Tôhoku Math. Journ.
21(1969), 65-83.

SOME REMARKS ON SATURATION PROBLEM IN THE LOCAL APPROXIMATION II

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(Received August 12, 1968)

1. Introduction. One of the authors ([8], [9]) has considered the problem of determining the order of saturation and its class in the local approximation by a special class of linear positive operators which includes the Bernstein polymomial [4], the Szász operator [10] and the V.A.Baskakov operator [1] as particular cases. Meyer-König and Zeller [6] deal with a power series which has approximation properties similar to those of the Bernstein polynomial.

Our object of this paper is to prove the local saturation theorem about the Meyer-König and Zeller operator in the C-space.

2. The operator of Meyer-König and Zeller. This operator*)

$$M_n(f;\, x)=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right)\binom{\nu+n}{\nu} x^{\nu}(1-x)^{n+1} \tag{1}$$

is constructed in correspondence with a function $f(x)\in C[0,1]$. The sequence of operators $\{M_n(f;x)\}$ converges uniformly in $[0,1]$ to the function $f(x)$, if $f(x)$ is continuous. In this section, we shall prove the following local saturation theorem for the operator $M_n(f;x)$.

THEOREM 1. *For any function $f(x)\in C[0,1]$, we get:*
(i) *If*

$$|M_n(f;\, x)-f(x)|<\frac{Mx(1-x)^2}{2n},\quad x\in[a,b]\ (n=1,2,\cdots), \tag{2}$$

then $f(x)$ has a derivative which belongs to $\mathrm{Lip}_M 1$ *on* $[a,b]$.
(ii) *If $f'(x)$ exists and belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1,b_1]$, *then*

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*) Actually in the definition of the operator, Meyer-König and Zeller have taken $f\left(\frac{\nu}{\nu+n+1}\right)$ in stead of $f\left(\frac{\nu}{\nu+n}\right)$.

$$|M_n(f;x)-f(x)|<\frac{Mx(1-x)^2}{2n}+o\left(\frac{1}{n}\right), \text{ uniformly on } x\in[a_2,b_2].$$

(iii) *If in addition to the assumption of* (i), *the relation*

$$M_n(f;x)-f(x)=o\left(\frac{1}{n}\right)$$

holds a.e. on $[a_1,b_1]$, *then* $f(x)$ *is linear on* $[a_1,b_1]$, *where*

$$0\leqq a<a_1<a_2<b_2<b_1<b\leqq 1.$$

2.1. Auxiliary theorems. The following theorems for linear positive operator are known by P.P.Korovkin [3], R.G.Mamedov [5] and F.Schurer [7].

THEOREM A (P.P.Korovkin) [3]). *Let* $\{L_n(f;x)\}$ *be an infinite sequence of linear positive operators, which satisfies the three conditions*

$$L_n(1;x)=1+\alpha_n(x),$$
$$L_n(t;x)=x+\beta_n(x),$$
$$L_n(t^2;x)=x^2+\gamma_n(x),$$

where $\alpha_n(x),\beta_n(x)$ *and* $\gamma_n(x)$ *being any functions uniformly tending to zero on* $[a,b]$ *as* $n\to\infty$. *Then* $L_n(f;x)$ *converges uniformly on* $[a,b]$ *to* $f(x)$, *if* $f(x)$ *is continuous in* $[a,b]$.

THEOREM B (R.G.Mamedov [5] and F.Schurer [7]). *The "weight function"* $\psi(x)$ *is a bounded, twice continuously differentiable function not equal to zero on* $[a,b]$. *If the three conditions*

$$L_n(1;x)=1, \qquad x\in[a,b],$$
$$L_n(t;x)=x, \qquad x\in[a,b],$$

and

$$L_n(t^2;x)=x^2+\frac{\psi(x)}{n}+o\left(\frac{1}{n}\right), \text{ uniformly on } [a,b],$$

are satisfied for a sequence of linear positive operators $\{L_n(f;x)\}$, *which have the property*

$$L_n\{(t-x)^4;x\}=o\left(\frac{1}{n}\right), \text{ uniformly on } [a,b],$$

then for each function $f(x)\in C^{(2)}[a,b]$, *we get*

$$L_n(f;x)-f(x)=\frac{\psi(x)f''(x)}{2n}+o\left(\frac{1}{n}\right),\ x\in[a,b].$$

As a slight modification of Theorem B, we have

THEOREM C. *Let a sequence of linear positive operators* $\{L_n(f;x)\}$ *satisfy the same conditions of Theorem* B, *then for each function* $f(x)\in C^{(2)}[a,b]$, *we get*

$$L_n(f;x)-f(x)=\frac{\psi(x)f''(x)}{2n}+o\left(\frac{1}{n}\right),\ \textit{uniformly on } [a_1,b_1],$$

where $a<a_1<b_1<b$.

