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# ON HYPERSURFACES SATISFYING A CERTAIN CONDITION ON THE CURVATURE TENSOR*

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If a Riemannian manifold $M$ is locally symmetric, then its curvature tensor $R$ satisfies

$$(*) \qquad R(X,Y)\cdot R = 0 \quad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X,Y)$ operates on $R$ as a derivation of the tensor algebra at each point of $M$. Conversely, does this algebraic condition (*) on the curvature tensor field $R$ imply that $M$ is locally symmetric (i.e. $\nabla R = 0$)? We conjecture that the answer is affirmative in the case where $M$ is irreducible and complete and $\dim M \geqq 3$. For partial and related results, see [4], p.11, [9], Theorem 8, and [6].

The main purpose of the present paper is to give an affirmative answer in the case where $M$ is a complete hypersurface in a Euclidean space. More precisely, we prove

THEOREM. *Let $M$ be an $n$-dimensional, connected, complete Riemannian manifold which is isometrically immersed in a Euclidean space $R^{n+1}$ so that the type number is greater than 2 at least at one point. If $M$ satisfies condition (*), then it is of the form $M = S^k \times R^{n-k}$, where $S^k$ is a hypersphere in a Euclidean subspace $R^{k+1}$ of $R^{n+1}$ and $R^{n-k}$ is a Euclidean subspace orthogonal to $R^{k+1}$.*

As a result, $M$ is, of course, symmetric. We have also

COROLLARY. *Let $M$ be an $n$-dimensional, connected compact Riemannian manifold which is isometrically immersed in $R^{n+1}$, where $n > 3$. If $M$ satisfies condition (*), it is a hypersphere.*

In the appendix, we shall show that slight modifications of our proof of

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the theorem above lead to the result of Hartman-Nirenberg [2] that a complete locally Euclidean hypersurface is actually imbedded as a cylinder built over a plane curve.

**1. Reduction of condition** (*). The following is a purely local argument. Let $U$ be a neighborhood of a point $x_0 \in M$ on which we choose a unit vector field $\xi$ normal to $M$. For any vector fields $X$ and $Y$ tangent to $M$, we have the formulas of Gauss and Weingarten:

$$D_X Y = \nabla_X Y + h(X, Y)\xi$$

$$D_X \xi = -AX,$$

where $D_X$ and $\nabla_X$ denote covariant differentiations for the Euclidean connection of $R^{n+1}$ and the Riemannian connection on $M$, respectively. $A$ is a field of symmetric endomorphisms which corresponds to the second fundamental form $h$, that is, $h(X, Y) = g(AX, Y)$ for tangent vectors $X$ and $Y$. The equation of Gauss expresses the curvature tensor $R$ of $M$ by means of $A$:

$$R(X, Y) = AX \wedge AY,$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps $Z$ upon $g(Z, Y)X - g(Z, X)Y$, $g$ being the Riemannian metric. The type number $k(x)$ at $x$ is, by definition, the rank of $A$ at $x$.

At a point $x \in M$, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $Ae_i = \lambda_i e_i$, $1 \leqq i \leqq n$. Then the equation of Gauss implies

$$R(e_i, e_j) = \lambda_i \lambda_j e_i \wedge e_j.$$

By computing

$$(R(e_i, e_j) \cdot R)(e_k, e_l) = [R(e_i, e_j), R(e_k, e_l)]$$
$$- R(R(e_i, e_j)e_k, e_l) - R(e_k, R(e_i, e_j)e_l),$$

we find that it is zero except possibly in the case where $k = i$ and $l \neq i, j$ $(i \neq j)$. For this case we have

$$(R(e_i, e_j) \cdot R)(e_i, e_l) = \lambda_i \lambda_j \lambda_l (\lambda_j - \lambda_i) e_j \wedge e_l.$$

Thus we see that condition (*) is equivalent to

$$(**) \qquad \lambda_i \lambda_j \lambda_l(\lambda_j - \lambda_i) = 0 \ \text{ for } l \neq i,j, \text{ where } i \neq j.$$

Suppose that the type number $k(x)$ is $\geqq 3$ at a point $x$ and assume that $\lambda_1, \cdots, \lambda_k$ are non-zero eigenvalues of $A$ at $x$ and $\lambda_{k+1} = \cdots = \lambda_n = 0$. For any $i$ and $j$ such that $1 \leqq i < j \leqq k$, we choose $l$ such that $1 \leqq l \leqq k$ and $l \neq i,j$. Then (**) implies $\lambda_i = \lambda_j$. In other words, all the non-zero eigenvalues $\lambda_1, \cdots, \lambda_k$ are equal to each other.

