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ON η-COHOMOLOGY OF η^{*}-INVARIANT FORMS IN A COMPACT SASAKIAN SPACE

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Dedicated to Professor Tannaka on his 60th Birthday.

Introduction. We have discussed the C-harmonic form and C^{*}-harmonic form in a compact Sasakian space M^{2m+1} in the recent papers and obtained especially the following theorem : The maximun number c_p of linearly independent C-harmonic p-forms $(p \leqq m)$ is given by $c_p = b_p + b_{p-2} + \cdots + b_{p-2r}$, where b_i means the i-th Betti number of M^{2m+1} and r is the integral part of $p/2$. Thus each de Rahm cohomology class contains generally more than one C-harmonic forms, which differs from the existence of just one harmonic form in each class. It comes into a question how to introduce a new equivalence relation in the space of closed differential forms so that each equivalence class can contain just one C-harmonic form. The harmonic form is characterized by its minimum length property in each de Rahm cohomology class. It is natural to expect the corresponding theorem for C-harmonic form. On the other hand, the C^{*}-harmonic form was introduced as the dual of the C-harmonic form and it is C-harmonic if and only if it is harmonic. We shall define in this paper the η-harmonic form, which is self-dual, containing the C-harmonic form and C^{*}-harmonic form as its special cases. This notion suggests us to introduce new differential operators d_η and δ_η, corresponding to usual d and δ, by which we can consider a cohomology of η^{*}-invariant differential forms.

Preliminary facts on differential forms in a Sasakian space are given in §1. We shall define in §2 d_η, δ_η and the η-harmonic forms and prove a theorem which asserts that the maximum number e_p $(p \leqq m)$ of η-harmonic p-forms is givn by $e_p = b_p + b_{p-1} + \cdots + b_0$, (Theorem 2.9). §3 will be devoted to the discussion on a cohomology of η^{*}-invariant differential forms to get the main Theorem 3.6.

We would like to express our gratitude to Professor Shigeo Nakano for his suggestion given to one of us so as to introduce new operators.

1. Preliminaries.[1] Let M be an n-dimensional Riemannian space with the

1) As to notations we follow Y. Ogawa [3].

metric tensor g. We denote by $\omega^{\#}$ the associated vector field of a 1-form ω with respect to the metric g, i. e.,

$$g(\omega^{\#}, X) = \omega(X),$$

for any vector X. The notation $X_{\#}$ for a vector field X means the similarly identified 1-form. If M admits a 1-form η and a tensor J of type $(1, 1)$ which satisfy the equations

$$g(\eta^{\#}, \eta^{\#}) = 1,$$

$$J^2 X = -X + \eta(X)\eta^{\#},$$

for any vector X, then it is called an almost contact space. It is well known that M has an odd dimension and is orientable. We define a 2-form Ω on M by

$$\Omega(X, Y) = g(JX, Y).$$

The inner product and exterior product with respect to η is denoted by $i(\eta)$ and $e(\eta)$ respectively. An almost contact space M with (J, η, g) is called a normal contact metric space or Sasakian space if the vector field $\eta^{\#}$ is Killing and the equations

$$JX = \nabla_X \eta^{\#},$$

$$\nabla_X \Omega = e(\eta) X_{\#}$$

are satisfied for any vector X.

Suppose that M is a compact Sasakian space of dimension $n = 2m+1$. We denote the dual form of a p-form u by $*u$, and denote the adjoint operator of exterior differential d by δ. Then we have for any p-form u

$$i(\eta)u = (-1)^{p-1} * e(\eta) * u,$$

$$\delta u = (-1)^p * d * u.$$

We define the operators L and Λ by

$$Lu = d\eta \wedge u, \quad \Lambda u = *L*u,$$

for any p-form u. The operator Λ annihilates 0- and 1-forms. We write for p-forms u and v the global inner product of u and v as

$$(u, v) = \int_M u \wedge *v,$$

then we get

$$(du, v) = (u, \delta v),$$

$$(e(\eta)u, v) = (u, i(\eta)v),$$

$$(Lu, v) = (u, \Lambda v).$$

We call a form u to be horizontal or effective if it satisfies $i(\eta)u = 0$ or $\Lambda u = 0$ respectively. The following results are known (see [3]).

