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# ON AUTOMORPHISM GROUPS OF SOME CONTACT RIEMANNIAN MANIFOLDS

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**1. Introduction.** Recently Dr. S. Tanno offered the author a conjecture that the dimension of the automorphism group of a contact Riemannian manifold of dimension $2n+1$ will not exceed $(n+1)^2$. It has been proved that the unit sphere $S^{2n+1}$ with the standard Sasakian structure has the automorphism group of dimension $(n+1)^2$. ([3]). In this paper, we shall examine the conjecture for some kinds of contact Riemannian manifolds. Especially, we shall prove that the dimension of the automorphism group of Euclidean space $E^{2n+1}$ with its standard contact metric structure is just $(n+1)^2$.

The author wishes to express his gratitude to Dr. S. Tanno for his suggestions.

**2. Expressions in adapted local coordinate systems.** A differentiable manifold $M$ of dimension $2n+1$ is called a contact Riemannian manifold, [2], if it admits a vector field $\xi$, a 1-form $\eta$, a 1-1 tensor field $\Phi$ and a Riemannian metric $G$ such that

$$\eta(\xi) = 1, \tag{2.1}$$

$$\Phi^2 + 1 = \xi \otimes \eta, \tag{2.2}$$

$$G(\xi, X) = \eta(X), \tag{2.3}$$

$$G(\Phi X, \Phi Y) = G(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$G(X, \Phi Y) = d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta[X, Y]\}, \tag{2.5}$$

where $X$ and $Y$ are vector fields.

Let $M$ be a contact Riemannian manifold of dimension $2n+1$ with $\xi$, $\eta$, $\Phi$ and $G$. A vector field $X$ on $M$ is called an infinitesimal automorphism if

$$L_X\eta = 0 \tag{2.6}$$

and

$$L_X G = 0, \tag{2.7}$$

where $L_X$ is the Lie derivative with respect to $X$. (2.6) indicates that $X$ is an infinitesimal strict contact transformation and (2.7) indicates that $X$ is a Killing vector field.

Let $P$ be a point of $M$. As is well known, we can take a coordinate system $x^1, \cdots, x^n, y^1, \cdots, y^n, z$, called an adapted coordinate system such that

$$\eta = dz - \Sigma y^\alpha dx^\alpha \tag{2.8}$$

and

$$x^1(p) = \cdots = x^n(p) = y^1(p) = \cdots = y^n(p) = z(p) = 0.$$

We express above (2.1). (2.7) by means of this coordinate system. For convenience, we write $\partial_\alpha$, $\partial_{\alpha^*}$, $\partial_\Delta$ for $\dfrac{\partial}{\partial x^\alpha}, \dfrac{\partial}{\partial y^\alpha}, \dfrac{\partial}{\partial z}$ and indices vary from 1 to $n$. At first by direct calculations we can see

$$\xi = \partial_\Delta. \tag{2.9}$$

Put

$$\begin{aligned} \Phi(\partial_\alpha) &= \phi_\alpha^\beta \partial_\beta + \phi_\alpha^{\beta^*} \partial_{\beta^*} + \phi_\alpha^\Delta \partial_\Delta, \\ \Phi(\partial_{\alpha^*}) &= \phi_{\alpha^*}^\beta \partial_\beta + \phi_{\alpha^*}^{\beta^*} \partial_{\beta^*} + \phi_{\alpha^*}^\Delta \partial_\Delta, \\ \Phi(\partial_\Delta) &= 0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} G(\partial_\alpha, \partial_\beta) &= g_{\alpha\beta}, & G(\partial_\alpha, \partial_{\beta^*}) &= g_{\alpha\beta^*}, \\ G(\partial_{\alpha^*}, \partial_{\beta^*}) &= g_{\alpha^*\beta^*}, & G(\partial_\alpha, \partial_\Delta) &= g_{\alpha\Delta}, \\ G(\partial_{\alpha^*}, \partial_\Delta) &= g_{\alpha^*\Delta}, & G(\partial_\Delta, \partial_\Delta) &= g_{\Delta\Delta}. \end{aligned} \tag{2.11}$$

