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SOME FUNCTION-THEORETIC NULL SETS

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1. Let E be a totally disconnected compact set in the complex z-plane and let G be the complementary domain of E with respect to the extended z-plane. Consider a domain in G whose relative boundary consists of at most a countable number of analytic curves clustering nowhere in G. Such a domain is called a subregion in G. If for any subregion in G there exists no non-constant single-valued bounded analytic function whose real part vanishes continuously on its relative boundary, then the set E is said to be in the class N_B^0.

It is known that if E is of logarithmic capacity zero, then E belongs to the class N_B^0 and that there exists a compact set of positive logarithmic capacity and belonging to N_B^0 (Kuroda [5]).

It is also known that there exists no non-constant single-valued bounded analytic function in the complementary domain of $E \in N_B^0$, that is, N_B^0 is a subclass of the class N_B in the sense of Ahlfors-Beurling [1].

If E is of logarithmic capacity zero, then there exists an Evans-Selberg's potential which is harmonic in G except at $z=\infty$ and whose boundary value at every point of E is positively infinite. Such a function plays an important role to study the covering property of meromorphic functions in G.

In this paper, we shall treat Noshiro's theorem on cluster sets [10] in detail. In §2, by the argument due to Matsumoto [7], we shall give a sufficient condition in order that there exists an analogous function to an Evans-Selberg's potential in the subregion inside G. As its application, in §3 we shall prove a theorem which is an improvement of Noshiro's theorem [10] on cluster sets under the so-called Hervé's condition. §4 is devorted to show that in the theorem, Hervé's condition can not be dropped. In Appendix, Kuroda's criterion for E to be in the class N_B^0 is proved in a correct form.

2. First we shall prove the following.

THEOREM 1. *If E is a compact set of the class N_B^0, then any closed subset E_0 of E is also in the class N_B^0.*

PROOF. Contrary to the assertion, we suppose that there exists a closed subset E_0 of E not belonging to N_B^0.

We denote by G and G_0 the complementary domains of E and E_0 with respect to the extended z-plane, respectively. Then there exist a subregion Δ_0 in G_0, whose boundary consists of a closed subset of E_0 and the relative boundary γ_0, and a non-constant single-valued bounded analytic function $f(z)$ in Δ_0 whose real part vanishes continuously on γ_0. We put

$$\gamma_0 - \gamma_0 \cap E = \gamma \quad \text{and} \quad \Delta_0 - \Delta_0 \cap E = \Delta .$$

It is obvious that Δ is a subregion in G with the relative boundary γ and the above function $f(z)$ is also non-constant, single-valued, bounded and analytic in Δ and the real part of $f(z)$ vanishes continuously on γ. Hence the set E does not belong to N_B^0, which is a contradiction.

Using Theorem 1, we can get the following theorem.

THEOREM 2. *If Δ is a subregion in G whose boundary consists of the relative boundary γ and a compact set E^* belonging to N_B^0 and if each point of E^* belongs to a non-degenerate boundary continuum of Δ, then there exists a positive harmonic function $u(z)$ in $\Delta \cup \gamma$ whose boundary value at each point of* E* *is positively infinite.*

PROOF. We denote by $\{D_n\}$ $(n = 1, 2, \cdots)$ the sequence of such complementary continua of Δ with respect to the extended z-plane that for each n, the boundary of D_n contains at least one point of E^*. Let Δ_n $(n=1, 2, \cdots)$ be the complementary domain of D_n with respect to the extended z-plane.

Since D_n is a non-degenerate continuum by our assumption, Δ_n is a simply connected domain of hyperbolic type containing Δ. The boundary of Δ_n consists of a part γ_n of γ and a compact subset E_n of E^* and clearly $E^* = \bigcup_{n=1}^{\infty} E_n$.