LEMMA 1.1. *In a Sasakian space, we have*

$$L = e(\eta)d + de(\eta),$$

$$\Lambda = i(\eta)\delta + \delta i(\eta).$$

LEMMA 1.2. *In a Sasakian space, the operator* L *(resp.* Λ*) commutes with the operators* $i(\eta)$, $e(\eta)$ *and* d *(resp.* $i(\eta)$, $e(\eta)$ *and* δ*).*

LEMMA 1.3. *In a Sasakian space, the Lie derivative* $\theta(\eta^{\#})$ *with respect to* $\eta^{\#}$ *satisfies the equation*

$$\theta(\eta^{\#}) = -\delta e(\eta) - e(\eta)\delta$$

and commutes with $e(\eta)$, L *and* Λ.

LEMMA 1.4. *In a Sasakian space, we have for a* p*-form* u

$$(\Lambda L - L\Lambda)u = 4(m-p)u + 4e(\eta)i(\eta)u.$$

A p-form u in a Sasakian space is called to be C-harmonic if it satisfies

$$du = 0, \quad \delta u = e(\eta)\Lambda u,$$

and to be C^*-harmonic if it satisfies

$$\delta u = 0, \quad du = i(\eta)Lu.$$

It is an easy fact that a p-form u is C^*-harmonic if and only if it is a dual form of a C-harmonic form. We know that

PROPOSITION 1.5. *In a compact* $(2m+1)$-*dimensional Sasakian space, any C-harmonic p-form u* $(p \leqq m)$ *is horizontal and can be written uniquely in the form*

$$u = \sum_{k=0}^{r} L^k \phi_{p-2k} ,$$

where ϕ_{p-2k} *is a harmonic* $(p-2k)$-*form and r is the integral part of* $p/2$.

Let H_p, C_p and C_p^* be the vector spaces of all harmonic, C-harmonic and C^*-harmonic p-forms respectively. Then we know

PROPOSITION 1.6. *In a compact Sasakian space, we have*

$$H_p = C_p \cap C_p^* ,$$

for any integer p.

PROPOSITION 1.7. *In a compact Sasakian space, the mapping* $e(\eta)|C_p$ *is an isomorphism into* C_{p+1}^*, *and* $i(\eta)|C_{p+1}^*$ *is a homomorphism onto* C_p, *if* $p \leqq m$.

2. η-harmonic forms. Let M be an n $(=2m+1)$-dimensional compact Sasakian space. We introduce operators d_η and δ_η by

$$d_\eta u = du - i(\eta)Lu ,$$

$$\delta_\eta u = \delta u - e(\eta)\Lambda u ,$$

for any p-form u and put $\Delta_\eta = d_\eta \delta_\eta + \delta_\eta d_\eta$. An η-harmonic form u is defined by the following two equations

$$d_\eta u = 0 \quad \text{and} \quad \delta_\eta u = 0 .$$

If a p-form u satisfies $d_\eta u = 0$, then we call it to be η-closed. The following lemmas are proved easily, taking account of

$$i(\eta)e(\eta) + e(\eta)i(\eta) = \text{identity} .$$

LEMMA 2.1. *In a Sasakian space, we have*

$$d_\eta e(\eta) + e(\eta)d_\eta = 0 ,$$

$$\delta_\eta i(\eta) + i(\eta)\delta_\eta = 0 .$$

LEMMA 2.2. *In a Sasakian space, we have*

$$d_\eta i(\eta) + i(\eta) d_\eta = \theta(\eta^\#) ,$$

$$\delta_\eta e(\eta) + e(\eta)\delta_\eta = -\theta(\eta^\#).$$

LEMMA 2.3. *In a Sasakian space, we have*

$$d_\eta^{\,2} = -L\theta(\eta^\#) ,$$

$$\delta_\eta^{\,2} = \Lambda\theta(\eta^\#) .$$

LEMMA 2.4. *In a compact Sasakian space, δ_η is an adjoint operator of d_η and Δ_η is a self adjoint operator.*

THEOREM 2.5. *In a compact Sasakian space, a form u is η-harmonic if and only if it satisfies*

$$\Delta_\eta u = 0 .$$

PROOF. It is evident that the following equation

$$(u, \Delta_\eta u) = (d_\eta u, d_\eta u) + (\delta_\eta u, \delta_\eta u)$$

holds good. Hence the theorem follows easily.