Then (2.2) is equivalent to

$$\begin{aligned} \phi_\alpha^\gamma \phi_\gamma^\beta + \phi_\alpha^{\gamma^*} \phi_{\gamma^*}^\beta &= -\delta_\alpha^\beta \qquad \text{Kronecker's delta}, \\ \phi_\alpha^\gamma \phi_\gamma^{\beta^*} + \phi_\alpha^{\gamma^*} \phi_{\gamma^*}^{\beta^*} &= 0, \\ \phi_{\alpha^*}^\gamma \phi_\gamma^\beta + \phi_{\alpha^*}^{\gamma^*} \phi_{\gamma^*}^\beta &= 0, \\ \phi_{\alpha^*}^\gamma \phi_\gamma^{\beta^*} + \phi_{\alpha^*}^{\gamma^*} \phi_{\gamma^*}^{\beta^*} &= -\delta_\alpha^\beta, \end{aligned} \tag{2.12}$$

$$\phi^{\gamma}_{\alpha}\phi^{\Delta}_{\gamma} + \phi^{\gamma^*}_{\alpha}\phi^{\Delta}_{\gamma^*} = -y^{\alpha},$$
$$\phi^{\gamma}_{\alpha^*}\phi^{\Delta}_{\gamma} + \phi^{\gamma^*}_{\alpha^*}\phi^{\Delta}_{\gamma^*} = 0$$

and (2. 3), (2. 5) are equivalent to

$$\begin{aligned}
&g_{\alpha\beta} = -\frac{1}{2}\phi^{\beta^*}_{\alpha} + y^{\alpha}y^{\beta}, \quad \phi^{\beta^*}_{\alpha} = \phi^{\alpha^*}_{\beta}, \\
&g_{\alpha\beta^*} = \frac{1}{2}\phi^{\beta}_{\alpha} = -\frac{1}{2}\phi^{\alpha^*}_{\beta^*}, \\
&g_{\alpha^*\beta^*} = \frac{1}{2}\phi^{\beta}_{\alpha^*}, \quad \phi^{\beta}_{\alpha^*} = \phi^{\alpha}_{\beta^*}, \\
&g_{\alpha\Delta} = -y^{\alpha}, \\
&g_{\alpha^*\Delta} = 0, \\
&g_{\Delta\Delta} = 1.
\end{aligned} \tag{2. 13}$$

Thus

$$G = \begin{pmatrix} -\frac{1}{2}\phi^{\beta^*}_{\alpha} + y^{\alpha}y^{\beta} & \frac{1}{2}\phi^{\beta}_{\alpha} & -y^{\alpha} \\ \frac{1}{2}\phi^{\alpha}_{\beta} & \frac{1}{2}\phi^{\beta}_{\alpha^*} & 0 \\ -y^{\beta} & 0 & 1 \end{pmatrix}.$$

Let $X = a^{\alpha}\partial_{\alpha} + a^{\alpha^*}\partial_{\alpha^*} + a^{\Delta}\partial_{\Delta}$ be a vector field. The condition $L_x\eta = 0$ is expressed as follows:

$$\begin{aligned}
&y^{\gamma}\partial_{\alpha}a^{\gamma} - \partial_{\alpha}a^{\Delta} + a^{\alpha^*} = 0, \\
&y^{\gamma}\partial_{\alpha^*}a^{\gamma} - \partial_{\alpha^*}a^{\Delta} = 0, \\
&y^{\gamma}\partial_{\Delta}a^{\gamma} - \partial_{\Delta}a^{\Delta} = 0.
\end{aligned} \tag{2. 14}$$

Differentiating (2. 14), we obtain

$$\begin{aligned}
&\partial_{\Delta}a^{\alpha} = 0, \quad \partial_{\Delta}a^{\alpha^*} = 0, \quad \partial_{\Delta}a^{\Delta} = 0 \\
&\partial_{\alpha}a^{\beta^*} = \partial_{\beta}a^{\alpha^*}, \\
&\partial_{\alpha^*}a^{\beta} = \partial_{\beta^*}a^{\alpha}, \\
&\partial_{\alpha}a^{\beta} + \partial_{\beta^*}a^{\alpha^*} = 0.
\end{aligned} \tag{2. 15}$$

(First three equations indicate that coefficients of $X$ do not depend on $z$.)