Since E_n belongs to N_B^0 from Theorem 1, the harmonic measure of E_n with respect to the simply connected domain Δ_n vanishes (cf. Kuroda [5]). Therefore, by virtue of a theorem due to F. and M. Riesz [11], there exists a function $u_n(z)$ such that $u_n(z)$ is positive and harmonic in $\Delta_n \cup \gamma_n$ and such that the boundary value of $u_n(z)$ at every point of E_n is positively infinite. Further, we can find a sequence $\{c_n\}$ $(n = 1, 2, \cdots)$ of positive numbers such that the series $\sum_{n=1}^{\infty} c_n u_n(z_0)$ converges at a fixed point z_0 in Δ.

By Harnack's principle, the series $\sum_{n=1}^{\infty} c_n u_n(z)$ converges uniformly to a limiting function $u(z)$ on any compact subset of $\Delta \cup \gamma$. It is evident that $u(z)$ satisfies the condition of the theorem.

3. Let D be a domain on the z-plane, Γ its boundary, E a totally disconnected compact set contained in Γ and z_0 a point of E such that $U(z_0) \cap (\Gamma - E) \neq \emptyset$ for every neighborhood $U(z_0)$ of z_0. Let $f(z)$ be a non-constant, single-valued and meromorphic function in D. Suppose that the set $\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is not empty. Here $C_D(f, z_0)$ and $C_{\Gamma-E}(f, z_0)$ are the interior cluster set and the boundary cluster set of $f(z)$ at z_0 (cf. Noshiro [9]).

The following was proved by Tsuji [13]:

If E is of logarithmic capacity zero, then Ω is an open set and $\Omega - R_D(f, z_0)$ is at most of logarithmic capacity zero. Here $R_D(f, z_0)$ is the range of values of $f(z)$ at z_0 (cf. Noshiro [9]).

Noshiro [10] considered the case of $E \in N_B^0$ and proved the following:

If E belongs to the class N_B^0, then Ω is an open set and $\Omega - R_D(f, z_0)$ is an at most countable union of sets of the class N_B.

Now we prove the following as an application of Theorem 2.

THEOREM 3. *If E belongs to the class N_B^0 and if each point of E belongs to a non-degenerate boundary continuum of D, then the set $\Omega - R_D(f, z_0)$ is of logarithmic capacity zero.*

REMARK. The second assumption that each point of E belongs to a non-degenerate boundary continuum of D, is called Hervé's condition for E (cf. Hervé [3]).

PROOF. We follow an argument due to Noshiro [9].

We denote by e_n $(n = 1, 2, \cdots)$ the set of values in Ω which $f(z)$ does not take in $\{z \mid |z - z_0| < 1/n\} \cap D$. Then it is easy to see that e_n is a closed set with respect to Ω, $e_n \subset e_{n+1}$ and $\Omega - R_D(f, z_0) = \bigcup_{n=1}^{\infty} e_n$. So, if we suppose the contrary to the assertion, then there exists a set e_n of positive logarithmic capacity.

We can find a point $w_0 \in e_n$ such that for any positive number ρ the part of e_n contained in the disc $|w - w_0| < \rho$ is of positive logarithmic capacity. We select a positive number r such that the circle K: $|z - z_0| = r$ does not intersect E and $f(z) \neq w_0$ on $K \cap D$ and such that w_0 does not belong to the closure M_r of $\bigcup_{\zeta} C_D(f, \zeta)$ for ζ belonging $(\Gamma - E) \cap (\overline{K})$, where $(\overline{K})$ denotes the closure of the interior (K) of K.

We can choose a positive number ρ_0 less than the distance of w_0 from M_r such that $|f(z)-w_0|>\rho_0$ on $K\cap D$. Since $w_0\in C_D(f,z_0)$, the function $w=f(z)$ takes a value belonging to (c): $|w-w_0|<\rho_0$ at $z_1\in(K)\cap D$. We consider the component Δ of the inverse image of (c) inside $(K)\cap D$ by $w=f(z)$ which contains the point z_1. Obviously, Δ is a subregion in the complementary domain of E with respect to the extended z-plane and the boundary of Δ consists of a closed subset E^* of E and at most a countable number of analytic curves γ.

Since, by the assumption, Δ satisfies the condition of Theorem 2, there exists a positive harmonic function $u(z)$ in $\Delta\cup\gamma$ having the positively infinite boundary value at each point of E^*.