We have

LEMMA 0. *If $k(x_0) \geqq 3$, then there is a neighborhood $U$ of $x_0$ on which the type number $k(x)$ is equal to a constant and the non-zero eigenvalue $\lambda(x)$ of $A$ is a differentiable function.*

PROOF. If $k(x_0) = n$, then obviously $k(x)$ is $n$ in a neighborhood of $x_0$. Assume that $3 \leqq k(x_0) < n$ and that $\lambda_1 = \cdots = \lambda_{k_0} = \lambda \neq 0$, $\lambda_{k_0+1} = \cdots = \lambda_n = 0$ are the eigenvalues of $A$ at $x_0$. By continuity of the eigenvalues of $A$, there is a neighborhood $U$ of $x_0$ on which $k_0$ eigenvalues of $A$ are of absolute value $> |\lambda|/2$ and $n-k_0$ eigenvalues are of absolute value $< |\lambda|/2$ (both counting the multiplicity). Since $k(x) \geqq k_0 \geqq 3$ for $x \in U$, we know that all the non-zero eigenvalues of $A$ at $x$ are equal. Hence the eigenvalues of absolute value $< |\lambda|/2$ must be 0. Thus $k(x) = k_0$ for every $x \in U$. The non-zero eigenvalue $\lambda(x)$ is a differentiable function on $U$, since $\lambda(x) = \text{trace } A/k_0$ and since trace $A$ is a differentiable function (where it is defined, that is, in a neighborhood of $x_0$ on which the unit normal field $\xi$ is defined).

**2. Lemmas.** In this section, we shall assume that $M$ is oriented (so that a unit normal field $\xi$ is defined on the whole $M$) and that the type number $k(x)$ is $\geqq 3$ everywhere on $M$. By the observations we made in **1**, the function $k(x)$ is locally constant and hence is a constant function, say, $k$, since $M$ is connected. We may also speak of the differentiable function $\lambda(x)$ which assings to each $x \in M$ the non-zero eigenvalue of $A$ at $x$.

Thus, at each $x \in M$, $\lambda(x)$ is the non-zero eigenvalue of $A$ with multiplicity $k$ and 0 is the eigenvalue with multiplicity $n-k$. We define two distributions on $M$ as follows:

$$T_0(x) = \{X \in T_x(M)\,;\ AX = 0\}$$

$$T_1(x) = \{X \in T_x(M)\,;\ AX = \lambda(x)X\}\,.$$

We have $T_x(M) = T_0(x) + T_1(x)$ (direct sum). For any $Z \in T_x(M)$, $Z_0$ and $Z_1$ will denote the components of $Z$ in $T_0(x)$ and $T_1(x)$. respectively.

LEMMA 1. *$T_0$ and $T_1$ are differentiable.*

PROOF. For any point $x_0 \in M$, let $\{X_1, \cdots, X_k\}$ be a basis of $T_1(x_0)$ and let $\{X_{k+1}, \cdots, X_n\}$ be a basis of $T_0(x_0)$. We extend $X_i$'s to vector fields on $M$ and define vector fields

$$Y_i = AX_i \qquad \text{for} \quad 1 \leqq i \leqq k$$

and

$$Y_j = (A - \lambda I) X_j \quad \text{for} \quad k+1 \leqq j \leqq n ,$$

where $I$ denotes the identity transformation. At $x_0$, we have $Y_i = \lambda X_i$ for $1 \leqq i \leqq k$ and $Y_j = -\lambda X_j$ for $k+1 \leqq j \leqq n$. Thus $Y_1, \cdots, Y_n$ are linearly independent at $x_0$ and hence in a neighborhood $U$ of $x_0$. At each point of $U$, we have

$$(A - \lambda I) Y_i = (A - \lambda I) AX_i = 0 \quad \text{for} \quad 1 \leqq i \leqq k$$

$$AY_j = A(A - \lambda I) X_j = 0 \quad \text{for} \quad k+1 \leqq j \leqq n .$$

Hence $Y_1, \cdots, Y_k$ form a basis of $T_1$ and $Y_{k+1}, \cdots, Y_n$ form a basis of $T_0$.

LEMMA 2. *$T_0$ and $T_1$ are involutive.*

PROOF. We recall the Codazzi equation

$$(\nabla_X A) Y = (\nabla_Y A)(X) .$$

Suppose that $X$ and $Y$ are vector fields belonging to $T_0$. Then

$$(\nabla_X A) Y = \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y) ,$$

and

$$(\nabla_Y A) X = -A(\nabla_Y X) .$$

Thus we get $A(\nabla_X Y) = A(\nabla_Y X)$, that is,

$$A([X, Y]) = A(\nabla_X Y - \nabla_Y X) = 0 ,$$

showing that $[X, Y]$ belongs to $T_0$. Thus $T_0$ is involutive.