The condition $L_XG = 0$ is expressed as follows:

$$
\begin{aligned}
&-\frac{1}{2} X\phi_\alpha^{\beta^*} + a^{\alpha^*}y^\beta + a^{\beta^*}y^\alpha = \frac{1}{2}\phi_\alpha^{\beta^*}\partial_\alpha a^r - \frac{1}{2}\phi_\beta^r\partial_\alpha a^{r^*} \\
&+\frac{1}{2}\phi_r^{\alpha^*}\partial_\beta a^r - \frac{1}{2}\phi_\alpha^r\partial_\beta a^{r^*} - y^r y^\beta\partial_\alpha a^r + y^\beta\partial_\alpha a^\Delta \\
&-y^r y^\alpha\partial_\beta a^r + y^\alpha\partial_\beta a^\Delta, \\
&\frac{1}{2} X\phi_\alpha^\beta = -\frac{1}{2}\phi_r^\beta\partial_\alpha a^r - \frac{1}{2}\phi_{r^*}^\beta\partial_\alpha a^{r^*} + \frac{1}{2}\phi_\alpha^{r^*}\partial_{\beta^*}a^r \\
&\qquad -\frac{1}{2}\phi_\alpha^r\partial_{\beta^*}a^{r^*} - y^\alpha(y^r\partial_{\beta^*}a^r - \partial_{\beta^*}a^\Delta), \\
&\frac{1}{2} X\phi_{\alpha^*}^\beta = -\frac{1}{2}\phi_r^\beta\partial_{\alpha^*}a^r - \frac{1}{2}\phi_{r^*}^\beta\partial_{\alpha^*}a^{r^*} - \frac{1}{2}\phi_r^\alpha\partial_{\beta^*}a^r \\
&\qquad -\frac{1}{2}\phi_{r^*}^\alpha\partial_{\beta^*}a^{r^*}, \\
&-a^{\alpha^*} = y^r\partial_\alpha a^r - \partial_\alpha a^\Delta + \frac{1}{2}\phi_r^{\alpha^*}\partial_\Delta a^r - \frac{1}{2}\phi_\alpha^r\partial_\Delta a^{r^*} \\
&\qquad - y^\alpha(y^r\partial_\Delta a^r - \partial_\Delta a^\Delta), \\
&0 = y^r\partial_{\alpha^*}a^r - \partial_{\alpha^*}a^\Delta - \frac{1}{2}\phi_r^\alpha\partial_\Delta a^{r^*} - \frac{1}{2}\phi_{r^*}^\alpha\partial_\Delta a^{r^*}, \\
&y^r\partial_\Delta a^r = \partial_\Delta a^\Delta .
\end{aligned}
\tag{2.16}
$$

Connecting (2.14), (2.15) and (2.16), we conclude that the conditions that an infinitesimal strict contact transformation $X$ is to be an infinitesimal automorphism are

$$
\begin{aligned}
X\phi_\alpha^{\beta^*} &= -\phi_r^{\beta^*}\partial_\alpha a^r + \phi_\beta^r\partial_\alpha a^{r^*} - \phi_r^{\alpha^*}\partial_\beta a^r + \phi_\alpha^r\partial_\beta a^{r^*}, \\
X\phi_\alpha^\beta &= -\phi_r^\beta\partial_\alpha a^r - \phi_{r^*}^\beta\partial_\alpha a^{r^*} + \phi_\alpha^{r^*}\partial_{\beta^*}a^r - \phi_\alpha^r\partial_{\beta^*}a^{r^*}, \\
X\phi_{\alpha^*}^\beta &= -\phi_r^\beta\partial_{\alpha^*}a^r - \phi_{r^*}^\beta\partial_{\alpha^*}a^{r^*} - \phi_r^\alpha\partial_{\beta^*}a^r - \phi_{r^*}^\alpha\partial_{\beta^*}a^{r^*}.
\end{aligned}
\tag{2.17}
$$

By Libermann [1], there is a one-to-one correspondence between infinitesimal strict contact transformations and differentiable functions of $x^1, \cdots, x^n, y^1, \cdots, y^n$. This correspondence is given by

$$
\begin{aligned}
f &\longrightarrow X_f = (\partial_{\alpha^*}f)\partial_\alpha - (\partial_\alpha f)\partial_{\alpha^*} + (y^\alpha\partial_{\alpha^*}f - f)\partial_\Delta, \\
X &\longrightarrow -\eta(X) = \Sigma a^\alpha y^\alpha - a^\Delta .
\end{aligned}
$$