Since $(c)\cap e_n$ is of positive logarithmic capacity, we can find a closed subset e of $(c)-e_n$ such that e is of positive logarithmic capacity. So there exists a positive bounded harmonic function $\omega(w)$ in $(c)-e$ which vanishes continuously on the circle $c:|w-w_0|=\rho_0$. We consider the composed function $\omega(f(z))$ in Δ.

By the maximum principle, we have

$$\omega(f(z))\leqq\frac{u(z)}{\lambda}$$

in Δ for any positive number λ, whence follows that $\omega(f(z))\equiv 0$ in Δ. Thus we arrive at a contradiction.

4. In the next section we shall show that Herve's condition in Theorem 3 can not be dropped.

For the purpose, first we prepare an example which guarantees the existence of a compact set E of positive logarithmic capacity which belongs to N_B^0 and of a single-valued meromorphic function $f(z)$ in the complementary domain D of E such that $f(z)$ has an essential singularity at every point of E and such that the set of exceptional values of $f(z)$ in Picard's sense at each point of E is of positive logarithmic capacity but belongs to N_B^0. This example was used for the other purpose in [6].

Consider a general Cantor set $E(p_1,p_2,\cdots)$ on the w-plane. This set is constructed as follows.

Let p_n $(n\geqq 1)$ be a positive number greater than 1 and delete an open interval with length $1-1/p_1$ from the closed interval $I_0=\left[-\frac{1}{2},\frac{1}{2}\right]$ on the real axis of the w-plane so that there remains the closed set I_1 which consists of two closed intervals I_1^i $(i=1,2)$ with equal length $l_1=1/2p_1$. In general, if I_n consists of closed intervals I_n^i $(i=1,2,\cdots,2^n)$ of equal length $l_n=1/2^np_1\cdots p_n$, we delete an open interval of length $l_n(1-1/p_{n+1})$ from every I_n^i so that there remain two closed intervals $I_{n+1}^{2i-1}, I_{n+1}^{2i}$ $(i=1, \cdots, 2^n)$ with equal length $1/(2^{n+1}p_1 \cdots p_{n+1})$.

The set $E(p_1, p_2, \cdots)$ is the set of intersection $\bigcap_{n=1}^{\infty} I_n$. It is known that $E(p_1, p_2, \cdots)$ is of positive logarithmic capacity if and only if

$$(1) \qquad \sum_{n=1}^{\infty} \frac{\log p_n}{2^n} < +\infty$$

(cf. Nevanlinna [8]).

Denote by F the complementary domain of $E(p_1, p_2, \cdots)$ with respect to the extended w-plane.

We describe circles

$$K_0^1: |w|=1, \ K_n^i: |w-w_n^i| = r_n \quad (n \geqq 1, 1 \leqq i \leqq 2^n)$$

in F where w_n^i is the middle point of I_n^i, $r_n = \dfrac{1}{2^n p_0 p_1 \cdots p_{n-1}}\left(1-\dfrac{1}{2p_n}\right)$ and $p_0 = 1$.

Clearly K_n^{2i-1} and K_n^{2i} are tangent outside each other and if

$$(2) \qquad 1 + 2p_{n-1}p_n > 3p_n \quad (n \geqq 2),$$

then K_n^{2i-1} and K_n^{2i} are enclosed by K_{n-1}^i $(1 \leqq n, \ 1 \leqq i \leqq 2^{n-1})$. Let F_n^i be the doubly connected domain surrounded by three circles K_n^{2i-1}, K_n^{2i} and K_{n-1}^i $(n \geqq 1)$ and let F_n be the domain bounded by $\bigcup_{i=1}^{2^n} K_n^i$ and containing the point $z=\infty$ in its interior. We make a slit L_n^i in every F_n^i such that L_n^i is contained in $|w-w_{n-1}^i| \leqq 2r_n$ $(w_0^1 = 0)$ and such that only one end point of L_n^i lies on $K_n^{2i-1} \cup K_n^{2i}$. We put

$$F^0 = F - \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_0^1,$$

$$F_k^1 = F - \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_1^k, \ (k=1, 2^m),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$F_k^m = F - \bigcup_{n=m+1}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_m^k, \ (k=1, \cdots, 2^m),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