Now let $0\leqq a<b\leqq 1$ be given. We consider the following class U of functions $u(x)$, $x\in[0,1]$: $u(x)=\psi(x)q(x)$ where $q(x)$ is twice continuously differentiable and vanishes outside of an interval (α,β) with $a<\alpha<\beta<b$. Auxiliary numbers a_i and $b_i(i=1,2)$ are chosen to satisfy $a<a_1<a_2<\alpha<\beta<b_2<b_1<b$. For the linear positive operators $\{L_n(f;x)\}$ and $f(x)\in C[a,b]$, let us define the linear functional $A_n(f)$ by

$$(3)\qquad A_n(f)=2\sum_{na<k<nb}\frac{L_n\left(f;\frac{k}{n}\right)-f\left(\frac{k}{n}\right)}{\psi\left(\frac{k}{n}\right)}u\left(\frac{k}{n}\right)$$

$$=2\sum_{na<k<nb}\left[L_n\left(f;\frac{k}{n}\right)-f\left(\frac{k}{n}\right)\right]q\left(\frac{k}{n}\right).$$

We assume that for each $u(x)\in U$, there is an absolute constant K such that

$$|A_n(f)|\leqq K\|f\|,\ \|f\|\equiv\max_{x\in[a,b]}|f(x)|.$$

THEOREM D (Y. Suzuki [8]). *For the operator which has the same properties as the hypothesis of Theorem* B, *and* $f(x)\in C[a,b]$, *we get*:

(i) *If there is an absolute constant* K *such that*

$$|A_n(g)|\leqq K\|g\|,\ \textit{for any } g(x)\in C[a,b],$$

and

$$|L_n(f;x)-f(x)| < \frac{M\psi(x)}{2n}, \quad x \in [a,b] \quad (n=1,2,\cdots),$$

then $f(x)$ *has a derivative which belongs to* $\mathrm{Lip}_M 1$ *on* $[a,b]$.

(ii) *If* $f'(x)$ *exists and belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1,b_1]$, *then*

$$|L_n(f;x)-f(x)| < \frac{M\psi(x)}{2n} + o\left(\frac{1}{n}\right), \text{ uniformly on } x \in [a_2,b_2].$$

(iii) *If in addition to the assumption of* (i), *the relation*

$$L_n(f;x)-f(x)=o\left(\frac{1}{n}\right)$$

holds a.e. on $[a_1,b_1]$, *then* $f(x)$ *is linear on* $[a_1,b_1]$.

The functional, $A_n(f)$ in (3) is used in order to consider the local saturation problem for the Baskakov operator, but it is convenient to modify the definition of $A_n(f)$ for the Meyer-König and Zeller operator $M_n(f,x)$. That is, let us define a linear functional $\tilde{A}_n(f)$ by

$$(4) \quad \tilde{A}_n(f)=2\sum_{a<\frac{k}{k+n}<b} \frac{n^2}{(k+1+n)(k-1+n)}\left[\frac{L_n\left(f;\frac{k}{k+n}\right)-f\left(\frac{k}{k+n}\right)}{\psi\left(\frac{k}{k+n}\right)}\right]u\left(\frac{k}{k+n}\right)$$

$$=2\sum_{a<\frac{k}{k+n}<b} \frac{n^2}{(k+1+n)(k-1+n)}\left[L_n\left(f;\frac{k}{k+n}\right)-f\left(\frac{k}{k+n}\right)\right]q\left(\frac{k}{k+n}\right).$$

Also, we assume momently that for each $u(x) \in U$, there is an absolute constant K such that

$$|\tilde{A}_n(f)| \leqq K\|f\|, \quad \|f\| \equiv \max_{x \in [a,b]} |f(x)|,$$

where the constant K is independent on $f(x)$. This is an essential point of our proof and its proof will be given in the section 2.2. Then we have

THEOREM 2. *For the linear positive operators* $\{L_n(f;x)\}$ *which have the same properties as the hypothesis of Theorem* B, *and* $f(x) \in C[a,b]$, *we get:*

(i) *If there is an absolute constant* K *such that*

$$|\widetilde{A}_n(g)| \leqq K\|g\|, \textit{ for any } g(x) \in C[a,b],$$

and

$$(5) \qquad |L_n(f;x)-f(x)| < \frac{M\psi(x)}{2n}, \ x \in [a,b] \ (n=1,2,\cdots),$$

then $f(x)$ *has a derivative which belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1, b_1]$.

(ii) *If* $f'(x)$ *exists and belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1, b_1]$, *then*

$$|L_n(f;x)-f(x)| < \frac{M\psi(x)}{2n} + o\left(\frac{1}{n}\right), \textit{ uniformly on } x \in [a_2, b_2].$$

(iii) *If in addition to the assumption of* (i), *the relation*

$$L_n(f;x)-f(x) = o\left(\frac{1}{n}\right)$$

holds a.e. on $[a_1, b_1]$, *then* $f(x)$ *is linear on* $[a_1, b_1]$.