THEOREM 2.6. *An η-harmonic form u in a compact Sasakian space is $\eta^\#$-invariant, that is, it satisfies*

$$\theta(\eta^\#) u = 0 .$$

PROOF. Using Lemma 2.2, we have

$$\theta(\eta^\#) u = d_\eta i(\eta) u = -\delta_\eta e(\eta) u ,$$

for any η-harmonic form u. Since δ_η is an adjoint operator to d_η, we have

$$\begin{aligned}(\theta(\eta^\#)u, \theta(\eta^\#)u) &= -(d_\eta^{\,2} i(\eta)u, e(\eta)u)\\ &= (L\theta(\eta^\#) i(\eta)u, e(\eta)u) .\end{aligned}$$

As $i(\eta)$ commutes with $\theta(\eta^{\#})$ and L, we see that the last term vanishes, hence it is concluded that $\theta(\eta^{\#})u=0$.

THEOREM 2.7. *For an η-harmonic form u in a compact Sasakian space, $e(\eta)i(\eta)u$ (resp. $i(\eta)e(\eta)u$) is a C^*(resp. C)-harmonic form.*

PROOF. By virtue of Lemma 1.1 and Theorem 2.6, we have

$$\begin{aligned} d(e(\eta)i(\eta)u) &= Li(\eta)u+e(\eta)i(\eta)du \\ &= Li(\eta)u = i(\eta)L(e(\eta)i(\eta)u), \end{aligned}$$

and

$$\begin{aligned} \delta(e(\eta)i(\eta)u) &= -e(\eta)\delta i(\eta)u \\ &= -e(\eta)\Lambda u + e(\eta)i(\eta)\delta u. \end{aligned}$$

As $\delta u = e(\eta)\Lambda u$, we see $e(\eta)i(\eta)\delta u$ equal $e(\eta)\Lambda u$, and that $\delta(e(\eta)i(\eta)u)$ vanishes. Therefore the form $e(\eta)i(\eta)u$ is C^*-harmonic. The fact that $i(\eta)e(\eta)u$ is C-harmonic can be proved similarly.

Let H_p^η be the vector space of all η-harmonic p-forms.

THEOREM 2.8. *In a compact Sasakian space, we have*

$$H_p^\eta = C_p \cup C_p^*,$$

for any integer p.

PROOF. Theorem 2.7 shows that H_p^η is contained in $C_p \cup C_p^*$. Conversely, if we take a C-harmonic form u and a C^*-harmonic form v, then we know that

$$\begin{aligned} Li(\eta)u &= 0, \\ \Lambda e(\eta)v &= 0 \end{aligned}$$

hold good (see [3]). Thus u and v satisfy

$$\begin{aligned} d_\eta u &= \delta_\eta u = 0, \\ d_\eta v &= \delta_\eta v = 0 \end{aligned}$$

and they are η-harmonic. This proves that $C_p \cup C_p^*$ is in H_p^η.

REMARK. An effective η-harmonic form is C^*-harmonic.

THEOREM 2.9. *In a compact Sasakian space, $dim H_p^\eta (= e_p)$ and the p-th Betti number $dim H_p (= b_p)$ satisfy the following relation*

$$e_p = b_p + b_{p-1} + \cdots + b_0 ,$$

where p dose not exceed m.

PROOF. From Proposition 1.6 and Theorem 2.8, we obtain

$$e_p = c_p + c_p^* - b_p ,$$

where $c_p = \dim C_p$ and $c_p^* = \dim C_p^*$. On the other hand, we know that under the condition $p \leqq m$ the relations

$$C_p = H_p \oplus LC_{p-2}, \quad C_p^* = H_p \oplus e(\eta)C_{p-1}$$

hold good. By virtue of Lemma 1.1 and Proposition 1.7, the operators $e(\eta)$: $C_{p-1} \to C_p^*$ and L : $C_{p-2} \to C_p$ are isomorphisms (see the proof of Theorem 3.3), hence we see that the equations

$$c_p^* - b_p = c_{p-1}, \quad c_p - b_p = c_{p-2}$$

are valid. Therefore we have

$$\begin{aligned} e_p &= c_p + c_{p-1} \\ &= b_p + c_{p-1} + c_{p-2} \\ &= b_p + b_{p-1} + \cdots + c_1 + c_0 . \end{aligned}$$