First we connect two replicas of F^0 with each other crosswise across the slit L_0^1 and denote by $\widehat{F^0}$ the resulting surface which has two free slits corresponding to every L_1^k $(k=1,2)$. Next we take a replica of F_k^1 and connect it with $\widehat{F^0}$ crosswise across a free slit corresponding to L_1^k $(k=1,2)$. Doing this for every free slits of $\widehat{F^0}$ corresponding to L_1^k $(k=1,2)$, we get the resulting surface $\widehat{F^1}$ which has $2(1+2)$ sheets and $2(1+2)$ free slits corresponding to each L_2^k $(k=1,\cdots,2^2)$. In general, we connect a replica of F_k^n with $\widehat{F^{n-1}}$ crosswise across a free slit corresponding to L_n^k and proceed this for all slits of $\widehat{F^{n-1}}$ corresponding to L_n^k $(k=1,\cdots,2^n)$. Thus we get the surface $\widehat{F^n}$ with $\prod_{i=0}^{n}(1+2^i)$ sheets.

Continuing the procedure infinitely, we obtain the surface $\widehat{F}$ of planar character which covers no point of the set $E(p_1, p_2, \cdots)$.

This surface $\widehat{F}$ is considered as a limiting surface of $\widehat{F^n}$ and every $\widehat{F^n}$ is a subdomain of $\widehat{F}$. Denote by $\widehat{F}_n$ the part of $\widehat{F^n}$ lying over F_{n+1}.

It is not so difficult to see that $\{\widehat{F}_n\}_{n=1}^{\infty}$ is an exhaustion of $\widehat{F}$ and that the number $N(n)$ of doubly connected components $\widehat{F}_n^i$ of $\widehat{F}_{n+1}-\overline{\widehat{F}_n}$ equals $2^n\prod_{i=0}^{n-1}(1+2^i)$.

Denote $\log\mu_n^i$ the harmonic modulus of $\widehat{F}_n^i$. Putting $\log\nu_n=\min_i\log\mu_n^i$, we easily have

$$\log\nu_n>\log\frac{r_n}{2r_{n+1}},$$

because $\widehat{F}_n^i$ contains the univalent annulus lying over $2r_{n+2}<|w-w_{n+1}^i|<r_{n+1}$.

Therefore, we have

$$\sum_{i=0}^{n}\log\nu_i-\log N(n)>\log(p_1p_2\cdots p_{n+1})-\frac{n(n+1)}{2}\log 2+\log\frac{1-\dfrac{1}{2p_1}}{1-\dfrac{1}{2p_{n+2}}}.$$

So, if we take p_n as such as

$$p_n=2^{(n+1)^2}, \tag{3}$$

then (1) and (2) are valid and

$$\lim_{n\to\infty}\left\{\sum_{i=0}^{n}\log\nu_i - \log N(n)\right\} = +\infty \quad\text{and}\quad \lim_{n\to\infty}\log\nu_n = +\infty\,.$$

Hence, by a criterion proved in Appendix, any subregion on the covering surface $\hat{F}$ carries no non-constant single-valued bounded analytic function with the real part vanishing continuously on its relative boundary provided that (3) holds.

Now we map $\hat{F}$ onto a domain G on the extended z-plane in a one-to-one conformal manner such that G contains the point $z=\infty$. Denote by $\hat{f}(z)$ the inverse of this conformal mapping.

By the definition the complementary set E of G with respect to extended z-plane belongs to N_B^0.

We denote by $w=\varphi(p)$ projection of $\hat{F}$ on the extended w-plane and we put $w=\varphi(\hat{f}(z))=f(z)$. It is easy to see that $w=f(z)$ has an essential singularity at every point of E and has the set $E(p_1, p_2, \cdots)$ as the set of exceptional values in Picard's sense in any neighborhood of its essential singularities.