Suppose now that $X$ and $Y$ belong to $T_1$. Then

$$(\nabla_X A) Y = \nabla_X(AY) - A(\nabla_X Y) = \nabla_X(\lambda Y) - A(\nabla_X Y)$$
$$= X\lambda \cdot Y + \lambda \nabla_X Y - A(\nabla_X Y) .$$

Interchanging $X$ and $Y$ here and using the Codazzi equation, we get

$$(X\lambda)Y - (Y\lambda)X + (\lambda I - A)[X, Y] = 0.$$

Since $(X\lambda)Y - (Y\lambda)X \in T_1$ and $(\lambda I - A)[X, Y] = \lambda[X, Y]_0$, we get

$$(X\lambda)Y - (Y\lambda)X = 0 \text{ and } [X, Y]_0 = 0.$$

The second identity shows that $[X, Y] \in T_1$, proving that $T_1$ is involutive. The first identity will establish

LEMMA 3. *If $X$ belongs to $T_1(x)$, then $X\lambda = 0$.*

PROOF. Since $\dim T_1(x) = k \geqq 3$, we may choose $Y \in T_1(x)$ such that $X$ and $Y$ are linearly independent. Extending $X$ and $Y$ to vector fields belonging to $T_1$, we have $(X\lambda)Y - (Y\lambda)X = 0$ at $x$. Thus $X\lambda = Y\lambda = 0$ at $x$.

REMARK. The function $\lambda$ is therefore constant on each maximal integral manifold of $T_1$. We shall later see that $\lambda$ is actually a constant on $M$ (for this, completeness of $M$ is essential).

We now let $X \in T_1$, $Y \in T_0$ and compute the both sides of the Codazzi equation:

$$\begin{aligned}
(\nabla_X A)Y &= \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y) = -\lambda(\nabla_X Y)_1, \\
(\nabla_Y A)X &= \nabla_Y(AX) - A(\nabla_Y X) = \nabla_Y(\lambda X) - A(\nabla_Y X) \\
&= Y\lambda \cdot X + \lambda(\nabla_Y X) - A(\nabla_Y X) \\
&= Y\lambda \cdot X + \lambda(\nabla_Y X)_0.
\end{aligned}$$

Therefore we have

$$(\nabla_Y X)_0 = 0, \quad \text{that is,} \quad \nabla_Y X \in T_1$$

and

$$(Y\lambda)X = -\lambda(\nabla_X Y)_1 = -A(\nabla_X Y).$$

We have hence

LEMMA 4. *If $X \in T_1$, $Y \in T_0$, then $A(\nabla_X Y) = -(Y\lambda)X$.*

LEMMA 5.

(i) *If* $Y \in T_0$, *then* $\nabla_Y(T_1) \subset T_1$.

(ii) *If* $Y \in T_0$, *then* $\nabla_Y(T_0) \subset T_0$.

(iii) *If* $Y \in T_0$, $X \in T_1$ *and* $[X, Y]=0$, *then* $\nabla_X Y \in T_1$.

PROOF. (i) has been already shown above. (ii) follows from (i) and from the fact that $T_0$ and $T_1$ are orthogonal complements to each other. (iii) follows from $\nabla_X Y = \nabla_Y X + [X,Y] = \nabla_Y X \in T_1$.

LEMMA 6. *If* $Y\lambda=0$ *for every* $Y \in T_0$, *then* $X \in T_1$ *implies* $\nabla_X(T_0) \subset T_0$ *and* $\nabla_X(T_1) \subset T_1$.

PROOF. Under the assumption, Lemma 4 implies $A(\nabla_X Y) = 0$, that is, $\nabla_X Y \in T_0$ for $X \in T_1$ and $Y \in T_0$. Thus $\nabla_X(T_0) \subset T_0$ for $X \in T_1$. Since $T_1$ is the orthogonal complement of $T_0$, we have $\nabla_X(T_1) \subset T_1$ as well.

LEMMA 7. *Let* $Y$ *and* $Z$ *be vector fields belonging to* $T_0$ *such that* $\nabla_Y Z = \nabla_Z Y = 0$. *If there is a non-vanishing vector field* $X$ *belonging to* $T_1$ *such that* $[X, Y] = [X, Z] = 0$, *then* $(YZ)\left(\frac{1}{\lambda}\right) = 0$.

PROOF. We know that $R(X, Y) = AX \wedge AY = 0$ since $AY = 0$. On the other hand, we have

$$R(X, Y) \cdot Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z = -\nabla_Y(\nabla_X Z)$$

in view of $\nabla_Y Z = 0$ and $[X, Y]=0$. By Lemma 4, we have $-(Z\lambda)X=A(\nabla_X Z)$. By Lemma 5, (iii), we have $A(\nabla_X Z) = \lambda(\nabla_X Z)$. Thus we get $\nabla_X Z = -\frac{Z\lambda}{\lambda} X$.