Since 0- and 1-C-harmonic forms coincide with 0- and 1-harmonic forms, we have $c_1 = b_1$ and $c_0 = b_0$. Hence we get

$$e_p = b_p + b_{p-1} + \cdots + b_0 .$$

3. **Cohomology of η-closed forms.** In a compact Riemannian space, it is well known that any p-form u can be decomposed as the form

$$u = u_0 + d\delta Gu + \delta dGu$$

where u_0 is a harmonic form and G is the Green's operator. u_0 is determined uniquely for u. Especially, for a closed form u, we can find a unique harmonic form u_0 such that

$$(*) \qquad u = u_0 + d\alpha$$

holds good for some α. If we denote by D_p and B_p the vector spaces of closed and derived differential p-forms, then the decomposition theorem of de Rahm and Kodaira asserts that H_p is isomorphic with D_p/B_p. In the following we consider a cohomology of $\eta^{\#}$-invariant forms in a Sasakian space analogous to d-cohomology theory.

THEOREM 3.1. *If a closed form u in a compact Sasakian space is $\eta^{\#}$-invariant, then the form α in the decomposition $(*)$ of u can be taken to be $\eta^{\#}$-invariant.*

PROOF. The decomposition of u can be written as

$$u = u_0 + d\delta\alpha',$$

where u_0 is harmonic. It is well known that a harmonic form is invariant by a Killing vector field $\eta^{\#}$. Since $\theta(\eta^{\#})$ commutes with d and δ, we see easily that the form $\theta(\eta^{\#})\delta\alpha' = \delta\theta(\eta^{\#})\alpha'$ is harmonic. Thus it must be zero and the form $\alpha = \delta\alpha'$ is the required $\eta^{\#}$-invariant form.

COROLLARY 3.2. *If a closed p-form u $(p \leqq m)$ in a compact Sasakian space is horizontal, then the form α taken in Theorem 3.1 satisfies*

$$di(\eta)\alpha = 0.$$

PROOF. From a theorem of [1], a harmonic p-form u_0 $(p \leqq m)$ is horizontal in a compact Sasakian space. Therefore it follows that

$$i(\eta)d\alpha = 0$$

in $(*)$. As a closed horizontal form is $\eta^{\#}$-invariant, we get, making use of Theorem 3.1,

$$di(\eta)\alpha = \theta(\eta^{\#})\alpha - i(\eta)d\alpha = 0.$$

THEOREM 3.3. *An η-closed p-form $(p < m)$ in a compact Sasakian space is $\eta^{\#}$-invariant.*

PROOF. Lemma 2.3 shows that the equation

$$L\theta(\eta^*)u = 0$$

holds good for an η-closed p-form u. We have

$$-(L \wedge \theta(\eta^*)u, \theta(\eta^*)u) = 4(m-p)(\theta(\eta^*)u, \theta(\eta^*)u) + 4(\theta(\eta^*)i(\eta)u, \theta(\eta^*)i(\eta)u)$$

by virtue of Lemma 1.4. Therefore if $m-p>0$, then $\theta(\eta^*)u$ is necessarily zero.

REMARK. For an η-closed m-form u, $\theta(\eta^*)u$ is effective and horizontal.

THEOREM 3.4. *In a compact Sasakian space, a closed horizontal p-form u ($p \leqq m$) can be decomposed as the following form*

$$u = v + d\alpha$$

where v is a C-harmonic form and α is a horizontal form.

PROOF. Let u be a closed horizontal form. Then by virtue of Corollary 3.2, we can decompose u as

$$u = \lambda_p + d\alpha_{p-1},$$

where λ_p is a harmonic p-form and $i(\eta)\alpha_{p-1}$ is closed. Since $i(\eta)\alpha_{p-1}$ is horizontal, we use Corollary 3.2 again and have

$$i(\eta)\alpha_{p-1} = \lambda_{p-2} + d\alpha_{p-3},$$

where λ_{p-2} and α_{p-3} satisfy the similar conditions as λ_p and α_{p-1}. We continue this process to get the equation

$$i(\eta)\alpha_{p-2r+1} = \lambda_{p-2r} + d\alpha_{p-2r-1},$$

where r is the integral part of $p/2$ and we set $\alpha_{-1} = 0$. As we have

$$\begin{aligned} d\alpha_{p-2k+1} &= de(\eta)i(\eta)\alpha_{p-2k+1} + di(\eta)e(\eta)\alpha_{p-2k+1} \\ &= L\lambda_{p-2k} + Ld\alpha_{p-2k-1} + di(\eta)e(\eta)\alpha_{p-2k+1}, \end{aligned}$$

we see that

$$d\alpha_{p-1} = \sum_{k=1}^{r} L^k\lambda_{p-2k} + di(\eta)e(\eta)\sum_{k=0}^{r} L^k\alpha_{p-2k-1}$$

holds good. Hence we obtain the relation

$$u = \sum_{k=0}^{r} L^k\lambda_{p-2k} + di(\eta)e(\eta)\sum_{k=0}^{r} L^k\alpha_{p-2k-1}\,.$$

