\{\xi_{h,k} - \xi_{h-1,k} - \xi_{h,k-1} + \xi_{h-1,k-1} < \alpha_{hk} - \xi_{h-1,k} - \xi_{h,k-1} + \xi_{h-1,k-1}\,; \\
&\qquad\qquad h = 1\,,\cdots,r\,;\ k = 1,\cdots,s\} \\
&= \int_{-\infty}^{\alpha_{rs}-\eta_{r-1,s}-\eta_{r,s-1}+\eta_{r-1,s-1}} \cdots \int_{-\infty}^{\alpha_{11}} \prod_{h=1}^{r}\prod_{k=1}^{s} P(\Delta_{hk})\, d\eta_{11}\cdots d(\eta_{rs}-\eta_{r-1,s}-\eta_{r,s-1}+\eta_{r-1,s-1}) \\
&= \int_{-\infty}^{\alpha_{rs}} \cdots \int_{-\infty}^{\alpha_{11}} \prod_{h=1}^{r}\prod_{k=1}^{s} P(\Delta_{hk})\, d\eta_{11}\cdots d\eta_{rs}
\end{aligned}
$$

THEOREM 2. *The Gaussian process of Theorem* 1 *has the following properties.*

(a) *Almost all sample functions are Lipschitz-β continuous for* $0 < \beta < \frac{1}{2}$.

(b) *Along any fixed coordinate, say,* t = *constant or* τ = *constant, the process is Brownian motion in* 1-*dimensional parameter space.*

PROOF. For a proof of property (a) refer to Lemma 1 which is stated below. For our purpose it suffices to show that $E(|\xi_{t,\tau} - \xi_{s,\sigma}|^2) \leqq K\sqrt{(t-s)^2 + (\tau-\sigma)^2}$ for some constant K. Thus observe that

$$
\begin{aligned}
E(|\xi_{t,\tau} - \xi_{s,\sigma}|^2) &= E(\xi_{t,\tau}^2) - 2E(\xi_{t,\tau}\xi_{s,\sigma}) + E(\xi_{s,\sigma}^2) \\
&\leqq |E(\xi_{t,\tau}^2) - E(\xi_{t,\tau}\xi_{s,\sigma})| + |E(\xi_{s,\sigma}^2) - E(\xi_{t,\tau}\xi_{s,\sigma})| \\
&= \frac{1}{2}|t\tau - \min(t,s)\min(\tau,\sigma)| + \frac{1}{2}|s\sigma - \min(t,s)\min(\tau,\sigma)| \\
&< |t-s| + |\tau-\sigma|\,.
\end{aligned}
$$

The last inequality is obtained by substituting all possible values of $\min(t,s)$ $\min(\tau,\sigma)$ into term on the left of that inequality. Hence it follows that

$$
E(|\xi_{t,\tau} - \xi_{s,\sigma}|^2) \leqq \sqrt{2}\sqrt{(t-s)^2 + (\tau-\sigma)^2}\,.
$$

Property (b) follows immediately upon observation that the covariance function of Theorem 1 reduces to the covariance function of the Brownian process in 1-dimensional parameter space upon substituting a constant for either the variable t or the variable τ.

LEMMA 1. *Let* $\{\xi_t\,; t \in R^N\}$ *be a Gaussian process,* $E(\xi_t) = 0$, *and* T *a compact subset of* R^N *(N-dimensional Euclidean space). If there are constants* $\alpha > 0$ *and* K *such that*

$$E(|\xi_t - \xi_s|^2) \leqq K\|t-s\|^\alpha \tag{6}$$

for t, s *in* R^N, *then almost all sample functions of the process are Lipschitz-*β *continuous in* T *for* $0 < \beta < \alpha/2$.

A statement and proof of the lemma is found in [2]. It should be noted here that in [2] the term "almost all" is used in a special sense but that it takes on the usual meaning if we assume the process in question is separable and is separated by the subset D of the parameter space consisting of all dyadic coordinates. Thus, in order to avoid unnecessary complications we should assume throughout this note the process $\{\xi_{t,\tau} : 0 \leqq t, \tau \leqq 1\}$ is separable and is separated by the set $D = \{(t,\tau) : 0 \leqq t, \tau \leqq 1$, and both t and τ are dyadic numbers$\}$. $\|\cdot\|$ denotes the Euclidean norm.

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