Therefore $\nabla_Y\left(\frac{Z\lambda}{\lambda} X\right) = 0$, which implies

$$\frac{\lambda(YZ\lambda)-(Y\lambda)(Z\lambda)}{\lambda^2} X + \frac{Z\lambda}{\lambda} \nabla_Y X = 0 .$$

Since $[X, Y] = 0$, we have $\nabla_Y X = \nabla_X Y$ and this is equal to $\frac{-Y\lambda}{\lambda} X$ (in the same way as for $\nabla_X Z = \frac{-Z\lambda}{\lambda} X$). Hence the equation above reduces to

$$(\lambda(YZ\lambda) - 2(Y\lambda)(Z\lambda))\, X = 0 .$$

Since $X$ is non-vanishing, we get

$$\lambda(YZ\lambda) - 2(Y\lambda)(Z\lambda) = 0.$$

A simple computation shows

$$YZ\left(\frac{1}{\lambda}\right) = -\frac{\lambda Y(Z\lambda) - 2(Y\lambda)(Z\lambda)}{\lambda^3} = 0.$$

**3. Proof of the theorem in the case where $k(x) \geqq 3$ everywhere.** We restate the assumptions explicitly. $M$ is an $n$-dimensional, connected and complete Riemannian manifold satisfying condition ($*$). $f: M \to R^{n+1}$ is an isometric immersion such that the type number $k(x)$ is $\geqq 3$ everywhere. We wish to prove that $M$ is the direct product $M_0 \times M_1$ and that $f$ is the direct product of $f_0: M_0 \to R^{n-k}$ and $f_1: M_1 \to R^{k+1}$, where $R^{n-k}$ and $R^{k+1}$ are Euclidean subspaces of $R^{n+1}$ which are orthogonal to each other, $f_0$ is an isometry and $f_1$ is an isometry of $M_1$ onto a sphere $S^k$ in $R^{k+1}$.

Let $\widetilde{M}$ be the universal covering of $M$ with projection $\pi: \widetilde{M} \to M$. The assumptions above are satisfied for $\widetilde{M}$ and its isometric immersion $\widetilde{f} = f \circ \pi$. If we know that $\widetilde{f}$ is an isometry of $\widetilde{M}$ onto $R^{n-k} \times S^k$ in the manner above, then it follows that $\pi$ is one-to-one, that is, $\widetilde{M} = M$. Thus it will be sufficient to prove the theorem for $\widetilde{M}$.

We shall therefore assume that $M$ is simply connected (and hence orientable).

In **2** we have introduced involutive distributions $T_0$ and $T_1$. For each $x \in M$, we denote by $M_0(x)$ and $M_1(x)$ the maximal integral manifolds through $x$ of $T_0$ and $T_1$, respectively.

PROPOSITION 1.

(i) *$M_0(x)$ is totally geodesic in $M$ and is complete.*

(ii) *The restriction of $f$ to $M_0(x)$ is an isometry of $M_0(x)$ onto a Euclidean subspace $R^{n-k}(x)$ of $R^{n+1}$.*

PROOF. (i) By Lemma 5, (ii), we know $\nabla_Y(T_0) \subset T_0$ for $Y \in T_0$. This means that $M_0(x)$ is totally geodesic in $M$. $M_0(x)$ is complete as a maximal integral manifold which is totally geodesic. Indeed, let $y(t)$ be a geodesic in $M_0(x)$. As a geodesic in $M$, it is infinitely extendible. Suppose $t_0 = \sup\{t_1;\ y(t) \in M_0(x)$ for $t < t_1\}$. Take local coordinates $\{x^1, \cdots, x^k, x^{k+1}, \cdots, x^n\}$ with origin $y(t_0)$ such that $\{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$ and $\{\partial/\partial x^{k+1}, \cdots, \partial/\partial x^n\}$ are local bases for $T_1$ and $T_0$. Since $y(t)$, $t < t_0$, is a geodesic lying in the $T_0$-direction, we have $y^i(t) = c^i$, $1 \leqq i \leqq k$, for $t_0 - \delta < t < t_0$, where $\delta_0 > 0$. As $t \to t_0$, we have $y^i(t) \to 0$, hence $c^1 = \cdots = c^k = 0$. Thus the geodesic continues to lie in $M_0(x)$.

(ii) Consider $f$ locally. If $X$ and $Y$ are vector fields tangent to $M_0(x)$, then

$$D_{f(X)}f(Y) = f(\nabla_X Y) + h(X, Y)\,\xi\,.$$

We have $h(X, Y)=0$ since $X, Y \in T_0$. We know that $\nabla_X Y$ is tangent to $M_0(x)$. This means that $f: M_0(x) \to R^{n+1}$ is totally geodesic (that is, a geodesic in $M_0(x)$ is mapped upon a straight line in $R^{n+1}$). Hence $f(M_0(x))$ is contained in an $(n-k)$-dimensional Euclidean subspace $R^{n-k}(x)$. Since $M_0(x)$ is complete, it follows that $f(M_0(x)) = R^{n-k}(x)$. By a well known result (cf. Theorem 4.6 of Chapter IV, [3]), $f$ is a covering map and hence an isometry of $M_0(x)$ onto $R^{n-k}(x)$.

We now come to the crucial step of the proof.