PROOF. We have only to prove (i), for the proof of (ii) and (iii) are the same as in Theorem D (see, [8]). We shall verify the relation

$$(6) \qquad \lim_{n\to\infty} \widetilde{A}_n(g) = \int_a^b g(x)u''(x)dx, \text{ for any } g(x) \in C[a,b] \text{ and } u(x) \in U.$$

Firstly let us suppose that $g(x) \in C^{(2)}[a,b]$, then we get

$$(7) \qquad L_n(g;x) - g(x) = \frac{\psi(x)g''(x)}{2n} + o\left(\frac{1}{n}\right),$$

uniformly on $x \in [a_1, b_1]$, for any $g(x) \in C^{(2)}[a,b]$.
From (4) and (7), it follows that

$$\widetilde{A}_n(g) = \sum_{a_1 < \frac{k}{k+n} < b_1} \frac{n}{(k+1+n)(k-1+n)} g''\left(\frac{k}{k+n}\right) u\left(\frac{k}{k+n}\right) + o(1)$$

$$\longrightarrow \int_a^b g''(x)u(x)dx = \int_a^b g(x)u''(x)dx \ (n\to\infty),$$

which is equivalent to (6). Since $C^{(2)}[a, b]$ is dense in $C[a, b]$ and there is a constant K such that

$$|\tilde{A}_n(g)| \leqq K\|g\|, \text{ for any } g(x) \in C[a, b],$$

the relation (6) is established for all $g(x) \in C[a, b]$. On the other hand, we can write

$$\tilde{A}_n(f) = \int_a^b u(x) d\lambda_n(x), \tag{8}$$

with the step function

$$\lambda_n(x) = 2\sum \frac{n^2}{(k+1+n)(k-1+n)} \frac{L_n\left(f; \frac{k}{k+n}\right) - f\left(\frac{k}{k+n}\right)}{\psi\left(\frac{k}{k+n}\right)},$$

where the summation runs over k such that $a < k/(k+n) < x$. We assume that $f(x)$ satisfies (5) for $x \in [a, b]$. Then the function $\lambda_n(x)$ has a total variation not greater than M, and an increment $|\lambda_n(x) - \lambda_n(y)|$ does not exceed the number of points $k/(k+n)$ in $[x, y]$ multiplied with M/n. By Helly's theorem [11], we can extract a subsequence $\{\lambda_{n_\nu}(x)\}$ which converges on $[a, b]$ to a function $\lambda(x)$ of bounded variation and we have

$$\lim_{p\to\infty} A_{n_p}(f) = \int_a^b u(x) d\lambda(x). \tag{9}$$

From (6) and (9)

$$\int_a^b f(x) u''(x) dx = \int_a^b \Lambda(x) u''(x) dx,$$

where $\Lambda(x)$ is an indefinite integral of $\lambda(x)$. Since this is true for all $u(x) \in U$, we have

$$f(x) = \Lambda(x) + gx + h, \quad x \in [a, b],$$

with some constants g and h. Hence $f'(x) = \lambda(x) + g$, $x \in [a, b]$.

For the completion of the proof of (i), we have only to verify that $\lambda(x)$ belongs to $\mathrm{Lip}_M 1$, which is trivial by the definition of $\lambda(x)$.

2.2. Some lemmas.

LEMMA 1. *If we set*

$$F_k(\nu)=\frac{\nu-k}{\nu-k+n}\,,\quad G_k(\nu)=\binom{\nu-k+n}{\nu-k},\ (k=0,1,2,\cdots),$$

then we obtain

$$F_k(\nu)G_k(\nu)=G_{k+1}(\nu)\,, \tag{10}$$

$$F_0(\nu)=\frac{n-k}{n+k}\,F_k(\nu)+\frac{k}{k+n}\{1+F_0(\nu)F_k(\nu)\}\,. \tag{11}$$

PROOF. From an easy calculation, we have

$$F_k(\nu)G_k(\nu)=\frac{\nu-k}{\nu-k+n}\binom{\nu-k+n}{\nu-k}=\frac{(\nu-k-1+n)(\nu-k-2+n)\cdots(n+1)}{(\nu-k-1)(\nu-k-2)\cdots1}$$

$$=\binom{\nu-(k+1)+n}{\nu-(k+n)}=G_{k+1}(\nu)\,.$$

The right hand of (11) is

$$\frac{1}{n+k}[k+F_k(\nu)\{(n-k)+kF_0(\nu)\}]=\frac{1}{n+k}\left[k+F_k(\nu)\frac{n^2+n\nu-kn}{\nu+n}\right]$$

$$=\frac{1}{n+k}\left[k+\frac{\nu-k}{\nu-k+n}\,\frac{n(\nu-k+n)}{\nu+n}\right]$$

$$=\frac{1}{n+k}\left[\frac{k(\nu+n)+n(\nu-k)}{\nu+n}\right]$$

$$=\frac{1}{n+k}\,\frac{\nu(n+k)}{\nu+n}$$

$$=F_0(\nu)\,.$$

Thus we obtain Lemma 1.