Further, as mentioned already, (1) implies that the set $E(p_1, p_2, \cdots)$ is of positive logarithmic capacity, so we see from Nevanlinna's theorem [8] that the set E is also of positive logarithmic capacity.

Thus we see that the set E and the function $f(z)$ satisfy the requirements stated in the begining of this section.

5. From the above example, we can show the fact that Theorem 3 does not hold if we exclude Herve's condition on E.

In fact, we take a circle $K_m^i = K$ in the above example and denote by S a component of $\hat{F}-\hat{\bar{F}}_{m-1}$ whose projection lies on the disc (K) bounded by K. The counter image D of S by $\hat{f}(z)$ is a subregion in G whose boundary consists of a countable number of closed analytic curves Γ and a compact subset E^* of E. Theorem 1 implies that E^* belongs to N_B^0. Each point z_0 of E^* does not satisfy Hervé's condition, because the circle K does not intersect with $E(p_1, p_1, \cdots)$.

Obviously, $C_D(f, z_0)$ is the closed disc $(\bar{K})$ and $C_{\Gamma-E^*}(f, z_0)$ is the circle K, so $\Omega=C_D(f, z_0)-C_{\Gamma-E^*}(f, z_0)$ is the open disc (K).

Further $\Omega-R_D(f, z_0)$ coincides with the compact set $(K)\cap E(p_1, p_2, \cdots)$ of positive logarithmic capacity.

REMARK. Hällström-Kametani's theorem [2], [4] can be formulated in the following form.

Let E be a compact set of logarithmic capacity zero contained in a domain D. Suppose that $w=f(z)$ is single-valued meromorphic in $D-E$ and has an essential singularity at every point z_0 of E. Then the complement of $R_{D-E}(f, z_0)$ is at most of capacity zero.

By our example, it is immediately seen that in the above Hällström-Kametani's theorem the condition "E is of logarithmic capacity zero" can not be replaced by the condition "E belongs to N_B^0.

APPENDIX

A sufficient condition for a compact set E to belong to N_B^0 was stated by Kuroda [5], however, as he pointed out, his statement and the proof were incorrect, so here we state a correct form and its proof given by himself. It is quite similar to the proof of a criterion for $E \in N_B$ given in Appendix I of Sario-Noshiro's book [12].

Let E be a totally disconnected compact set in the complex z-plane and let F be the complementary domain of E with respect to the extended z-plane.

Let $\{F_n\}$ $(n=0, 1, \cdots)$ be an exhaustion of F such thch that F_n is compact with respect to F and the boundary Γ_n of F_n consists of a finite number of analytic curves in F and such that each connected component of $F-\overline{F}_n$ is non-compact and further such that $F_n \cup \Gamma_n \subset F_{n+1}$.

The open set $F_n - \overline{F}_{n-1}$ $(n \geqq 1)$ consists of a finite number of connected components F_n^k $(k=1, 2, \cdots, N(n))$. We denote by $\log \mu_n^k$ the harmonic modulus of F_n^k and we put $\max_{1 \leqq k \leqq N(n)} \log \mu_n^k = \log \nu_n$.

THEOREM A. *If there exists an exhaustion of the complementary domain of E such that, for a positiv constant δ,*

$$\log \nu_j > \delta \quad (j = 1, 2, \cdots)$$

and

$$\limsup_{n \to \infty} \left\{ \sum_{i=1}^{n} \log \nu_i - \log N(n) \right\} = +\infty,$$

then E belongs to N_B^0.

PROOF. We denote by $\log \mu_n$ the harmonic modulus of $F_n - \overline{F}_{n-1}$ and consider the graph $0 < u(z) < R = \sum_{n=1}^{\infty} \log \mu_n$, $0 < v(z) < 2\pi$ associated with the exhaustion $\{F_n\}$ $(n=0, 1, \cdots)$ in the sense of Noshiro [9].

The niveau curve $\gamma_r: u(z)=r$ $(0<r<R)$ consists of a finite number of analytic closed curves γ_r^i $(i=1, 2, \cdots, m(r))$.