PROPOSITION 2. *For any $Y \in T_0$, we have $Y\lambda = 0$.*

PROOF. For a point $x \in M$, let $\{y^1, \cdots, y^k, y^{k+1}, \cdots y^n\}$ be a coordinate system with origin $x$ in a neighborhood $U$ of $x$ such that $\{\partial/\partial y^1, \cdots, \partial/\partial y^k\}$ and $\{\partial/\partial y^{k+1}, \cdots, \partial/\partial y^n\}$ are local bases for $T_1$ and $T_0$ (cf. Lemma, [3], p. 182). Since $M_0(x)$ is isometric to a Euclidean space by Proposition 1, we may assume that the restriction of $\{y^{k+1}, \cdots, y^n\}$ to $M_0(x) \cap U$ is rectangular, that is

$$g(\partial/\partial y^\alpha, \partial/\partial y^\beta) = \delta_{\alpha\beta} \quad \text{for} \quad k+1 \leqq \alpha, \beta \leqq n.$$

We show that the restriction of $\{y^{k+1}, \cdots, y^n\}$ to $M_0(y) \cap U$ for any $y \in M_1(x) \cap U$ is rectangular. By setting $g_{\alpha\beta}(y^1, \cdots, y^n) = g(\partial/\partial y^\alpha, \partial/\partial y^\beta)$, $k+1 \leqq \alpha$, $\beta \leqq n$, we have

$$\frac{\partial g_{\alpha\beta}}{\partial y_i} = g(\nabla_{\partial/\partial y^i}(\partial/\partial y^\alpha), \partial/\partial y^\beta) + g(\partial/\partial y^\alpha, \nabla_{\partial/\partial y^i}(\partial/\partial y^\beta))\,.$$

But Lemma 5, (iii), implies $\nabla_{\partial/\partial y^i}(\partial/\partial y^\alpha) \in T_1$ for $1 \leqq i \leqq k$. Hence

$$g(\nabla_{\partial/\partial y^i}(\partial/\partial y^\alpha), \partial/\partial y^\beta) = 0$$

and, similarly, $g(\partial/\partial y^\alpha, \nabla_{\partial/\partial y^i}(\partial/\partial y^\beta)) = 0$. We have thus $\partial g_{\alpha\beta}/\partial y^i = 0$, that is,

$$g_{\alpha\beta}(y^1, \cdots, y^k, y^{k+1}, \cdots, y^n) = g_{\alpha\beta}(0, \cdots, 0, y^{k+1}, \cdots, y^n) = \delta_{\alpha\beta}\,.$$

Now let $Y=\partial/\partial y^\alpha$, where $k+1 \leqq \alpha \leqq n$, and $X=\partial/\partial y^i$, where $1 \leqq i \leqq k$. Since $\{y^{k+1}, \cdots, y^n\}$ is rectangular on each $M_0(y) \cap U$, which is totally geodesic in $M$, we have $\nabla_Y Y = 0$. Applying Lemma 7 to $X, Y$ and $Z = Y$, we have $Y^2(1/\lambda) = 0$.

If $L$ is a straight line in $M_0(x)$, let $Y$ be the parallel vector field in the direction of $L$ on the Euclidean space $M_0(x)$. For any point of $L$, we may choose suitable local coordinates $\{y^1, \cdots, y^n\}$ and show by the argument above that $Y^2(1/\lambda) = 0$. This means that if $s$ is the length parameter of $L$, then $\frac{d^2}{ds^2}\left(\frac{1}{\lambda}\right) = 0$. Thus

$$\frac{1}{\lambda} = as + b ,$$

where $a$ and $b$ are certain constants. If $a$ is not 0, then $1/\lambda$ will be 0 for $s = -b/a$, which is a contradiction. We have thus shown that $\lambda$ is equal to a constant on $L$. Since $L$ can be an arbitrary straight line in $M_0(x)$ starting from $x$, we conclude that $\lambda$ is equal to a constant on $M_0(x)$. Thus $Y\lambda = 0$ for any $Y \in T_0$.

REMARK. Since $X\lambda = 0$ for any $X \in T_1$, it follows that $Z\lambda = 0$ for any tangent vector $Z$. Thus $\lambda$ is a constant function on $M$.

We now prove

PROPOSITION 3.

(i) *$M_1(x)$ is totally geodesic in $M$ and is complete.*
(ii) *For any point $o$, let $M_0 = M_0(o)$ and $M_1 = M_1(o)$. Then $M$ is isometric to the direct product of $M_0$ and $M_1$.*
(iii) *The Euclidean subspaces $R^{n-k}(x) = f(M_0(x))$, $x \in M_1$, in Proposition 1 are all parallel to $R^{n-k} = R^{n-k}(o)$.*
(iv) *The restriction $f_1$ of $f$ to $M_1$ is an isometry of $M_1$ onto a sphere $S^k$ in the Euclidean subspace $R^{k+1}$ which is perpendicular to $R^{n-k}$.*
(v) *If $f_0$ is the restriction of $f$ to $M_0$, then $f = f_0 \times f_1$, that is,*

$$f(y, x) = (f_0(y), f_1(x)) \in R^{n-k} \times S^k .$$

*for every $(y, x) \in M_0 \times M_1 = M$.*

PROOF. (i) By Proposition 2 and Lemma 6, we know that $\nabla_X(T_1) \subset T_1$ for any vector field $X$ belonging to $T_1$ This means that $M_1(x)$ is totally geodesic. The completeness can be proved in the same way as for $M_0(x)$.