LEMMA 2. *We have*

(12) $$\sum_{\nu=0}^{\infty} F_0^2(\nu)G_0(\nu)x^\nu(1-x)^{n+1} = x^2 + \frac{x(1-x)^2}{n+1} + o\left(\frac{1}{n}\right),$$

(13) $$\sum_{\nu=0}^{\infty} F_0^3(\nu)G_0(\nu)x^\nu(1-x)^{n+1} = x^3 + \frac{3x^2(1-x)^2}{n+1} + o\left(\frac{1}{n}\right),$$

(14) $$\sum_{\nu=0}^{\infty} F_0^4(\nu)G_0(\nu)x^\nu(1-x)^{n+1} = x^4 + \frac{6x^3(1-x)^2}{n+1} + o\left(\frac{1}{n}\right).$$

PROOF. It is sufficient to show (12) only, for (13) and (14) are verified analogously.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} F_0^2(\nu)G_0(\nu)x^\nu(1-x)^{n+1} \\
&= x(1-x)^{n+1}\sum_{\nu=1}^{\infty} F_0(\nu)G_1(\nu)x^{\nu-1} \\
&= x(1-x)^{n+1}\sum_{\nu=1}^{\infty}\left[\frac{n-1}{n+1}F_1(\nu) + \frac{1}{n+1}\{1+F_0(\nu)F_1(\nu)\}\right]G_1(\nu)x^{\nu-1} \\
&= \frac{n-1}{n+1}x^2 + \frac{1}{n+1}x + \frac{x^2}{n+1}\sum_{\nu=2}^{\infty} F_0(\nu)G_2(\nu)x^{\nu-2}(1-x)^{n+1} \\
&= x^2 + \frac{x}{n+1}(1-2x) + \frac{x^2}{n+1}\sum_{\nu=2}^{\infty} F_0(\nu)G_2(\nu)x^{\nu-2}(1-x)^{n+1},
\end{aligned}$$

$$\begin{aligned}
&\sum_{\nu=2}^{\infty} F_0(\nu)G_2(\nu)^{\nu-2}(1-x)^{n+1} \\
&= (1-x)^{n+1}\sum_{\nu=2}^{\infty}\left[\frac{n-2}{n+2}F_2(\nu) + \frac{1}{n+2}(1+F_0(\nu)F_2(\nu))\right]G_2(\nu)x^{\nu-2} \\
&= \frac{n-2}{n+2}x + \frac{1}{n+2}\left[1 + x\sum_{\nu=3}^{\infty} F_0(\nu)G_3(\nu)x^{\nu-3}(1-x)^{n+1}\right] \\
&= x + \frac{1}{n+2}\left[1 + 4x + x\sum_{\nu=3}^{\infty} F_0(\nu)G_3(\nu)x^{\nu-3}(1-x)^{n+1}\right].
\end{aligned}$$

Hence

$$\sum_{\nu=0}^{\infty} F_0^2(\nu)G_0(\nu)x^\nu(1-x)^{n+1} = x^2 + \frac{x(1-x)^2}{n+1} + o\left(\frac{1}{n}\right),$$

where we used the estimation:

$$0<\sum_{\nu=3}^{\infty} F_0(\nu)G_3(\nu)x^{\nu-3}(1-x)^{n+1}<\sum_{\nu=3}^{\infty} G_3(\nu)x^{\nu-3}(1-x)^{n+1}=1.$$

LEMMA 3. *It holds that*

$$M_n\{(t-x)^4; x\} = o\left(\frac{1}{n}\right).$$

PROOF. Using Lemma 1 and Lemma 2, we have

$$\begin{aligned}
&M_n\{(t-x)^4\}\\
&\quad = -M_n(t^4; x) - 4xM_n(t^3;x) + 6x^2M_n(t^2; x) - 3x^4\\
&\quad = \sum_{\nu=0}^{\infty} F_0^4(\nu)G_0(\nu)x^\nu(1-x)^{n+1} - 4x\sum_{\nu=0}^{\infty} F_0^3(\nu)G_0(\nu)x^\nu(1-x)^{n+1}\\
&\quad\quad + 6x^2\sum_{\nu=0}^{\infty} F_0^2(\nu)G_0(\nu)x^\nu(1-x)^{n+1} - 3x^4\\
&\quad = \left\{x^4+\frac{6x^3(1-x)^2}{n+1}\right\} - \left\{4x^4+\frac{12x^3(1-x)^2}{n+1}\right\}\\
&\quad\quad + \left\{6x^4+\frac{6x^3(1-x)^2}{n+1}\right\} - 3x^4 + o\left(\frac{1}{n}\right)\\
&\quad = o\left(\frac{1}{n}\right).
\end{aligned}$$

LEMMA 4. *If $f''(x)$ exists and is bounded, we have*

$$M_n(f; x)=f(x) + \frac{f''(x)x(1-x)^2}{2n} + o\left(\frac{1}{n}\right),\ x\in[a,b],\ 0\leqq a<b\leqq 1.$$

This holds uniformly on the interval $[a_1, b_1]$ if $0\leqq a<a_1<b_1<b\leqq 1$ and if $f(x)$ is twice continuosly differentiable on $[a, b]$.

PROOF. This follows from Theorems B, C and Lemma 3.