Substituting $X_f$ for $X$ in (2.17), we obtain

$$
\begin{aligned}
&X_f\phi_\alpha^{\beta^*} = (-\phi_r^{\beta^*}\partial_\alpha\partial_{r^*} - \phi_\beta^r\partial_\alpha\partial_r - \phi_r^{\alpha^*}\partial_\beta\partial_{r^*} - \phi_\alpha^r\partial_\beta\partial_r)f,\\
(2.18)\qquad &X_f\phi_\alpha^{\beta} = (-\phi_r^{\beta}\partial_\alpha\partial_{r^*} + \phi_{r^*}^{\beta}\partial_\alpha\partial_r + \phi_\alpha^{r^*}\partial_{\beta^*}\partial_{r^*} + \phi_\alpha^r\partial_{\beta^*}\partial_r)f,\\
&X_f\phi_{\alpha^*}^{\beta} = (-\phi_r^{\beta}\partial_{\alpha^*}\partial_{r^*} + \phi_{r^*}^{\beta}\partial_{\alpha^*}\partial_r - \phi_r^{\alpha}\partial_{\beta^*}\partial_{r^*} + \phi_{r^*}^{\alpha}\partial_{\beta^*}\partial_r)f.
\end{aligned}
$$

3. **Calculations of dimensions**. We impose on $\Phi$ following conditions for the convenience of calculations. Let $v$ be an everywhere non-zero function of $x^1, \cdots, x^n, y^1, \cdots, y^n, z$. Let

$$
\begin{aligned}
(3.1)\qquad &\phi_\alpha^{\beta^*} = -\delta_{\alpha\beta}v,\quad \phi_{\alpha^*}^{\beta} = \delta_{\alpha\beta}\frac{1}{v},\quad \phi_\alpha^\beta = 0,\\
&\phi_{\alpha^*}^{\beta^*} = 0,\qquad \phi_{\alpha^*}^{\Delta} = \frac{1}{v}y^\alpha,\quad \phi_\alpha^{\Delta} = 0,
\end{aligned}
$$

and so

$$
G = \begin{pmatrix} \frac{\delta_{\alpha\beta}}{2}v + y^\alpha y^\beta & 0 & -y^\alpha \\ 0 & \frac{\delta_{\alpha\beta}}{2v} & 0 \\ -y^\beta & 0 & 1 \end{pmatrix}.
$$

By this (2,12) still holds and hence we have a contact Riemannian structure.

Remark 1. The case $v \equiv 1$ is just the standard contact Riemannian structure of $E^{2n+1}$.

Now (2.18), in this case, reduces to

$$
\begin{aligned}
(3.2)\qquad -\delta_{\alpha\beta}X_f v &= v(\partial_\alpha\partial_{\beta^*} + \partial_\beta\partial_{\alpha^*})f,\\
0 &= (-\partial_\alpha\partial_\beta + v^2\partial_{\alpha^*}\partial_{\beta^*})f.
\end{aligned}
$$

CASE $1^0$. When $v$ is non-zero constant. In this case, (3.2) is

$$\partial_\alpha \partial_{\beta^*} f = -\partial_\beta \partial_{\alpha^*} f, \qquad (3.3)$$
$$\partial_\alpha \partial_\beta f = v^2 \partial_{\alpha^*} \partial_{\beta^*} f.$$

Differentiating the last equation and using the first equation,

$$\partial_\alpha \partial_\beta \partial_\gamma f = v^2 \partial_\alpha \partial_{\beta^*} \partial_{\gamma^*} f = -v^2 \partial_\beta \partial_{\alpha^*} \partial_{\gamma^*} f = -\partial_\alpha \partial_\beta \partial_\gamma f.$$