(ii) Lemmas 5 and 6 together imply that $T_0$ and $T_1$ are parallel. Since $M$ is simply connected and complete, our conclusion is a standard result (cf. Theorem 6.1 of Chapter IV, [3]).

(iii) Let $Y \in T_0(o)$ and let $Y_t$ be the family of tangent vectors parallel to $Y$ along a curve $x(t)$ in $M_1$. By (ii) we have $Y_t \in T_0(x(t))$. Considering $f$ locally, we get (denoting by $\vec{x}_t$ the tangent vector of the curve $x(t)$)

$$D_{f(\vec{x}_t)} f(Y_t) = f(\nabla_{\vec{x}_t} Y_t) + h(\vec{x}_t, Y_t)\,\xi = 0\,,$$

since $\nabla_{\vec{x}_t} Y_t = 0$ and $h(\vec{x}_t, Y_t) = 0$. Thus $f(Y_t)$ is parallel in $R^{n+1}$. This proves that $f(T_0(x))$ are parallel in $R^{n+1}$. Since the Euclidean subspace $R^{n-k}(x) = f(M_0(x))$ has $f(T_0(x))$ as the tangent space at $f(x)$, we conclude that $R^{n-k}(x)$, $x \in M_1$, are parallel.

(iv) Consider the $R^{n+1}$-valued vector function $x \to \xi_x + \lambda f(x)$ on $M_1$. For any tangent vector $X$ to $M_1$ we have

$$D_{f(X)}(\xi + \lambda f) = f(-AX + \lambda X) = 0\,,$$

which shows that $\xi + \lambda f$ is equal to a constant vector, say, $\alpha$, in $R^{n+1}$. Hence

$$\| f(x) - \alpha/\lambda \| = |1/\lambda| \quad \text{on} \quad M_1,$$

showing that $f(M_1)$ lies on the hypersphere $S^n$ with center $\alpha/\lambda$ and radius $|1/\lambda|$. On the other hand, $f(M_1)$, is perpendicular to $f(M_0(x)) = R^{n-k}(x)$, $x \in M_1$, at each point of $f(M_1)$, and $R^{n-k}(x)$ are all parallel to $R^{n-k}$. It follows that $f(M_1)$ lies in the Euclidean subspace $R^{k+1}$ through $f(o)$ that is perpendicular to $R^{n-k}$. Hence $f(M_1)$ lies in the sphere $S^k = S^n \cap R^{k+1}$. Again by Theorem 4.6, Chapter IV, [3], it follows that $f_1 : M_1 \to S^k$ is a covering map and hence an isometry.

(v) Let $(y, x) \in M_0 \times M_1$. Let $y = \exp_o sY_0$, where $Y_0$ is a unit vector in $T_0(o)$. Then the point $(y, x)$ is equal to $\exp_x sY$, where $Y$ is the unit vector in $T_0(x)$ which is parallel to $Y_0$. By (iii) we know that $f(Y_0)$ and $f(Y)$ are parallel in $R^{n+1}$. Since $f$ maps geodesics in $M_0(x)$ upon straight lines in $R^{n-k}(x)$, we see that $f(y, x) = \exp_{f_1(x)} sf(Y)$ and this is equal to $(f_0(y), f_1(x))$, since $f_0(y) = \exp_{f(o)} sf(Y_0)$. We have thus shown $f(y, x) = (f_0(y), f_1(x))$.

With Proposition 3 the main theorem has been proved under the assumption that $k(x) \geqq 3$ everywhere.

**4. Proof of the theorem**. We now prove the theorem under the weaker assumption that the type number $k(x)$ is $\geqq 3$ at some point, say, $o \in M$. As in the beginning of **3**, we may assume that $M$ is simply connected.

Let $W = \{x\,;\, k(x) \geqq 3\}$, which is an open set. Let $W_0$ be the connected component of $o$ in $W$. As before, we know that $k(x)$ is a constant on $W_0$, $\lambda(x)$ is a differentiable function, and the distributions $T_0$ and $T_1$ defined on $W_0$ are differentiabe and involutive. All the lemmas are valid.

Let $M_0$ and $M_1$ be the maximal integral manifolds of $T_0$ and $T_1$, respectively, through $o$.

PROPOSITION 4.