LEMMA 5. *For $0\leqq x<1$, let us write*

$$T_{nr}(x) = \sum_{\nu=0}^{\infty} \{(n+1)x - (1-x)\nu\}^r p_{n\nu}(x),$$

$$T_n^*(x) = \sum_{\nu=0}^{\infty} |(n+1)x - (1-x)\nu)| p_{n\nu}(x),$$

where $p_{n\nu}(x) = \binom{\nu+n}{\nu} x^\nu (1-x)^{n+1}$, *then we have*

(15) $$T_{n0}(x) = 1,\ T_{n1}(x) = 0,\ T_{n2}(x) = (n+1)x$$

and

(16) $$T_n^*(x) \leqq \sqrt{(n+1)x}.$$

PROOF. Since $U_{n0}(x) \equiv \sum_{\nu=0}^{\infty} p_{n\nu}(x) = 1$, we shall calculate the following quantities:

$$U_{nr}(x) = \sum_{\nu=0}^{\infty} \nu^r p_{n\nu}(x), \quad (r = 1, 2)$$

$$\begin{aligned} U_{n1}(x) &= \sum_{\nu=0}^{\infty} \nu \binom{\nu+n}{\nu} x^\nu (1-x)^{n+1} \\ &= x \sum_{\nu=1}^{\infty} (\nu+n) \binom{\nu-1+n}{\nu-1} x^{\nu-1}(1-x)^{n+1} \\ &= x \sum_{\nu=0}^{\infty} \{(n+1)+(\nu-1)\} \binom{\nu-1+n}{\nu-1} x^{\nu-1}(1-x)^{n+1} \\ &= (n+1)x + U_{n1}(x)x. \end{aligned}$$

Hence we get

$$U_{n1}(x) = \frac{(n+1)x}{1-x}.$$

$$\begin{aligned} U_{n2}(x) &= \sum_{\nu=0}^{\infty} \nu^2 \binom{\nu+n}{\nu} x^\nu (1-x)^{n+1} \\ &= x \sum_{\nu=1}^{\infty} \nu\{(n+1)+(\nu-1)\} \binom{\nu-1+n}{\nu-1} x^{\nu-1}(1-x)^{n+1} \end{aligned}$$

$$= x(n+1)\sum_{\nu=1}^{\infty} \nu \binom{\nu-1+n}{\nu-1} x^{\nu-1}(1-x)^{n+1}$$

$$+ x\sum_{\nu=1}^{\infty} \{1+(\nu-1)\}(\nu-1)\binom{\nu-1+n}{\nu-1} x^{\nu-1}(1-x)^{n+1}$$

$$= x(n+1)(U_{n0}(x)+U_{n1}(x)) + x(U_{n1}(x)+U_{n2}(x)),$$

$$(1-x)U_{n2}(x) = x(n+1)\left\{1+\frac{(n+1)x}{1-x}\right\} + \frac{(n+1)x^2}{1-x} = \frac{x(n+1)(1+nx+x)}{1-x}.$$

Thus we have

$$U_{n2}(x) = \frac{(n+1)^2x^2+(n+1)x}{(1-x)^2}.$$

Therefore it follows

$$T_{n0}(x) = \sum_{\nu=0}^{\infty} p_{n\nu}(x) = U_{n0}(x) = 1,$$

$$\begin{aligned} T_{n1}(x) &= \sum_{\nu=0}^{\infty} \{(n+1)x-(1-x)\nu\} p_{n\nu}(x) \\ &= (n+1)xU_{n0}(x) - (1-x)U_{n1}(x) \\ &= (n+1)x-(n+1)x \\ &= 0, \end{aligned}$$

$$\begin{aligned} T_{n2}(x) &= \sum_{\nu=0}^{\infty} \{(n+1)x-(1-x)\nu\}^2 p_{n\nu}(x) \\ &= (n+1)^2x^2U_{n0}(x) - 2(n+1)x(1-x)U_{n1}(x) + (1-x)^2U_{n2}(x) \\ &= (n+1)^2x^2 - 2(n+1)^2x^2 + (n+1)^2x^2 + (n+1)x \\ &= (n+1)x, \end{aligned}$$

and

$$T_n^*(x) = \sum_{\nu=0}^{\infty} |(n+1)x-(1-x)\nu|\, p_{n\nu}(x)$$

$$\leqq \left\{\sum_{\nu=0}^{\infty} \{(n+1)x-(1-x)\nu\}^2 p_{n\nu}(x)\right\}^{1/2} \left\{\sum_{\nu=0}^{\infty} p_{n\nu}(x)\right\}^{1/2}$$

$$=\sqrt{(n+1)x}\,.$$

LEMMA 6. *For given* $0<a<b<1$, *there exist constants* $C_r (r=1,2)$ *such that for polynomials*

$$R_n(x)=\sum_{\nu=0}^{\infty} a_\nu p_{n\nu}(x), \ |a_\nu| \leqq L,$$

one has

$$|R_n^{(r)}(x)| \leqq C_r L(n+1)^{r/2}, \ a \leqq x \leqq b\,. \tag{17}$$

PROOF. Let us set $X=[x(1-x)]^{-1}$. Then

$$p'_{n\nu}(x)=-\{(n+1)x-(1-x)\nu\}Xp_{n\nu}(x), \tag{18}$$

$$p''_{n\nu}(x)=-(n+1+\nu)Xp_{n\nu}(x)+\{(n+1)x-(1-x)\nu\}^2X^2p_{n\nu}(x) \tag{19}$$
$$+(1-2x)\{(n+1)x-(1-x)\nu\}X^2p_{n\nu}(x)\,.$$