Hence $\partial_\alpha \partial_\beta \partial_\gamma f = \partial_\alpha \partial_{\beta^*} \partial_{\gamma^*} f = 0$. In the same way, $\partial_{\alpha^*} \partial_{\beta^*} \partial_{\gamma^*} f = \partial_{\alpha^*} \partial_\beta \partial_\gamma f = 0$. Thus, if we expand $f$ in formal power series, terms of degree more than two vanish. On the other hand, if we put

$$f = f_0 + f_\alpha x^\alpha + f_{\alpha^*} y^\alpha + \frac{1}{2} f_{\alpha\beta} x^\alpha x^\beta + f_{\alpha\beta^*} x^\alpha y^\beta + \frac{1}{2} f_{\alpha^*\beta^*} y^\alpha y^\beta,$$
$$f_{\alpha\beta} = f_{\beta\alpha} \text{ and } f_{\alpha^*\beta^*} = f_{\beta^*\alpha^*},$$

then by (3.3), $f_{\alpha\beta^*} = -f_{\beta\alpha^*}$, $f_{\alpha\beta} = v^2 f_{\alpha^*\beta^*}$. As $f_0$, $f_\alpha$, $f_{\alpha^*}$ are arbitrary constants, the dimension of the space of such $f$'s is

$$1 + 2n + \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = (n+1)^2.$$

Thus,

THEOREM 1. *The dimension of the automorphism group of Euclidean space $E^{2n+1}$ with a contact Riemannian structure such that $\Phi$ is given by* (3.1), *where $v$ is non-zero constant, is* $(n+1)^2$.

CASE 2°. When $v$ and $\xi v$ are everywhere non-zero functions. Differentiating the last equation of (3.2) by $\partial = \xi$, we obtain

$$0 = 2v\partial\ v\partial_{\alpha^*}\partial_{\beta^*} f.$$

By assumption $v \neq 0$ and $\partial\ v \neq 0$, $\partial_{\alpha^*}\partial_{\beta^*} f = 0$ and so $\partial_\alpha \partial_\beta f = 0$. These indicate that if we expand $f$ in formal power series, terms including $x^\alpha x^\beta$ or $y^\alpha y^\beta$ vanish. On the other hand, if we put

$$f = f_0 + f_\alpha x^\alpha + f_{\alpha^*} y^\alpha + f_{\alpha\beta^*} x^\alpha y^\beta,$$

then

$$f_{\alpha\beta^*} = -f_{\beta\alpha^*} \quad \text{if } \alpha \neq \beta,$$

(3. 4) $$2vf_{ss^*} = -(f_{\alpha^*} + f_{r\alpha^*}x^r)\partial_\alpha v + (f_\alpha + f_{\alpha r^*}y^r)\partial_{\alpha^*}v + (f_0 + f_\alpha x^\alpha)\partial_\Delta v, \qquad s = 1, \cdots, n,$$

are the first equation of (3.2).

If we expand $v$ in formal power series

$$v = v_0 + v_{1\Delta}z + v_\alpha x^\alpha + v_{\alpha^*}y^\alpha + \frac{1}{2}v_{2\Delta}z^2 + \cdots,$$

then by above last condition,

$$2v_0 f_{ss^*} = -f_{\alpha^*}v_\alpha + f_\alpha v_{\alpha^*} + f_0 v_{1\Delta} \qquad s = 1, \cdots, n.$$

But $v_0 \neq 0$, $f_{ss^*} = \frac{1}{2v_0}(-f_{\alpha^*}v_\alpha + f_{\alpha\alpha^*} + f_0 v_{1\Delta})$, $s = 1, \cdots, n$. Thus the dimension of the space of such $f's$ does not exceed $1 + 2n + \frac{n(n-1)}{2} = \frac{(n+1)(n+2)}{2}$ *which is less than* $(n+1)^2$.

THEOREM 2. *The dimension of the automorphism group of Euclidean space $E^{2n+1}$ with a contact Riemannian structure such that $\Phi$ is given by* (3.1), *where $v$ and $\xi v$ are everywhere non-zero analytic functions, does not exceed* $\frac{(n+1)(n+2)}{2}$.

In the case when $v$ does not depend on $x^1, \cdots, x^n, y^1, \cdots y^n$, but depends only on $z$ and $\partial_\Delta v \neq 0$, the dimension is determined definitely. For, in this case, (3. 4) reduce to

$$f_{\alpha\beta^*} = -f_{\beta\alpha^*}, \qquad \alpha \neq \beta,$$
$$2vf_{ss^*} = f_0 \partial_\Delta v + f_\alpha x^\alpha \partial_\Delta v, \qquad s = 1, \cdots, n.$$

Differentiating the last equation by $x^\alpha$, $f_\alpha = 0$ and so $2vf_{ss^*} = \partial_\Delta v f_0$. If $f_{ss^*} \neq 0$, then $\frac{\partial_\Delta v}{2v}$ must be a constant. Hence,

$v = Ae^{BZ}$ where $A$ and $B$ are non-zero constants.