(i) *$M_0$ is totally geodesic in $M$ and is locally Euclidean.*

(ii) *On a geodesic $L(s)$ in $M_0$ with arc length parameter $s$, we have*
$$\lambda(s) = \frac{1}{as+b}.$$

(iii) *$M_0$ is complete and $\lambda$ is a constant on $M_0$.*

(iv) *The type number $k(x)$ is, in fact, $\geqq 3$ everywhere on $M$.*

PROOF. (i) $M_0$ is totally geodesic by Lemma 5, (ii). Hence the curvature tensor of $M_0$ is the restriction of the curvature tensor $R$ of $M$ to $M_0$. We have $R(X, Y) = AX \wedge AY = 0$ for $X$ and $Y$ tangent to $M_0$. Thus $M_0$ is locally Euclidean.

(ii) For any geodesic $L(s)$ in $M_0$ with arc length parameter $s$, we may show that $\frac{d^2}{ds^2}\left(\frac{1}{\lambda}\right) = 0$ by using the essentially same argument as for Proposition 2.

(iii) Let $L(s)$ be a geodesic in $M_0$ starting from $o$. As a geodesic in $M$, it is infinitely extendible. If this entire geodesic does not lie in $W_0$, let $s_0$ be such that $L(s) \in W_0$ (hence $L(s) \in M_0$) for $s < s_0$ but $L(s_0) \notin W_0$. We derive a contradiction by showing that the type numder at $L(s_0)$ is $\geqq 3$. The characteristic polynomial of $A$ at $L(s)$, $s < s_0$, is $(t-\lambda(s))^k t^{n-k}$. That of $A$ at $L(s_0)$ is therefore the limit as $s \to s_0$, namely, $(t-\lambda(s_0))^k t^{n-k}$. But $\lambda(s_0) = \lim_{s\to s_0} \lambda(s) = \lim_{s \to s_0} \frac{1}{as+b}$ cannot be 0. This shows that the type number of $A$ at $L(s_0)$ is $k \geqq 3$. It follows that $L(s_0) \in W_0$ and hence $L(s_0) \in M_0$. Thus $M_0$ is complete. We also see that the constant $a$ has to be 0 (as in the proof of Proposition 2), namely, $\lambda$ is a constant on $M_0$.

(iv) Since $\lambda$ is constant on any maximal integral manifold of $T_0$ (defined on $W_0$), we have $Y\lambda = 0$ for $Y \in T_0$. By Lemma 3, we have $X\lambda = 0$ for $X \in T_1$. Thus we see that $\lambda$ is a constant function on $W_0$. We now show that $W_0$ is actually equal to $M$. Suppose $W_0 \neq M$ and let $x$ be a point of $\bar{W}_0 - W_0$. By the continuity argument for the characteristic polynomial of $A$, we see that the type number at $x$ is again $k \geqq 3$. Thus $W_0$ is open and closed so that $W_0 = M$, completing the proof of Proposition 4.

Proposition 4 shows that the assumption that the type number is $\geqq 3$ at one point actually implies that it is $\geqq 3$ everywhere on $M$. Thus our main theorem has been proved.

The Corollary follows easily from the fact that for an $n$-dimensional compact Riemannian manifold $M$ isometrically immersed in $R^{n+1}$ there is a point $x \in M$ where the type number is $n$ (for example, a point $x \in M$ where the distance from an arbitrarily fixed point in $R^{n+1}$ attains a maximum).

**5. Appendix**. Let $M$ be an $n$-dimensional, connected, locally Euclidean and complete Riemannian manifold and let $f: M \to R^{n+1}$ be an isometric immersion. The result of Hartman-Nirenberg [2] says that $f(M)$ is of the form $\gamma \times R^{n-1}$, where $R^{n-1}$ is a Euclidean subspace of $R^{n+1}$ and $\gamma$ is a curve: $-\infty < s < \infty \to \gamma(s)$ in a plane $R^2$ perpendicular to $R^{n-1}$. We indicate a proof of this result.

First assume that $M$ is moreover simply connected (so that $M$ is isometric to a Euclidean space $R^n$). Since its curvature tensor is identically zero, the eigenvalues of $A$ are 0 except possibly one of them, say, $\lambda$. If $\lambda$ is also identically 0, then obviously $f(M)$ is a hyperplane in $R^{n+1}$ and $f$ is an isometry of $M$ onto the hyperplane.

Assume that $\lambda$ is not identically zero. Let $W$ be the set of points where $\lambda$ is not 0 and let $W = \bigcup_\alpha W_\alpha$ be the decomposition of $W$ into the connected components. On each $W_\alpha$ we may define two distributions $T_0 = \{X; AX=0\}$ and $T_1 = \{X; AX=\lambda X\}$, for which all the lemmas are valid except Lemma 3 (for Lemma 3, $\dim T_1 \geqq 2$ is needed, whereas here $\dim T_1 = 1$). For each point $x \in W_\alpha$, we may show, as in Proposition 4, that the maximal integral manifold $M_0(x)$ of $T_0$ through $x$ is totally geodesic in $M$ and is complete, that $\lambda$ is a constant on $M_0(x)$, and that $f$ induces an isometry of $M_0(x)$ onto an $(n-1)$-dimensional subspace $R^{n-1}$ of $R^{n+1}$. $M$ being isometric with $R^n$, we may identify $M_0(x)$ with a hyperplane, say $H(x)$, of $R^n = M$. The hyperplanes $H(x)$ are parallel for all points $x$ in one component $W_\alpha$, because if $H(x)$ and $H(y)$ are distinct, they have no common point as the distinct maximal integral manifolds of $T_1$. We also see that the maximal integral manifold $M_1(x)$ of $T_1$ through each point $x$ is a geodesic in $W_\alpha$, hence part of a straight line in $M = R^n$.