From (18), we get

$$\begin{aligned}|R'_n(x)| &\leqq LX\sum_{\nu=0}^{\infty} |(n+1)x-(1-x)\nu|\,p_{n\nu}(x) \\ &\leqq L\frac{\sqrt{x}}{x(1-x)}\sqrt{n+1} \\ &\leqq C_1L(n+1)^{1/2}\,.\end{aligned} \tag{20}$$

By (19), we have

$$\begin{aligned}R''_n(x) &= \sum_{\nu=0}^{\infty} a_\nu p''_{n\nu}(x) \\ &= -X\sum_{\nu=0}^{\infty} a_\nu(n+1+\nu)p_{n\nu}(x)+X^2\sum_{\nu=0}^{\infty} a_\nu\{(n+1)x-(1-x)\nu\}^2p_{n\nu}(x) \\ &\quad + (1-2x)X^2\sum_{\nu=0}^{\infty} a_\nu\{(n+1)x-(1-x)\nu\}p_{n\nu}(x)\,.\end{aligned}$$

Since

$$\sum_{\nu=0}^{\infty} a_\nu\{(n+1)x-(1-x)\nu\}\, p_{n\nu}(x)=(n+1)x\sum_{\nu=0}^{\infty} a_\nu p_{n\nu}(x)-(1-x)\sum_{\nu=0}^{\infty} a_\nu \nu p_{n\nu}(x),$$

we get

$$\begin{aligned}(21)\qquad |R_n''(x)| &\leqq L\{(n+1)X+(n+1)x^2X^2+(n+1)xX^2+2(n+1)x|1-2x|X^2\}\\ &= L(n+1)\left\{\frac{1}{x(1-x)}+\frac{1}{(1-x)^2}+\frac{1+2|1-2x|}{x(1-x)^2}\right\}\\ &\leqq C_2L(n+1).\end{aligned}$$

From (20) and (21) we obtain Lemma 6.

LEMMA 7. *For arbitrary $\delta>0$, there is a constant $C(\delta)$ such that*

$$(22)\qquad \sum_{\left|\frac{\nu}{\nu+n}-x\right|\geqq\delta} p_{n\nu}(x)\leqq C(\delta)n^{-1}, \qquad x\in[a,b].$$

PROOF.

$$\begin{aligned}\sum_{\left|\frac{\nu}{\nu+n}-x\right|\geqq\delta} p_{n\nu}(x) &\leqq \delta^{-2}\sum_{\left|\frac{\nu}{\nu+n}-x\right|\geqq\delta}\left(\frac{\nu}{\nu+n}-x\right)^2 p_{n\nu}(x)\\ &\leqq n^{-2}\delta^{-2}\sum_{\nu=0}^{\infty}[-\{(n+1)x-(1-x)\nu\}+x]^2 p_{n\nu}(x)\\ &= n^{-2}\delta^{-2}\{T_{n2}(x)-2xT_{n1}(x)+x^2\}\\ &= n^{-1}\delta^{-2}x\left\{\left(1+\frac{1}{n}\right)+\frac{x}{n}\right\}\leqq C(\delta)\frac{1}{n}.\end{aligned}$$

LEMMA 8. *Let $Q_n(x)$ be a sequence of twice continuously differentiable functions on* $[0,1]$; (A) *let the maximum μ_n of $|Q_n(x)|$ on the intervals $(0,a_2)$ and $(b_2,1)$ be $\mu_n=O(n^{-1})$, and* (B) *the maximum M_n of $|Q_n''(x)|$ on (a_1,b_1) be $M_n=O(n)$.*

Then

$$(23)\qquad \sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}Q_n\left(\frac{k}{k+n}\right)-n\int_0^1 Q_n(x)dx=O(1).$$

PROOF. Let k_1 be the smallest and k_2 the largest value of k satisfying the condition $a_1<\frac{k}{k+n}<b_1$. For large n,

$$\frac{k_1}{k_1+n}<a_2, \quad \text{and} \quad \frac{k_2}{k_2+n}>b_2.$$

For these n the difference (23) is equal to

$$(24) \qquad n\sum_{k=k_1}^{k_2}\left\{\frac{1}{2}\left(\frac{k+1}{k+1+n}-\frac{k}{k+n}\right)\left[Q_n\left(\frac{k}{k+n}\right)\right.\right.$$
$$\left.\left.+Q_n\left(\frac{k+1}{k+1+n}\right)\right]-\int_{\frac{k}{k+n}}^{\frac{k+1}{k+1+n}} Q_n(x)dx\right\},$$

with an error not exceeding $(n+1)\mu_n=O(1)$. The curled bracket in (24) equals to

$$\frac{1}{12}\left\{\frac{n}{(k+1+n)(k+n)}\right\}^3 Q_n''(\xi_k), \quad \frac{k}{k+n}\leqq\xi_k\leqq\frac{k+1}{k+1+n},$$

by the remainder formula of the trapezoid approximation. This order is $O(n^{-2})$, and the order of the whole sum in (24) is $O(1)$.