If $v$ is not of this form, then $f_{ss^*} = 0$ and $f_0 = 0$,

$$f = f_0 + f_{\alpha^*}y^\alpha + \sum_{\alpha \neq \beta} f_{\alpha\beta^*}y^\alpha y^\beta + \frac{\partial_\Delta v}{2v} f_0\Big(\sum_{s=1}^{n} x^s y^s\Big) \quad \textit{if } v = Ae^{BZ},$$

and

$$f=f_{\alpha^*}y^{\alpha}+\sum_{\alpha\neq\beta}f_{\alpha\beta^*}y^{\alpha}y^{\beta}\qquad \textit{if } v\neq Ae^{BZ}.$$

Thus

THEOREM 3. *The dimension of the automorphism group of Euclidean space $E^{2n+1}$ with a contact Riemannian structure such that $\Phi$ is given by* (3.1), *where $v=Ae^{BZ}$, A and B being non-zero constants, is $n(n+1)/2+1$. If $v$ is not of the form $Ae^{BZ}$ but still is an analytic function of $z$ such that $\frac{\partial v}{\partial z}$ is everywhere non-zero, then the dimension is $n(n+1)/2$.*

CASE $3^0$. When $v$ is an everywhere non-zero function such that $\xi v\equiv 0$. Though in this case calculations are a little difficult, we can see the dimension of automorphism group is still less than $(n+1)^2$. At first we expand $v$ and $f$ in formal power series:

$$v=v_0+v_r x^r+v_{r^*}y^r+\frac{1}{2}v_{r\delta}x^r x^\delta+v_{r\delta^*}x^r y^\delta+\frac{1}{2}v_{r^*\delta^*}y^r y^\delta+\cdots,$$

$$f=f_0+f_r x^r+f_{r^*}y^r+\frac{1}{2}f_{r\delta}x^r x^\delta+f_{r\delta^*}x^r y^\delta+\frac{1}{2}f_{r^*\delta^*}y^r y^\delta+\cdots.$$

Using these expansion, the first equation of (3.2) is expressed in this case as the following infinite number of equations;

$$\begin{aligned}&f_{\alpha\beta^*}=-f_{\alpha^*\beta},\ f_{\alpha\beta^*r}=-f_{\alpha^*\beta r},\ f_{\alpha\beta^*r^*}=-f_{\alpha^*\beta r^*},\\ &f_{\alpha\beta^*r}=-f_{\alpha^*\beta r},\ f_{\alpha\beta^*r\delta^*}=-f_{\alpha^*\beta r\delta^*},\ f_{\alpha\beta^*r^*\delta^*}=-f_{\alpha^*\beta r^*\delta^*},\\ &\cdots\cdots\cdots\cdots\cdots\qquad \textit{for } \alpha\neq\beta,\end{aligned}\tag{3.5}$$

*and*

$$\begin{aligned}&2v_0f_{ss^*}=f_\alpha v_{\alpha^*}-f_{\alpha^*}v_\alpha,\\ &2v_0f_{ss^*r}+2v_rf_{ss^*}=v_{\alpha^*}f_{\alpha r}-v_\alpha f_{\alpha^*r}+f_{\alpha^*r}f_\alpha-v_{\alpha r}f_{\alpha^*},\\ &2v_0f_{ss^*r^*}+2v_{r^*}f_{ss^*}=v_{\alpha^*}f_{\alpha r^*}-v_\alpha f_{\alpha^*r^*}+v_{\alpha^*r^*}f_\alpha-v_{\alpha r^*}f_{\alpha^*},\\ &\cdots\cdots\cdots\cdots\cdots,\ \textit{for } s=1,\cdots,n,\end{aligned}\tag{3.6}$$