We now choose an arbitrary point $o \in W$ and extend the geodesic $M_1(o)$ as a straight line, say, $L$ of $M=R^n$. We have the following situations:

1) For each point $x$ of $W_\alpha$, we have assigned a hyperplane $H(x) \subset W_\alpha$ and $\lambda$ is constant on $H(x)$.

2) All the hyperplanes $H(x)$, $x \in W$, are parallel. In fact, if $x, y \in W_\alpha$, then $H(x)$ and $H(y)$ are parallel as we already know. Suppose $x \in W_\alpha$,

$y \in W_\beta$ $(\alpha \neq \beta)$. If there is a point $z \in H(x) \cap H(y)$, then, since $\lambda$ is a constant on $H(x)$, $z \in W_\alpha$ and, similarly, $z \in W_\beta$, which is a contradiction. Thus $H(x)$ and $H(y)$ are disjoint, that is, parallel.

3) The straight line $L$ is perpendicular to $H(x)$ at every point $x \in L \cap W$. Indeed, if $\lambda(x) \neq 0$, then $x$ belongs to $W_\alpha$ for some $\alpha$ and the hyperplane $H(x)$, which is the maximal integral manifold of $T_0$ through $x$, is parallel to $H(o)$. Since $L$ is perpendicular to $H(o)$, we see that $L$ is perpendicular to $H(x)$.

4) For each $x$ on $L-W$, we define $H(x)$ to be the hyperplane through $x$ which is parallel to $H(o)$. Then $\lambda(y)=0$ for every $y \in H(x)$. Indeed, suppose there is a point $y \in H(x)$ with $\lambda(y) \neq 0$. Then $H(y)$, being parallel to $H(o)$, must coincide with $H(x)$. Since $\lambda$ is constant on $H(y)$, we must have $\lambda(x) \neq 0$, which is a contradiction.

We now show how $f$ maps all $H(x)$ into $R^{n+1}$. Let $Y_t$ be a vector field along $L=L_t$ which is parallel to $Y \in T_0(o)$. We have locally

$$D_{f(\vec{L}_t)} f(Y_t) = f(\nabla_{\vec{L}_t} Y_t) + h(\vec{L}_t, Y_t)\,\xi = h(\vec{L}_t, Y_t)$$

since $\nabla_{\vec{L}_t} Y_t = 0$. If $\lambda(L_t) \neq 0$, then, in a neighborhood, $Y_t$ belongs to $T_0$ and $\vec{L}_t$ belongs to $T_1$. Thus $h(\vec{L}_t, Y_t)=0$. If $\lambda(L_t)=0$, this means that $h$ is identically 0 at the point $L_t$. Hence $h(\vec{L}_t, Y_t)=0$. In either case, that is, for each point of $L$, we have $D_{f(\vec{L}_t)} f(Y_t)=0$. This means that $f(Y_t)$ is parallel in $R^{n+1}$. It follows that $f(H(x))$, $x \in L$, are all parallel to the subspace $R^{n-1}=f(H(o))$.

Since $L$ is perpendicular to all $H(x)$ and since $f$ is isometric, we see that $\gamma=f(L)$ is a curve on a plane perpendicular to $R^{n-1}$. From the fact that $f(Y_t)$ is parallel whenever $Y_t$ is parallel along $L$, it follows, as in Proposition 3, (iii), that

$$f(L_t, Y) = (f_1(L_t), f_0(y))$$

for all $(L_t, y) \in L \times H(o) = M$, where $f_1$ and $f_0$ are the restrictions of $f$ to $L$ and $H(o)$, respectively.

We have thus proved that $M=R^n$, which is the direct product of the straight line $L$ and the hyperplane $H(o)$, is mapped onto the cylinder $\gamma \times R^{n-1}$.

In the case where $M$ is not simply connected, let $\widetilde{M}$ be the universal covering of $M$ with projection $\pi : \widetilde{M} \to M$. From the result for $\widetilde{M}$ and its immersion $\widetilde{f}=f \circ \pi$, we see that $\widetilde{f}(\widetilde{M})=f(M)$ is a cylinder in the sense above.

We note that the result of Hartman-Nirenberg was earlier proved under weaker differentiability assumptions by A. Pogorelov [8]. Also for the case of a 2-dimensional surface, see Massey [5]. As a matter of fact, our proof of the main theorem is an adaptation of Massey's arguments for a higher-dimensional case. For extensions of the cylinder theorem, see O'Neill [7] and Hartman [1].

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