Since $v = \sum_{k=0}^{r} L^k\lambda_{p-2k}$ is C-harmonic and $\alpha = i(\eta)e(\eta)\sum_{k=0}^{r} L^k\alpha_{p-2k-1}$ is horizontal, the decomposition $u = v + d\alpha$ is a required one.

LEMMA 3.5. *For an η-closed p-form $(p<m)$ in a compact Sasakian space, $e(\eta)u$ is η-closed too, and $i(\eta)u$ is closed and horizontal.*

PROOF. By virtue of Lemma 2.1, we have $d_\eta e(\eta)u = 0$ for an η-closed form u. Since $i(\eta)du = i(\eta)Li(\eta)u = 0$, the second half of the lemma comes from Theorem 3.3.

Consider the case $p<m$ and put

A_p = the vector space of $\eta^{\#}$-invariant p-forms,

B_p^η = the vector space of forms $d_\eta\gamma$ with $\gamma \in A_{p-1}$,

D_p^η = the vector space of η-closed p-forms.

By Theorem 3.3 and Lemma 2.3 it holds that

$$A_p \supset D_p^\eta \supset B_p^\eta\,.$$

We shall introduce an equivalence relation in D_p^η, i.e., we call $u_1 \in D_p^\eta$ to be η-cohomologous to $u_2 \in D_p^\eta$ if $u_1 - u_2 \in B_p^\eta$. Each equivalence class is called an η-cohomology class. Then we have

THEOREM 3.6. *There exists one and only one η-harmonic p-form $(p<m)$ in each η-cohomology class of a $(2m+1)$-dimensional compact Sasakian space.*

PROOF. Let u be η-closed. Then $i(\eta)u$ is closed and horizontal by Lemma 3.5. Therefore we can take a C-harmonic form v and a horizontal form α such that the form $i(\eta)u$ can be written as

$$i(\eta)u = v + d\alpha\,.$$

Since $e(\eta)u$ is also η-closed, we have a C-harmonic form w and a horizontal form β satisfying

$$i(\eta)e(\eta)u = w + d\beta .$$

Therefore we have

$$\begin{aligned} u &= i(\eta)e(\eta)u + e(\eta)i(\eta)u \\ &= w + e(\eta)v + d\beta + e(\eta)d\alpha . \end{aligned}$$

Setting $\gamma = \beta - e(\eta)\alpha$, we can easily see that the equation

$$d_\eta\gamma = d\beta + e(\eta)d\alpha$$

holds good because of the horizontal property of α and β. On the other hand, since w is C-harmonic and $e(\eta)v$ is C^*-harmonic, the form $u_0 = w + e(\eta)v$ is η-harmonic by virtue of Theorem 2.8. Taking account of $i(\eta)d\alpha = i(\eta)d\beta = 0$ and $\theta(\eta^\#)e(\eta) = e(\eta)\theta(\eta^\#)$, we know that $\gamma \in A_{p-1}$. Thus the wanted decomposition $u = u_0 + d_\eta\gamma$ is obtained. The uniqueness of u_0 can be easily verified.

The statement of Theorem 3.6 is equivalent to the fact that the vector space H_p^η is isomorphic with the vector space D_p^η/B_p^η. Corresponding to the relation between the cohomology class of closed forms and the representative harmonic forms, we can get the following

THEOREM 3.7. *The η-harmonic p-form ($p < m$) in a $(2m+1)$-dimensional compact Sasakian space is characterized by the minimum length property in each η-cohomology class.*

PROOF. Let u be any η-closed p-form ($p < m$), which is written as $u = u_0 + d_\eta\gamma$ with the η-harmonic u_0 in the class of u and $\gamma \in A_{p-1}$. Then we have

$$\begin{aligned} (u, u) &= (u_0 + d_\eta\gamma, u_0 + d_\eta\gamma) \\ &= (u_0, u_0) + (d_\eta\gamma, d_\eta\gamma) \geqq (u_0, u_0) . \end{aligned}$$

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