2.3. The proof of Theorem 1. We can rewrite (4) in the form

$$\tilde{A}_n(f)=2\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left\{M_n\left(f;\frac{k}{k+n}\right)-f\left(\frac{k}{k+n}\right)\right\}q\left(\frac{k}{k+n}\right)$$

$$=2\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left\{\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{\nu+n}\right)p_{n\nu}\left(\frac{k}{k+n}\right)-f\left(\frac{k}{k+n}\right)\right\}q\left(\frac{k}{k+n}\right)$$

$$=2\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{\nu+n}\right)\left\{\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}p_{n\nu}\left(\frac{k}{k+n}\right)q\left(\frac{k}{k+n}\right)\right.$$
$$\left.-\frac{n^2}{(\nu+1+n)(\nu-1+n)}q\left(\frac{\nu}{\nu+n}\right)\right\}.$$

From the Taylor expansion for $q(k/(k+n))$, it follows

$$q\left(\frac{k}{k+n}\right)=q\left(\frac{\nu}{\nu+n}\right)+\left(\frac{k}{k+n}-\frac{\nu}{\nu+n}\right)q'\left(\frac{\nu}{\nu+n}\right)+\frac{1}{2}\left(\frac{k}{k+n}-\frac{\nu}{\nu+n}\right)^2 q''(\xi_{k\nu}),$$

where $\xi_{k\nu}$ are between $k/(k+n)$ and $\nu/(\nu+n)$. Therefore we get

$$\begin{aligned}\widetilde{A}_n(f)=2\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{\nu+n}\right)\Bigg[\Bigg\{&\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\,p_{n\nu}\left(\frac{k}{k+n}\right)\\&-\frac{n^2}{(\nu+1+n)(\nu-1+n)}\Bigg\}\,q\left(\frac{\nu}{\nu+n}\right)\\&+\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left(\frac{k}{k+n}-\frac{\nu}{\nu+n}\right)p_{n\nu}\left(\frac{k}{k+n}\right)q'\left(\frac{\nu}{\nu+n}\right)\\&+\frac{1}{2}\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left(\frac{k}{k+n}-\frac{\nu}{\nu-n}\right)^2 p_{n\nu}\left(\frac{k}{k+n}\right)q''(\xi_{k\nu})\Bigg].\end{aligned}$$

Since $q''(x)$ is bounded, the statement will follow if we can prove the three sums

$$\begin{aligned}S_n^{(1)}=\sum_{\nu=0}^{\infty}\Bigg|&\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}p_{n\nu}\left(\frac{k}{k+n}\right)\\&-\frac{n^2}{(\nu+1+n)(\nu-1+n)}\Bigg|\left|q\left(\frac{\nu}{\nu+n}\right)\right|,\end{aligned}$$

$$S_n^{(2)}=\sum_{\nu=0}^{\infty}\left|\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left(\frac{k}{k+n}-\frac{\nu}{\nu+n}\right)p_{n\nu}\left(\frac{k}{k+n}\right)\right|\left|q'\left(\frac{\nu}{\nu+n}\right)\right|,$$

and

$$S_n^{(3)}=\sum_{\nu=0}^{\infty}\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left(\frac{k}{k+n}-\frac{\nu}{\nu+n}\right)^2 p_{n\nu}\left(\frac{k}{k+n}\right)$$

are bounded. For the third sum we have, using the estimation (15) for $T_{nr}(x)$ in Lemma 5,

$$(25)\quad S_n^{(3)} \leqq \frac{1}{n^2} \sum_{a_1 < \frac{k}{k+n} < b_1} \frac{n^2}{(k+1+n)(k-1+n)} \sum_{\nu=0}^{\infty} \left\{\left[(n+1)\frac{k}{k+n} - \left(1-\frac{k}{k+n}\right)\nu\right] - \frac{k}{k+n}\right\}^2 p_{n\nu}\left(\frac{k}{k+n}\right) + O(1)$$

$$\leqq \frac{1}{n^2} \sum_{a_1 < \frac{k}{k+n} < b_1} \left\{T_{n2}\left(\frac{k}{k+n}\right) - 2\frac{k}{k+n} T_{n1}\left(\frac{k}{k+n}\right) + \left(\frac{k}{k+n}\right)^2\right\} + O(1)$$

$$= \frac{1}{n^2} \sum_{a_1 < \frac{k}{k+n} < b_1} \left\{(n+1)\frac{k}{k+n} - \left(\frac{k}{k+n}\right)^2\right\} + O(1)$$

$$= O(1).$$

To estimate $S_n^{(1)}$, we can rewrite it in the from

$$S_n^{(1)} = \sum_{\nu=0}^{\infty} q_{n\nu} \left\{ \sum_{a_1 < \frac{k}{k+n} < b_1} \frac{n^2}{(k+1+n)(k-1+n)} p_{n\nu}\left(\frac{k}{k+n}\right) - \frac{n^2}{(\nu+1+n)(\nu-1+n)} \right\},$$