We put

$$\Lambda_i(r)=\int_{\gamma_r^i} dv\,, \qquad \max_{1\leqq i\leqq m(r)} \Lambda_i(r)=\Lambda(r) \quad \text{and} \quad \tau_n=\sum_{j=1}^{n}\log\mu_j\,.$$

Suppose that there exists a non-compact subregion Δ on F with the relative boundary C and a non-constant single-valued bounded analytic function $f(z)$ in Δ whose real part $U(z)$ vanishes continuously at every point on C.

We denote by Δ_r the open subset of Δ, where $u(z)<r$. The part θ_r of the niveau curve γ_r contained in Δ consists of a finite number of components θ_r^i $(i=1, 2, \cdots, n(r))$. We set $\Theta(r)=\max\limits_{1\leqq i\leqq n(r)}\int_{\theta_r^i} dv$.

If we denote by $D(r)$ the Dirichlet integral of $f(z)$ taken over Δ_r, then the argument of Kuroda [5] yields

$$e^{2\pi\int_0^r \frac{dr}{\Theta(r)}}\leqq\frac{D(r)}{D(0)}\,.$$

Since $\Theta(r)\leqq\Lambda(r)$, it follows that

$$e^{2\pi\int_0^r \frac{dr}{\Lambda(r)}}\leqq\frac{D(r)}{D(0)}\,.$$

On the other hand, it holds that

$$\frac{d}{dr}\left(\int_{\theta_r} U^2 dv\right)=2\int_{\theta_r} U\frac{\partial U}{\partial u}\,ds=2D(r)$$

for $\tau_{n-1}<r<\tau_n$ $(n=1, 2, \cdots)$, whence follows that

$$\int_{\tau_{n-1}}^{\tau_n} D(r)\,dr=\lim_{\to\tau_n-0}\int_{\theta_r} U^2 dv-\lim_{r\to\tau_{n-1}+0}\int_{\theta_r} U^2 dv\leqq 2\pi M^2\,,$$

where $M=\max\limits_{\Delta}|U|$.

Therefore, we have

$$(*)\qquad \int_{\tau_{n-1}}^{\tau_n} e^{2\pi\int_0^r\frac{dr}{\Lambda(r)}}\,dr\leqq\frac{2\pi M^2}{D(0)}\,.$$

It is evident that

$$\Lambda_i(r) \leqq 2\pi \frac{\log \mu_j}{\log \mu_j^k} \leqq 2\pi \frac{\log \mu_j}{\log \nu_j}$$

for $\gamma_r^i \subset F_j^k$. Hence, we have

$$\int_0^r \frac{dr}{\Lambda(r)} \geqq \frac{1}{2\pi} \sum_{j=1}^{n-1} \log \nu_j + \frac{1}{2\pi} \frac{\log \nu_n}{\log \mu_n} (r - \tau_{n-1})$$

for $\tau_{n-1} < r < \tau_n$, and

$$\int_{\tau_{n-1}}^{\tau_n} e^{2\pi\int_0^r \frac{dr}{\Lambda(r)}} dr \geqq \frac{\log \mu_n}{\log \nu_n} e^{\sum_{j=1}^{n} \log \nu_j} (1 - e^{-\log \nu_n}).$$

Since
$$\frac{1}{\log \mu_n} = \sum_{k=1}^{n} \frac{1}{\log \mu_n^k} \leqq \frac{N(n)}{\log \nu_n},$$

(**)
$$\int_{\tau_{n-1}}^{\tau_n} e^{2\pi\int_0^r \frac{dr}{\Lambda(r)}} \geqq e^{\sum_{j=1}^{n} \log \nu_j - \log N(n)} (1 - e^{-\delta}).$$

By (*), (**) and the assumption of theorem, E belongs to N_B^0.

From the proof of Theorem A, we can get easily the following theorem.

THEOREM B. *If there exsists an exhaustion of the complementary domain of E such that*

$$\limsup_{n\to\infty} \int_{\tau_{n-1}}^{\tau_n} e^{2\pi\int_0^r \frac{dr}{\Lambda(r)}} dr = +\infty,$$

then E belongs to N_B^0.

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