(in general, $f_{ss^*}\cdots=\frac{1}{2v_0}$(a linear combination of $f's$ of indices less than the left)), and the last equation of (3.2) is expressed as follows;

$$
(3.7)\qquad
\begin{aligned}
&f_{\alpha\beta} = v_0^2 f_{\alpha^*\beta^*},\\
&f_{\alpha\beta r} = v_0^2 f_{\alpha^*\beta^* r} + 2v_0 v_r f_{\alpha^*\beta},\\
&f_{\alpha\beta r^*} = v_0^2 f_{\alpha^*\beta^* r^*} + 2v_0 v_{r^*} f_{\alpha^*\beta^*},\\
&f_{\alpha\beta r\delta} = v_0^2 f_{\alpha^*\beta^* r} + 2(v_r v_\delta + v_0 v_{r\delta}) f_{\alpha^*\beta^*} + 2v_0 v_\delta f_{\alpha^*\beta^* r} + 2v_0 v_r f_{\alpha^*\beta^*\delta},\\
&\cdots\cdots\cdots\cdots,
\end{aligned}
$$

(in general, $f_{\alpha\beta\ldots} = v_0^2 f_{\alpha^*\beta^*\ldots} +$ (a linear combination of $f's$ of indices less than the first term).).
By (3.6), we can see that coefficients of the expansion of $f$ of the form $f_{ss^*}\cdots$ are expressed by linear combinations of other coefficients of lower indices, and in (3.7), if $\beta \neq r$, then by (3.5)

$$
\begin{aligned}
f_{\alpha\beta r}\cdots &= v_0^2 f_{\alpha^*\beta^* r}\cdots + \text{lower terms}\\
&= -\, v_0^2 f_{\alpha^*\beta r^*}\cdots + \text{lower terms}\\
&= -\, f_{\alpha\beta r} \qquad + \text{lower terms}.
\end{aligned}
$$

Hence $f_{\alpha\beta r}\cdots = \frac{1}{2}$ (a linear combination of $f's$ of indices less than the left.), and hence $f_{\alpha^*\beta^* r}\cdots$ is also expressed by a linear combination of $f's$ of indices iless than itself. In the same way, if $\beta \neq r$, $f_{\alpha^*\beta^* r^*\ldots}$ and $f_{\alpha\beta r^*\ldots}$ are expressed by linear combinations of $f's$ of indices less than itself.

Thus, the coefficients of the expansion of $f$ except

$$f_0,\ f_\alpha,\ f_{\alpha^*},\ f_{\alpha\beta} \text{ and } f_{\alpha\beta^*} (\alpha \neq \beta)$$

are all expressed by linear combinations of these. Hence

THEOREM 4. *The dimension of the automorphism group of Euclidean space $E^{2n+1}$ with a contact Riemannian structure such that $\Phi$ is given by* (3.1), *where $v$ is an everywhere non-zero analytic function such that $\xi v \equiv 0$, is less than* $(n+1)^2$.

REMARK 2. Examining (3.5), (3.6) and (3.7) carefully, we can see that in Case $3^0$, if $n>1$ and if the dimension of the automorphism group is just $(n+1)^2$, then $v$ must be a constant. For $n=1$, however, it does not hold.

Summarizing cases $1^0 \sim 3^0$ in terms of manifolds,

THEOREM 5. *The dimension of the automorphism group of an analytic,* *complete, simly connected, contact Riemannian manifold M of dimension* $2n+1$ *such that for every point of M there exists an adapted local coordinate system such that* $\phi$ *is given by* (3.1), *where*

1°) $v$ *is a non-zero constant, is just* $(n+1)^2$,

2°) $v$ *and* $\xi v$ *are everywhere non-zero analytic functions, does not exceed* $(n+1)^2$,

3°) $v$ *is an everywhere non-zero analytic function such that* $\xi v \equiv 0$, *does not exceed* $(n+1)^2$.

REMARK 3. As is easily seen by (2.14) and (2.16), $\xi$ is always an infinitesimal strict contact transformation, and $\xi$ is an infinitesimal automorphism if and only if $\phi_\alpha^\beta$, $\phi_\alpha^{\beta^*}$, $\phi_{\alpha^*}^\beta$ do not depend on $z$. In this case, the manifold is called a $K$-contact manifold. Above Case 2° treats the cases of non $K$-contact manifolds, while Case 3° treats the cases of $K$-contact manifolds.

## BIBLIOGRAPHY

[1] P. LIBERMANN, Sur les automorphismes infinitéismau des structures symplectiques et des structures de contact, Colloque de géometrie différentile globale, (1958), 37-59.
[2] S. SASAKI, Almost contact manifolds, Lecture note, 1965.
[3] S. TANNO, The Automorphism Groups of Contact Riemannian manifolds and Kählerian manifolds, (unpublished).

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