where $q_{n\nu} = \pm q\left(\frac{\nu}{\nu+n}\right)$ and $q_{n\nu} = 0$ for $\frac{\nu}{\nu+n} < \alpha$ or $\frac{\nu}{\nu+n} > \beta$. Thus the $q_{n\nu}$ are bounded. If we put

$$(26)\qquad Q_n(x) = \sum_{\nu=0}^{\infty} q_{n\nu} p_{n\nu}(x) = \sum_{a_1 < \frac{\nu}{\nu+n} < b_1} q_{n\nu} p_{n\nu}(x),$$

then since $\int_0^1 p_{n\nu}(x)dx = \frac{n+1}{(\nu+1+n)(\nu+2+n)}$, we have

$$n\int_0^1 Q_n(x)dx = \sum_{\nu=0}^{\infty} \frac{n(n+1)}{(\nu+1+n)(\nu+2+n)} q_{n\nu}$$

$$= \sum_{\nu=0}^{\infty} \frac{n^2}{(\nu+1+n)(\nu-1+n)} q_{n\nu} - \sum_{\nu=0}^{\infty} \frac{n(2n-\nu+1)}{(\nu+1+n)(\nu-1+n)(\nu+2+n)} q_{n\nu}.$$

Therefore

$$S_n^{(1)}=\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}Q_n\left(\frac{k}{k+n}\right)-n\int_0^1 Q_n(x)dx+O(1).$$

For the function (26), the condition (A) in Lemma 8 is checked by means of Lemma 7 and (B) by means of Lemma 6. Hence using Lemma 8, we get

$$S_n^{(1)}=O(1). \tag{27}$$

To estimate $S_n^{(2)}$, we write it in the form

$$S_n^{(2)}=\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\left\{\frac{k}{k+n}\overline{Q}_n\left(\frac{k}{k+n}\right)-\overline{\overline{Q}}_n\left(\frac{k}{k+n}\right)\right\}, \tag{28}$$

where

$$\overline{Q}_n(x)=\sum_{\nu=0}^{\infty}q'_{n\nu}p_{n\nu}(x),\quad \overline{\overline{Q}}_n(x)=\sum_{\nu=0}^{\infty}\frac{\nu}{\nu+n}q'_{n\nu}p_{n\nu}(x),$$

$$q'_{n\nu}=\pm q'\left(\frac{\nu}{\nu+n}\right)\text{and } q'_{n\nu}=0 \text{ for } \frac{\nu}{\nu+n}<\alpha \text{ or } \frac{\nu}{\nu+n}>\beta.$$

Since $\int_0^1 xp_{n\nu}(x)dx=\frac{(n+1)(\nu+1)}{(\nu+1+n)(\nu+2+n)(\nu+3+n)}$, we have

$$\left|\int_0^1 x\overline{Q}_n(x)dx-\int_0^1\overline{\overline{Q}}_n(x)dx\right| \tag{29}$$

$$=\left|\sum_{\nu=0}^{\infty}q'_{n\nu}\left\{\int_0^1 xp_{n\nu}(x)dx-\frac{\nu}{\nu+n}\int_0^1 p_{n\nu}(x)dx\right\}\right|$$

$$=\left|\sum_{\nu=0}^{\infty}q'_{n\nu}\left\{\frac{(n+1)(\nu+1)}{(\nu+1+n)(\nu+2+n)(\nu+3+n)}-\frac{(n+1)\nu}{(\nu+1+n)(\nu+2+n)(\nu+n)}\right\}\right|$$

$$=\left|\sum_{\nu=0}^{\infty}q'_{n\nu}\frac{(n+1)(n-2\nu)}{(\nu+n)(\nu+1+n)(\nu+2+n)(\nu+3+n)}\right|$$

$$\leqq \text{const.}\sum_{a_1<\frac{\nu}{\nu+n}<b_1}\frac{1}{n^2}$$

$$=O\left(\frac{1}{n}\right).$$

Like the function (26), also the functions $\overline{Q}_n(x)$ and $\overline{\overline{Q}}_n(x)$ satisfy the conditions (A) and (B) of Lemma 8. For functions $x\overline{Q}_n(x)$ this follows from the fact

$$|\{x\overline{Q}_n(x)\}''|\leqq 2|\overline{Q}_n'(x)|+|\overline{Q}_n''(x)|, 0\leqq x\leqq 1,$$

and Lemma 6 with $r=1, 2$. Applying (29) and Lemma 8 to the sum (28), we obtain

$$|S_n^{(2)}|\leqq\left|\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\frac{k}{k+n}\overline{Q}_n\left(\frac{k}{k+n}\right)-n\int^1 x\overline{Q}_n(x)dx\right|$$

$$+n\left|\int_0^1 x\overline{Q}_n(x)dx-\int_0^1\overline{\overline{Q}}_n(x)dx\right|$$

$$+\left|\sum_{a_1<\frac{k}{k+n}<b_1}\frac{n^2}{(k+1+n)(k-1+n)}\overline{\overline{Q}}_n\left(\frac{k}{k+n}\right)-n\int_0^1\overline{\overline{Q}}_n(x)dx\right|$$

$$=O(1).$$

Consequently, from (25), (27) and the estimation for $S_n^{(2)}$, we complete the proof of Theorem 1.

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